Small ball inequality and low discrepancy constructions

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Uniform Distribution Theory and Applications

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The small ball inequality

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$n^{\frac{d-2}{2}} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

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- Sharpness: random signs/Gaussians.
- $\frac{d-1}{2}$ follows from an L^2 estimate.
- Connected to probability, approximation, discrepancy.
- Known: $\frac{d-1}{2} + \eta(d)$ for $d \geq 3$
  (DB, Lacey, Vagharshakyan, 2008)

Small ball inequality: "signed" version

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For dimensions $d \geq 2$, if all $\varepsilon_R = \pm 1$, we have

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- $\#\mathbb{H}_n^d \approx n^{d-1}$

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_2 \approx \Big(\#\mathbb{H}_n^d \cdot 2^n \cdot 1 \cdot 2^{-n} \Big)^{\frac{1}{2}} \approx n^{\frac{d-1}{2}}$$

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- **Thus**

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- **Thus**

$$\|\mathcal{H}_n\|_\infty \geq \langle \mathcal{H}_n, \Psi \rangle = \sum_{R:\,|R|=2^{-n}} 2^{-n} \approx n$$

Structure of the Riesz product

$$\Psi \stackrel{\text{def}}{=} \prod_{k=0}^{n} (1+f_k) = \begin{cases} 2^{n+1} & \text{if } f_k = +1 \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

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  $(k+1)^{st}$ binary digit of x_1 = $(n-k+1)^{st}$ digit of x_2.
- if this holds for all $k = 0, 1, ..., n$:
  Van der Corput set with $N = 2^{n+1}$ points, i.e. the set of all points of the form

$$\left(0.x^{(1)}x^{(2)}\dots x^{(n)}x^{(n+1)}, 0.x^{(n+1)}x^{(n)}\dots x^{(2)}x^{(1)}\right).$$

Riesz product

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Sidon's theorem

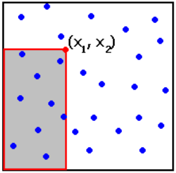
If a bounded 1-periodic function f has lacunary Fourier series $\sum_{k=1}^{\infty} a_k \sin(2\pi n_k x)$, $n_{k+1}/n_k > \lambda > 1$, then

$$\|f\|_\infty \gtrsim \sum_{k=1}^{\infty} |a_k|$$

- Riesz product: $P_K(x) = \prod_{k=1}^{K} (1 + \varepsilon_k \cos n_k x)$

Discrepancy function

Consider a set $\mathcal{P}_N \subset [0,1]^d$ consisting of N points:



Define the discrepancy function of the set $\mathcal{P}_N$ as

$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - N x_1 x_2 \dots x_d$$

L^p estimates, $1 < p < \infty$

Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0,1]^d$ *with* $\#\mathcal{P}_N = N$*:*

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

- Main idea:

$$D_N \approx \sum_{R:\,|R| \approx \frac{1}{N}} \frac{\langle D_N, h_R \rangle}{|R|} h_R$$

Theorem (Davenport, 1956 ($d = 2$, $p = 2$); Roth, 1979 ($d \geq 3$, $p = 2$); Chen, 1983 ($p > 2$); Chen, Skriganov, 2000's)

There exist sets $\mathcal{P}_N \subset [0,1]^d$ *with*

$$\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$$

L^∞ estimates

Conjecture

$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$

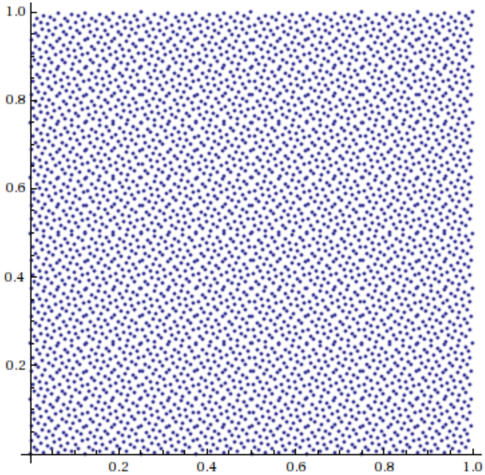
Theorem (Schmidt, 1972; Halász, 1981)

