Polynomial Corners in Finite Fields

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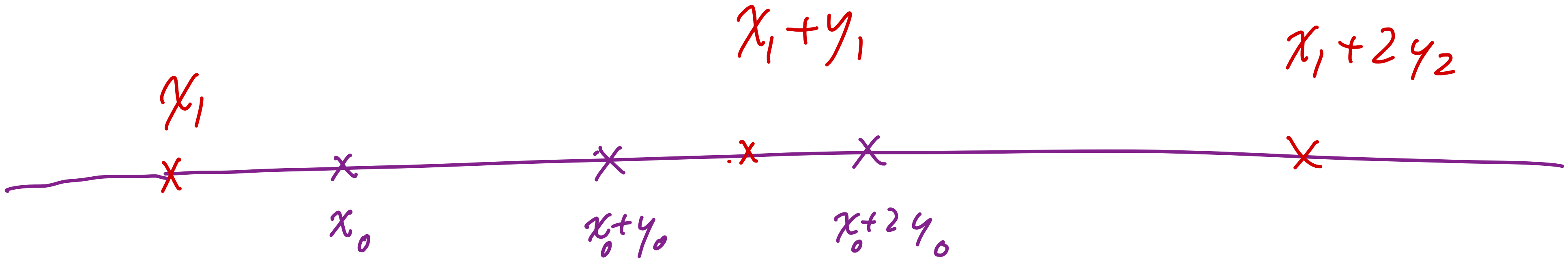
Roth Theorem

The largests cardinality set $A \subset \{1, \ldots, N\}$ that does not contain

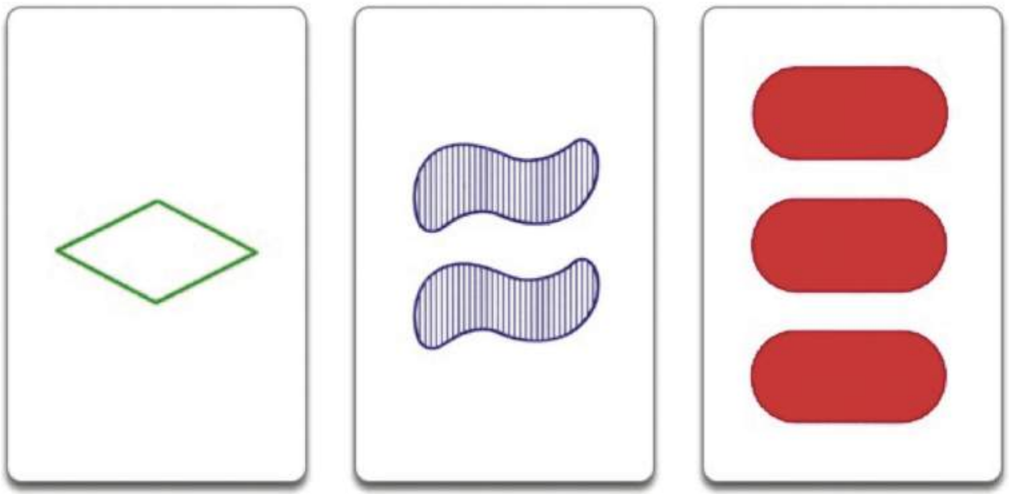
3 pts $x, \; x+y, \; x+2y, \quad y \neq 0$



