OPUC: Simon's Hausdorff Dimension Conjecture

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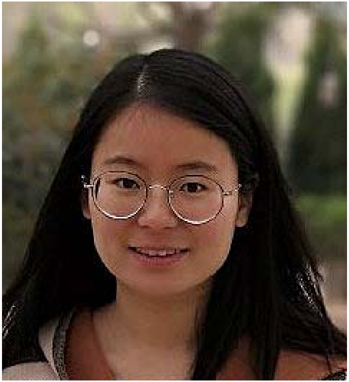

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Simon's Hausdorff Dimension Conjecture

Hausdorff Dimension Conjecture (Simon 2005)

Suppose that μ is a non-trivial probability measure on $\partial\mathbb{D}$ whose Verblunsky coefficients satisfy

$$\sum_{n=0}^{\infty} n^{\gamma} |\alpha_n|^2 < \infty$$

for $\gamma \in (0,1)$. Then there is a set $S \subset \partial\mathbb{D}$ of Hausdorff dimension at most $1-\gamma$ so that for $z \in \partial\mathbb{D} \setminus S$, the associated Szegő matrices $T_n(z)$ are bounded:

$$\sup_{n \geq 0} \| T_n(z) \| < \infty.$$

In particular, μ_{sing} is supported by a set of dimension at most $1-\gamma$.

We will proceed as follows:

- define all objects that appear in the statement of the conjecture
- explain why the statement is natural and interesting
- explain why the statement is optimal
- describe the general approach to the proof
- briefly discuss the steps of the proof

Probability Measures on the Circle and Their Verblunsky Coefficients

Suppose μ is a non-trivial (i.e., not finitely supported) probability measure on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

By the non-triviality assumption, the functions $1, z, z^2, \cdots$ are linearly independent in the Hilbert space

$$\mathcal{H} = L^2(\partial\mathbb{D}, d\mu)$$

and hence one can form, by the Gram-Schmidt procedure, the monic orthogonal polynomials $\Phi_n(z)$, whose Szegő dual is defined by

$$\Phi_n^* = z^n \overline{\Phi_n(1/\overline{z})}$$

There are constants $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, called the Verblunsky coefficients, so that

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z), \qquad \text{for } n \in \mathbb{Z}_+,$$

which is the so-called Szegő recurrence.

Conversely, every sequence $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ in $\mathbb{D}$ arises as the sequence of Verblunsky coefficients for a suitable nontrivial probability measure on $\partial\mathbb{D}$.

The Szegő Matrices

If we consider instead the orthonormal polynomials

$$\varphi(z,n) = \frac{\Phi_n(z)}{\|\Phi_n(z)\|},$$

it is easy to see that with $\rho_n = (1 - |\alpha_n|^2)^{1/2}$, the Szegő recurrence becomes

$$\rho_n \varphi(z,n+1) = z\varphi(z,n) - \overline{\alpha}_n \varphi^*(z,n)$$

The Szegő recurrence can be written in a matrix form as follows:

$$\begin{pmatrix} \varphi(z,n+1) \\ \varphi^*(z,n+1) \end{pmatrix} = \frac{1}{\rho_n} \begin{pmatrix} z & -\overline{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \varphi(z,n) \\ \varphi^*(z,n) \end{pmatrix}$$

Alternatively, one can consider a different initial condition and derive the orthogonal polynomials of the second kind, by setting $\psi(z,0) = 1$ and then

$$\begin{pmatrix} \psi(z,n+1) \\ -\psi^*(z,n+1) \end{pmatrix} = \frac{1}{\rho_n} \begin{pmatrix} z & -\overline{\alpha}_n \\ -\alpha_n z & 1 \end{pmatrix} \begin{pmatrix} \psi(z,n) \\ -\psi^*(z,n) \end{pmatrix}$$

Define the associated Szegő matrices by

$$T_n(z) = \frac{1}{2} \begin{pmatrix} \varphi_n(z) + \psi_n(z) & \varphi_n(z) - \psi_n(z) \\ \varphi_n^*(z) - \psi_n^*(z) & \varphi_n^*(z) + \psi_n^*(z) \end{pmatrix}$$

Simon's Hausdorff Dimension Conjecture

Hausdorff Dimension Conjecture (Simon 2005)

Suppose that μ is a non-trivial probability measure on $\partial\mathbb{D}$ whose Verblunsky coefficients satisfy

$$\sum_{n=0}^{\infty} n^{\gamma} |\alpha_n|^2 < \infty$$

for $\gamma \in (0,1)$. Then there is a set $S \subset \partial\mathbb{D}$ of Hausdorff dimension at most $1-\gamma$ so that for $z \in \partial\mathbb{D} \setminus S$, the associated Szegő matrices $T_n(z)$ are bounded:

$$\sup_{n \geq 0} \| T_n(z) \| < \infty.$$

In particular, μ_{sing} is supported by a set of dimension at most $1-\gamma$.