In dimension $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$

$d = 2$: Lerch, 1904; van der Corput, 1934

There exist $\mathcal{P}_N \subset [0,1]^2$ with $\|D_N\|_\infty \approx \log N$

Low discrepancy sets

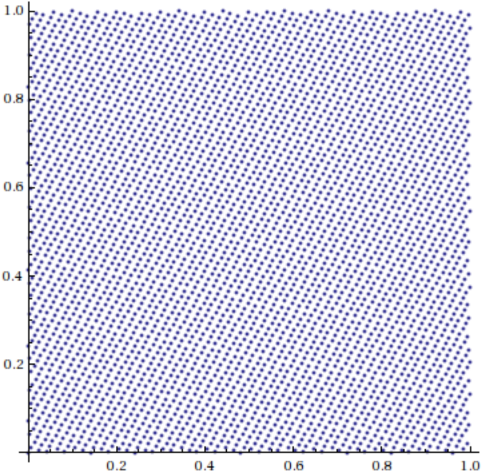


The van der Corput set with $N = 2^n$ points (here $n = 12$)

$$\big(0.x_1 x_2 \ldots x_n,\ 0.x_n x_{n-1} \ldots x_2 x_1\big), \quad x_k = 0 \text{ or } 1.$$

Discrepancy $\approx \log N$

Low discrepancy sets



The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$$\big(n/N, \{n\alpha\}\big), \quad n = 0, 1, ..., N-1.$$

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$d \geq 3$, Halton, Hammersley (1960):

There exist $\mathcal{P}_N \subset [0,1]^d$ with $\|D_N\|_\infty \lesssim (\log N)^{d-1}$

Conjectures and results

Conjecture 1

$\|D_N\|_\infty \gtrsim (\log N)^{d-1}$

Conjecture 2

$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

Theorem (J. Beck, 1989)

In dimension $d = 3$ for all N-point sets in $[0,1]^3$

$$\|D_N\|_\infty \gtrsim \log N \cdot (\log \log N)^{\frac{1}{8}-\varepsilon}.$$

Theorem (DB, M.Lacey, A.Vagharshakyan, 2008)

For $d \geq 3$ there exists $\eta = \eta(d) > 0$ such that

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}.$$

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For dimensions $d \geq 2$, we have for all choices of α_R

$$n^{\frac{1}{2}(d-2)} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

The small ball conjecture and discrepancy

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Conjecture 2

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- In both conjectures one gains a square root over the L^2 estimate.

Signed Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of $\alpha_R = \pm 1$

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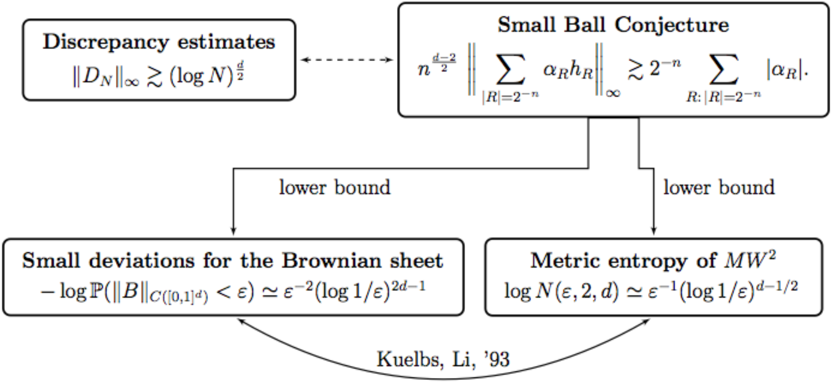
Discrepancy estimates	**Small Ball inequality (signed)**
Dimension $d = 2$	
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81)	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n$ (Talagrand, '94; Temlyakov, '95)
Higher dimensions, L^2 bounds	
$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_2 \gtrsim n^{(d-1)/2}$
Higher dimensions, conjecture	
$\|D_N\|_\infty \gtrsim (\log N)^{d/2}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}$
Higher dimensions, known results	
$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d-1}{2}+\eta}$

Discrepancy function $D_N(x) = \#\{\mathcal{P}_N \cap [0,x)\} - Nx_1x_2$	**Lacunary Fourier series** $f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_k x$, $\frac{n_{k+1}}{n_k} > \lambda > 1$
$\|D_N\|_2 \gtrsim \sqrt{\log N}$ (Roth, '54)	$\|f\|_2 \equiv \sqrt{\sum \|c_k\|^2}$
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81) Riesz product: $\prod (1 + cf_k)$	$\|f\|_\infty \gtrsim \sum \|c_k\|$ (Sidon, '27) Riesz product: $\prod \big(1 + \cos(n_k x + \phi_k)\big)$
$\|D_N\|_1 \gtrsim \sqrt{\log N}$ (Halász, '81) Riesz product: $\prod \big(1 + i \cdot \frac{c}{\sqrt{\log N}} f_k\big)$	$\|f\|_1 \gtrsim \|f\|_2$ (Sidon, '30) Riesz product: $\prod \big(1 + i \cdot \frac{\|c_k\|}{\|f\|_2} \cos(n_k x + \theta_k)\big)$