is $o(N)$

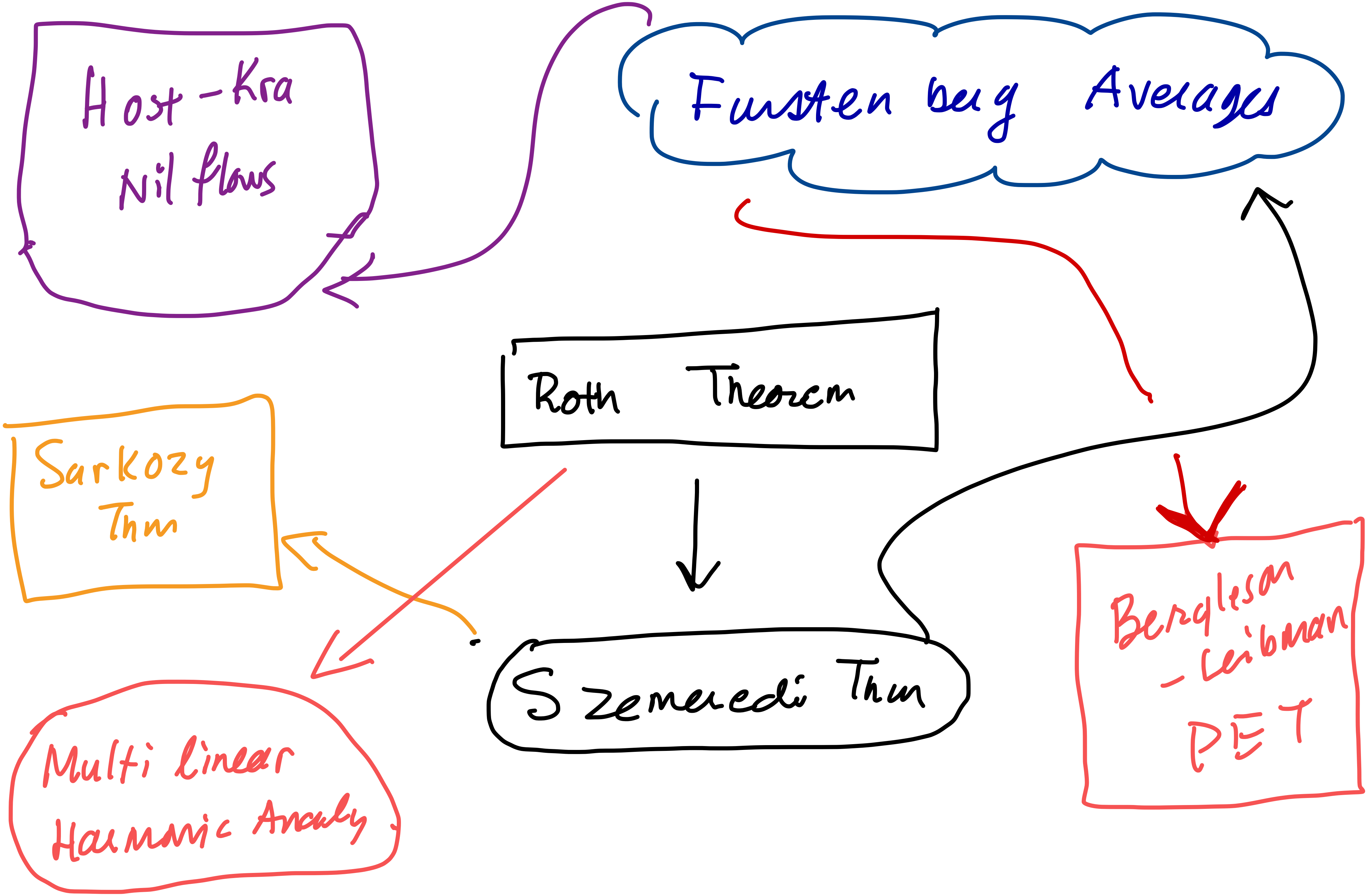
Thm (Cap Sets) Croot - Lev - Pach,
Ellenberg, Gijswijt

The largest set $A \subset \mathbb{F}_3^n$ not containing a 3 term AP is at most $|\mathbb{F}_3^n|^{1-\delta}$



Thm (Bloom - Sissak)