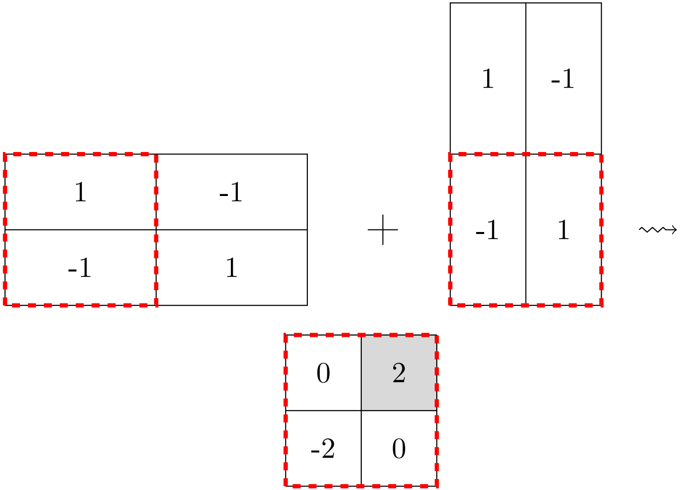
Table: Discrepancy function and lacunary Fourier series

Connections between problems



A new proof in $d = 2$: signed case

DB, N. Feldheim 2015



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- For each $k = \frac{n+1}{2}, \dots, n-1, n$,

$$F_k(x) = \sum_{R \in \mathcal{D}_k^2} \varepsilon_R h_R(x) + \sum_{R \in \mathcal{D}_{n-k}^2} \varepsilon_R h_R(x)$$

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- "Zoom in" into these cubes and iterate $k \to k+1$.
- In the end we have 2^{n+1} cubes Q_j of size $2^{-(n+1)} \times 2^{-(n+1)}$, on which all $F_k = +2$. Then on each Q_j

$$\sum_{|R|=2^{-n}} \varepsilon_R h_R(x) = \sum_{k=\frac{n+1}{2}}^{n} F_k(x) = \frac{n+1}{2} \cdot 2 = n+1.$$

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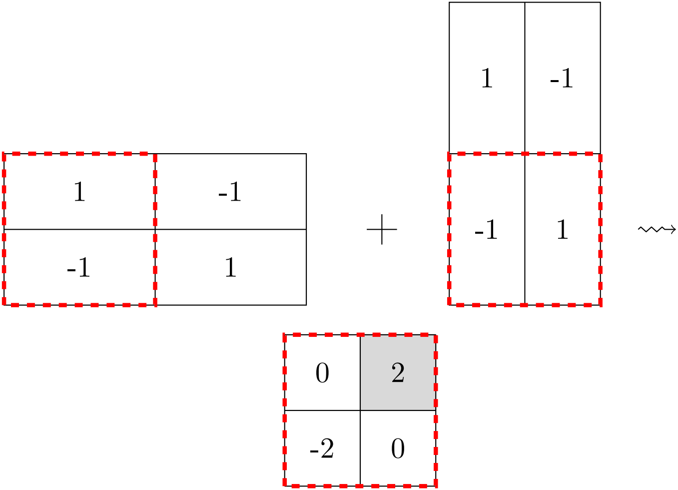

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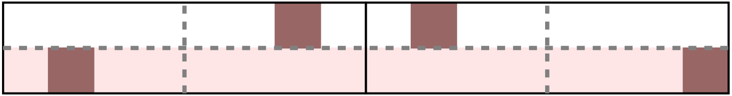
- At each following step every rectangle R will contain exactly two previously chosen squares.



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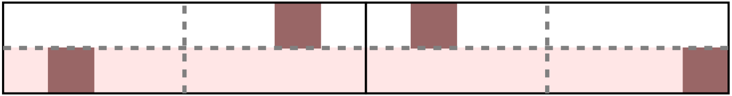

- We further choose a sub square in each of those and they have to lie in the opposite quarters of R.

Connection to binary (t, m, d)-nets

Definition

A set $\mathcal{P}$ of $N = b^m$ points in $[0,1)^d$ is called a (t, m, d)-net in base b if every b-adic box of volume b^{-m+t} contains exactly b^t points of $\mathcal{P}$.