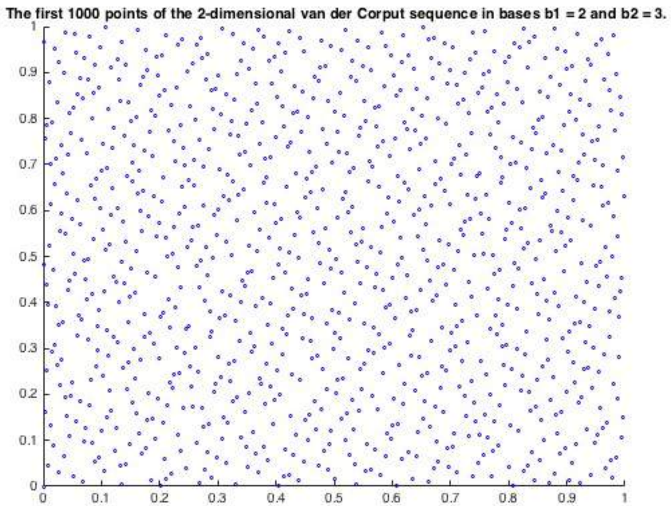
If $A \subset \{1, \ldots, N\}$ contains no 3 term AP's then $|A| < \dfrac{N}{(\log N)^{1+\delta}}$.


Host-Kra
Nil flows
Furstenberg Averages
Roth Theorem
Sarkozy
Thm
Szemeredi Thm
Bergelson
-Leibman
PET
Multi linear
Harmonic Analy

Two Thms of Roth

3 term AP

Discrepancy



Bourgain & Chang

Thm $\exists\, \delta > 0$.

$\forall\, p$ prime $\forall\, A \subset \mathbb{F}_p$

$$|A| \geq p^{1-\delta}$$

A contains 3 terms

$$x,\ x+y,\ x+y^2$$

$$y \neq 0$$

Non-linear setting leads to a huge improvement over the linear setting of arithmetic progressions

$\boxed{\text{Local Smoothing Estimate}}$

$$A(f,g)(x) = \mathbb{E}_p\, f(x+y)\, g(x+y^2)$$

$$\| A(f,g) - \mathbb{E}f \cdot \mathbb{E}g \|_2 \le c\, p^{-\frac{1}{10}} \|f\|_2 \cdot \|g\|_2$$

Also improving

Xiao chun Li / Dong / Sawin

Two linearly indep. polynomials

Sarah Peluse

Longer Polynomial Progressions

Peluse/Prendiville

$\forall\ k\ \exists\ C$
$\forall\ p_1, \dots, p_k$

integral polynom of different degrees If $A \subset \{1, \dots, N\}$ contains $\geq N/(\log N)^c$ points