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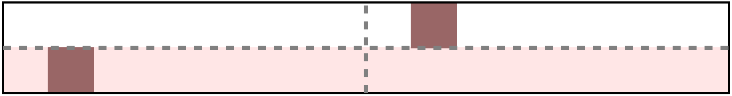
- Since every dyadic R with $|R| = 2^{-n}$ contains exactly two of the 2^{n+1} chosen squares, the extremal set is a $(1, n+1, 2)$-net in base $b = 2$.



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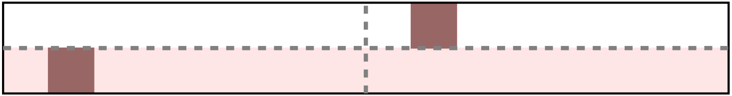

- Since in every such R these points lie in opposite quarters, it is actually a $(0, n+1, 2)$-net in base $b = 2$.

Examples of two-dimensional nets

- When $\varepsilon_R = +1$ for all $R \in \mathcal{D}^2$ with $|R| = 2^{-n}$:
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- Each dyadic $(0, m, 2)$-net $\mathcal{P}$ may be obtained this way
- The total number of different binary $(0, m, 2)$-nets is

$$2^{m2^{m-1}}$$

(Xiao, 1996)

A new proof in $d = 2$: general case

- At each step choose the subcube, where

$$F_k(x) = |\alpha_{R'}| + |\alpha_{R''}|.$$

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$$\Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty = \max_{j=1,\dots,2^{n+1}} \sum_{R \supset Q_j} |\alpha_R|$$

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Extension to b-adic nets

- A box $R \in \mathcal{D}_b^2$ of dimensions $b^{-m_1} \times b^{-m_2}$ is a union of a $b \times b$ array of b-adic boxes of dimensions $b^{-(m_1+1)} \times b^{-(m_2+1)}$.

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- Define the family of functions $\mathcal{H}_R$. The function $\phi_R \in \mathcal{H}_R$ iff
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- $\#\mathcal{H}_R = b!$.
- If $b = 2$, then $\mathcal{H}_R = \{\pm h_R\}$ and $\#\mathcal{H}_R = 2$.

Small ball inequality and b-adic nets

Theorem

Fix the scale $m \in \mathbb{N}$ and an integer base $b \geq 2$. For each b-adic box $R \in \mathcal{D}_b^2$ with $|R| = b^{-(m-1)}$, choose a function $\phi_R \in \mathcal{H}_R$.

(i) *A b-adic analogue of the signed small ball inequality holds:*

$$\max_{x \in [0,1)^2} \sum_{|R|=b^{-(m-1)}} \phi_R(x) \;= m.$$

(ii) *The set on which the maximum above is achieved has the form*

$$\mathcal{P} + \left[0, b^{-m}\right)^2,$$

where $\mathcal{P}$ is a standard $(0,m,2)$-net in base b.

(iii) *Each $(0,m,2)$-net $\mathcal{P}$ in base b may be obtained this way.*

(iv) *The number of different $(0,m,2)$-nets in base b is $(b!)^{mb^{m-1}}$.*

Dimension reduction: "signed" case

Lemma

Let $d \geq 2$. Assume that in dimension $d' = d - 1 \geq 1$ for all coefficients $\varepsilon_R = \pm 1$ we have the following inequality:

$$\Big\| \sum_{|R| \geq 2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d'+1}{2}} = n^{\frac{d}{2}}.$$

Then in dimension $d \geq 2$ for all coefficients $\varepsilon_R = \pm 1$ we have

$$\Big\| \sum_{|R| = 2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d}{2}}.$$

- In dimension $d = 2$ equivalent.
- $\big\| \sum_{|R| \geq 2^{-n}} \varepsilon_R h_R \big\|_2 \gtrsim n^{d'/2}$
- In $d = 1$ the bound $\big\| \sum_{|I| \geq 2^{-n}} \varepsilon_I h_I \big\|_\infty \geq n$ is trivial.

Dimension reduction: general case

In dimension $d' = 1$ a proper analog would be:

$$\Big\| \sum_{I \in \mathcal{D}:\, |I| \geq 2^{-n}} \alpha_I h_I \Big\|_\infty \gtrsim \sum_{|I| \geq 2^{-n}} |\alpha_I| \cdot |I|.$$

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- This would imply the signed small ball inequality in ALL dimensions $d \geq 2$!!!