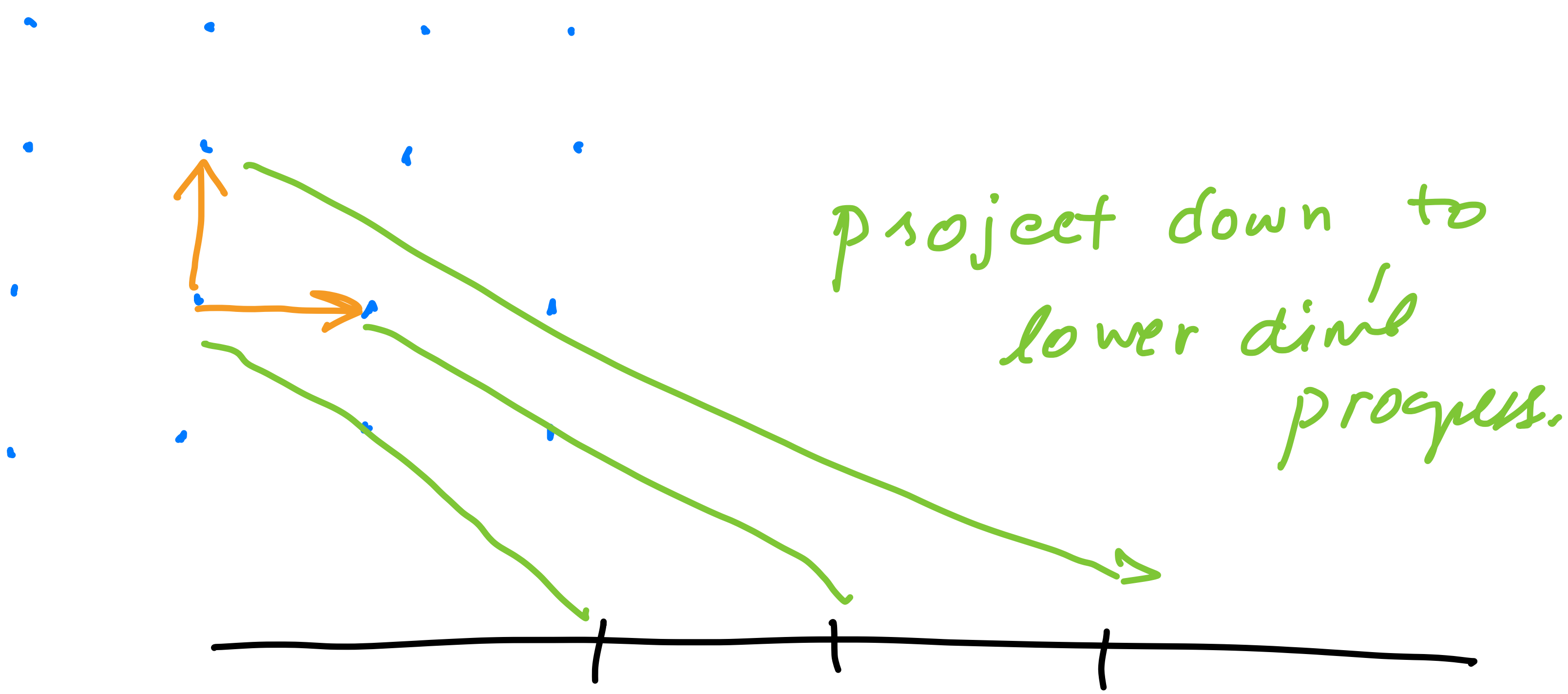
then it contains

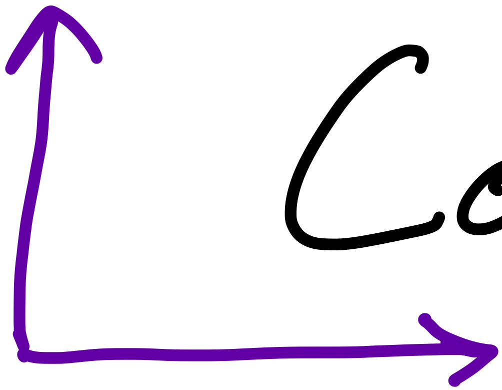
$$x,\ x + p_1(y),\ \dots,\ x + p_k(y)$$

$$y \neq 0$$

Corners

Progressions are of form $x,\ x+\begin{pmatrix} y \\ 0 \end{pmatrix};\ x+\begin{pmatrix} 0 \\ y \end{pmatrix}$





Corners

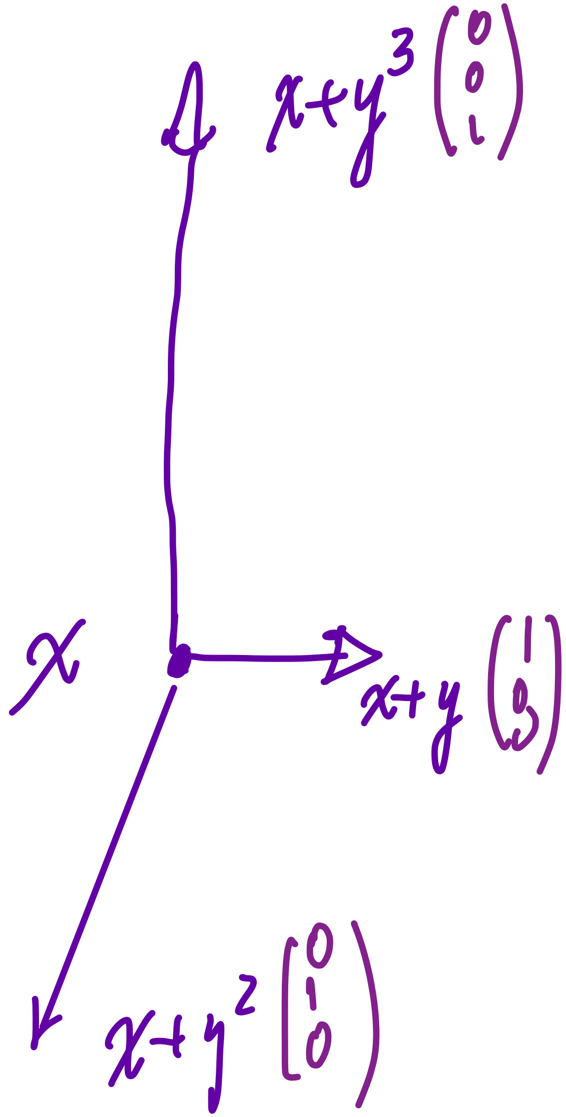
Thm (Bergelson Liebman)

of different degrees

Let $p_1, \dots, p_k$ be integral poly. The largest set $A \subset \{1, \dots, N\}^k$ not containing k poly. corners is $o(N^k)$

Poly. Corners

$$\boxed{x, \quad x + p_1(y)\vec{e_1}, \; \dots, \; x + p_k(y)\vec{e_k}}$$


$x+y^3 \begin{pmatrix}0\\0\\1\end{pmatrix}$
x
$x+y \begin{pmatrix}1\\0\\0\end{pmatrix}$
$x+y^2 \begin{pmatrix}0\\1\\0\end{pmatrix}$

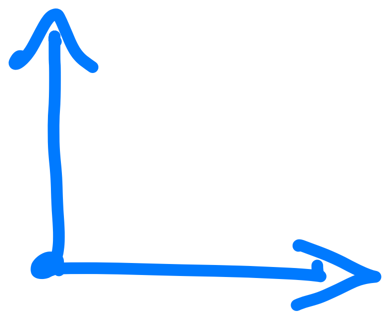
Quantitative Bounds

Thm (Shkredov) $\forall \; A \subset \{1, \dots, N\}^2$

w/ $|A| > N^2/(\log\log N)^c$

contains 3 pts , $y \neq 0$

$$x, \; x + ye_1, \; x + ye_2$$



Ergodic Theory of Roth Thm is of a single transformation T

$A \subset \mathbb{Z}$, positive density

$$\xrightarrow[\text{Coorespondence}]{\text{Furstenberg}} \quad \{T^k\}$$

The ergodic theory for Corners of 2 commuting transforms.

Much less is understood.

E.g. The finite field version of Shkredov's Thm is still double logarithmic.

Thm (Polynomial Roth Thm for Corners)

Rui Han, M. Lacey, Fan Yang

Let P_1, P_2 be two integer poly w/ distinct degrees. $\exists\ \delta > 0$ $\forall$ primes p

$\forall\ A \subset \mathbb{F}_p \times \mathbb{F}_p$ w/ $|A| > p^{2-\delta}$

contains three pts

$$x,\quad x + P_1(y)e_1,\quad x + P_2(y)e_2 \quad y \neq 0$$

If $A \subset \mathbb{F}_p^2$ contains $\gg p^{2-\delta}$ points,

it contains about as many poly. corners as you would expect

$$\left(\frac{|A|}{p^2}\right)^3 \times p \quad \text{corners}$$

Count # of corners

$\in \mathbb{F}_p^2$

$$T(f_1, f_2, f_3)(x) = \mathbb{E}_{\substack{y \\ \in \mathbb{F}_p}} f_3(x)\, f_1(x + P_1(y)\, e_1)$$
$$f_2(x + P_2(y)\, e_2)$$

$$= \langle f_3, \underbrace{A(f_1, f_2)}_{} \rangle$$

<u>Bilinear Avg.</u>

Bilinear Avg

$$A(f_1, f_2)(x_1, x_2) = \mathbb{E}_y f_1(x_1 + P_1(y), x_2)\, f_2(x_1, x_2 + P_2(y))$$

Lemma 1

(Hard!)