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- This implies the general small ball inequality in $d = 2$
- This would imply the signed small ball inequality in ALL dimensions $d \geq 2$!!!
- Unfortunately this estimate is NOT true in general!!!!

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This would easily imply small ball inequality in $d = 2$. Fix x_2:

$$\begin{aligned} & \Big\| \sum_{|R| = 2^{-n}} \alpha_R h_R \Big\|_{L^\infty(x_1)} \\ &= \Big\| \sum_{|R_1| \geq 2^{-n}} \Big(\sum_{|R_2| = \frac{2^{-n}}{|R_1|}} \alpha_{R_1 \times R_2} h_{R_2}(x_2) \Big) h_{R_1}(x_1) \Big\|_{L^\infty(x_1)} \\ &\geq \sum_{|R_1| \geq 2^{-n}} \Big| \sum_{|R_2| = 2^{-n}/|R_1|} \alpha_{R_1 \times R_2} h_{R_2}(x_2) \Big| \cdot |R_1|. \end{aligned}$$

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Replace the sup by the average in x_2:

$$\begin{aligned} \Big\| \sum_{|R| = 2^{-n}} \alpha_R h_R \Big\|_\infty &\geq \sum_{|R_1| \geq 2^{-n}} \Big\| \sum_{|R_2| = 2^{-n}/|R_1|} \alpha_{R_1 \times R_2} h_{R_2} \Big\|_{L^1(x_2)} \cdot |R_1| \\ &= \sum_{|R_1| \geq 2^{-n}} \sum_{|R_2| = 2^{-n}/|R_1|} |\alpha_{R_1 \times R_2}| \cdot |R_2| \cdot |R_1| \\ &= 2^{-n} \sum_{|R| = 2^{-n}} |\alpha_R|. \end{aligned}$$

Dimension reduction: general case

Actually, this one-dimensional bound

$$\Big\| \sum_{I \in \mathcal{D}:\, |I| \geq 2^{-n}} \alpha_I h_I \Big\|_\infty \gtrsim \sum_{|I| \geq 2^{-n}} \big|\alpha_I\big| \cdot |I|.$$

would imply the "signed" SBI in ALL dimensions $d \geq 2$.

Denote $H_n = \sum_{|R| = 2^{-n}} \varepsilon_R h_R$. Then

$$\|H_n\|_1 \approx \|H_n\|_2 \approx n^{\frac{d-1}{2}}.$$

Write $x \in [0,1)^d$ as $x = (x_1, x')$, where $x_1 \in [0,1)$, $x' \in [0,1)^{d-1}$.

Write $R = R_1 \times R' \in \mathcal{D}^d$ in a similar way.

Dimension reduction: general case

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_{L^\infty(x_1)} \geq \sum_{|R_1|\geq 2^{-n}} \Big| \sum_{|R'|=2^{-n}/|R_1|} \varepsilon_{R_1\times R'} h_{R'}(x') \Big| \cdot |R_1|$$

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_{\infty} \geq \sum_{|R_1|\geq 2^{-n}} \Big\| \sum_{|R'|=2^{-n}/|R_1|} \varepsilon_{R_1\times R'} h_{R'}(x') \Big\|_{L^1(x')} \cdot |R_1|$$

$$\gtrsim \sum_{k=0}^{n} \sum_{|R_1|=2^{-k}} (n-k)^{\frac{d-2}{2}} \cdot 2^{-k}$$

$$= \sum_{k=0}^{n} (n-k)^{\frac{d-2}{2}} \approx n^{d/2}.$$

Similar ideas

- Skriganov, 2014:
  For any N-point set $\mathcal{P} \subset [0,1)^d$ there is a digit shift σ such that
  $$\|D_{\mathcal{P}\oplus\sigma}\|_\infty \gtrsim (\log N)^{d/2}.$$
- Karslidis, 2015
  In any dimension $d \geq 2$, if the coefficients have product structure $\varepsilon_R = \varepsilon_{R_1} \cdot \varepsilon_{R'}$, then the signed SBI holds
  $$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}.$$
- L^∞ estimate in 1 dimension
  $\oplus$ equivalence of L^1 and L^2 in other $d-1$ dimensions

Dimension reduction: general case FAILS

Unfortunately, this one dimensional bound

$$\Bigg\| \sum_{I \in \mathcal{D}\colon |I| \geq 2^{-n}} \alpha_I h_I \Bigg\|_\infty \gtrsim \sum_{|I| \geq 2^{-n}} \big|\alpha_I\big| \cdot |I|.$$

IS NOT TRUE!!!