$$\| A(f_1, f_2) - \mathbb{E}_{x_1} f_1 \cdot \mathbb{E}_{x_2} f_2 \|_2 \leq p^{-\frac{1}{8}} \|f_1\|_4 \|f_2\|_4$$

Lemma 2

(Easy!)

$$\mathbb{E}\left(1_A(x_1,x_2)\, \mathbb{E}_{x_1} 1_A(\cdot\,, x_2)\, \mathbb{E}_{x_2} 1_A(x_1, \cdot)\right)$$

$$\gtrsim (\mathbb{E}\, 1_A)^3 .$$

Pf of Thm is then

$$T(1_A, 1_A, 1_A) \gtrsim (\mathbb{E}\, 1_A)^3 - p^{-1/8} (\mathbb{E}\, 1_A)$$

So A has abundant poly progressions if

$$\mathbb{E}\, 1_A \geq p^{-\frac{1}{16}} .$$

Main steps are lean on the approach of

Xiaochun Li, Dong Dong, Will Sawin

Nick Katz' Exponential Sum Estimates

Proof is Fourier Analytic:

Express $A(f_1, f_2)$ in Fourier variables, and analysize the kernel

$\mathbb{F}_p^2$ has unit mass counting measure

$\widehat{\mathbb{F}_p^2}$ has counting measure

$$\hat{f}(\underset{\substack{\uparrow \\ \widehat{\mathbb{F}_p^2}}}{m}) = \mathbb{E}_{\mathbb{F}_p^2} f(x)\, e^{-\frac{2\pi i}{p} \underbrace{x \cdot m}_{\text{2 dim vectors}}}$$

$e(-x \cdot m)$ abbrev.

$$f(x) = \sum_m \hat{f}(m)\, e(x \cdot m)$$

$$A(f_1, f_2)(x_1, x_2) = \mathbb{E}_y \, f_1(x_1 + y; x_2) \, f_2(x_1, x_2 + p(y))$$

$$= \sum_{\substack{m, n \\ \in \mathbb{F}_p^2}} \hat{f}_1(m) \, \hat{f}_2(n) \, e((m+n)x) \, \underbrace{\mathbb{E}_y \, e(m_1 y + n_2 p(y))}_{= K(m_1, n_2)}$$

<u>A. Weil</u>: $(m_1, n_2) \neq (0,0) \Rightarrow |K(m_1, n_2)| \lesssim \frac{1}{\sqrt{p}}$

(Gauss, if $p(y) = y^2$)

Term to subtract from $A(f_1, f_2)$ is

$$\sum_{m_2, n_1} \hat{f}_1(0, m_2)\, \hat{f}_2(n_1, 0)\, e((n_1, m_2)\cdot x)$$

$$= \left(\mathbb{E}_{x_1} f_1\right)(x_2) \left(\mathbb{E}_{x_2} f_2\right)(x_1)$$

Hard term is

$$J = \sum_{m_1} \sum_{n : n_2 \neq 0} \hat{f}_1(m-n)\, \hat{f}_2(n)\, K(m_1 - n_1, n_2)$$

Bound reduces to uniform estimates over exponential sums. !

* Steps are easy — Plancherel; H-Y, etc.
* Until get to a deep inequality on exponential sums.

Weil estimates

$$\left| \sum_{x \in \mathbb{F}_p} e^{2\pi i f(x)/p} \right| \lesssim \sqrt{p}$$

Li - Dong - Sawin recognized sums

$$S = \frac{1}{p^3} \sum_{\substack{y_1, y_2, y_3, y_4 \\ G(y_1, y_2, y_3, y_4) = 0}} e\left(H(y_1, y_2, y_3, y_4)\right)$$

$$G = G_{p_1, p_2} \qquad H = H_{p_1, p_2}$$

N. Katz extends Weil estimates to certain algebraic varieties

$$|S| \lesssim \frac{1}{p^{3/2}}$$

subject to conditions on non-degeneracy of G & H.

Further work

- New proof of $\mathbb{F}_p^2$ result
- Higher dim'l corners
- $\mathbb{Z}^2$ corners