- it is well known that μ_{sing} is supported by
$$S = \left\{ z \in \partial\mathbb{D} : \sup_{n \geq 0} \| T_n(z) \| = \infty \right\}$$
- the statement is also true for the endpoints $\gamma = 0$ and 1, but this was already known when the conjecture was made
- of course the conclusion for $\gamma = 0$ is vacuous, but this endpoint case is interesting anyway due to the optimality of the result
- indeed, the fact that in the case $\gamma = 0$, μ_{sing} can be one-dimensional is one of the most important consequences of Szegő's Theorem

Szegő's Theorem – The Case $\gamma = 0$

Verblunsky's form (1936) of Szegő's theorem (1920/21) reads

$$\prod_{n=0}^{\infty}(1 - |\alpha_n|^2) = \exp\left(\int \log(w(\theta))\,\frac{d\theta}{2\pi}\right),$$

where w denotes the Radon-Nikodym derivative of the absolutely continuous part of μ with respect to the normalized Lebesgue measure on the unit circle.

This identity implies in particular that

$$\sum_{n=0}^{\infty}|\alpha_n|^2 < \infty \;\Leftrightarrow\; \int \log(w(\theta))\,\frac{d\theta}{2\pi} > -\infty.$$

Observe that the singular part of μ is entirely unrestricted by this condition (other than by having weight less than 1), and hence one can have measures μ whose Verblunsky coefficients satisfy $\sum n^\gamma |\alpha_n|^2 < \infty$ with $\gamma = 0$ and singular part of Hausdorff dimension $1 = 1 - \gamma$.

Super-Coulomb Decay – The Case $\gamma = 1$

Golinskii and Ibragimov established in 1971 the following statement:

$$\sum_{n=0}^{\infty} n|\alpha_n|^2 < \infty \;\Rightarrow\; \mu_{\rm sing} = 0$$

We see in particular that if $\sum n^\gamma |\alpha_n|^2 < \infty$ with $\gamma = 1$, then $\mu_{\rm sing}$ is supported by a set of dimension $0 = 1 - \gamma$.

Thus, the Hausdorff dimension conjecture asks for a linear interpolation of the known endpoint results.

There is another reason why this particular interpolation is expected to be the correct one: it is known, due to a 2005 paper by Denisov and Kupin, that any general upper bound for the dimension of a support of $\mu_{\rm sing}$ cannot be smaller than $1 - \gamma$!

Related Results for Schrödinger Operators

There are some results in the same spirit for Schrödinger operators

$$[H\psi](x) = -\psi''(x) + V(x)\psi(x), \quad \psi(0) = 0$$

in $L^2(0,\infty)$. Write

$$S := \{E \in (0,\infty) : \text{ all solutions of } Hu = Eu \text{ are bounded}\}$$

Theorem (Remling 1998)

Suppose that

$$|V(x)| \le \frac{C}{(1+x)^\alpha}$$

for some $\alpha \in [\frac{1}{2}, 1]$. Then, $\mathbf{dim}\, S \le 4(1-\alpha)$.

Remark

(a) The bound is non-trivial only for $\alpha \in (\frac{3}{4}, 1]$.

(b) The α-assumption implies the γ-assumption ($(1+x)^{\frac{\gamma}{2}} V(x) \in L^2$) for every $\gamma < 2\alpha - 1$. Thus, the corresponding non-trivial γ-range is $(\frac{1}{2}, 1)$.

(c) As in all works on problems of this type, the boundedness of solutions is proved by establishing WKB asymptotics for them. Thus, the actual result is somewhat stronger than mere boundedness.

Related Results for Schrödinger Operators

Theorem (Remling 2000)

Suppose that

$$|V(x)| \leq \frac{C}{(1+x)^{\alpha}}$$

for some $\alpha \in [\frac{1}{2}, 1]$. *Then,* $\mathbf{dim}\, S \leq 2(1-\alpha)$.

Theorem (Christ-Kiselev 2001)

Suppose that

$$\int (1+x)^{\gamma} |V(x)|^2 \, dx < \infty$$

for some $\gamma \in (0, 1]$. *Then,* $\mathbf{dim}\, S \leq 1-\gamma$.

Remark

(a) The optimality of the bound $1-\gamma$ follows again from the 2005 work of Denisov and Kupin.

(b) The Simon conjecture asks for the direct OPUC analog of the Christ-Kiselev Schrödinger operator result.

(c) On the other hand, the method Remling uses seems to be easier to carry over to the OPUC setting. However, note that he does not prove the Schrödinger version of Simon's conjecture with his method. Worse yet, it is not clear whether this is in principle even possible.

The Main Result

In the remainder of the talk we want to discuss the following result, which simply states that Simon's OPUC Hausdorff dimension conjecture is true:

Theorem (D.-Guo-Ong)

Suppose that μ is a non-trivial probability measure on $\partial\mathbb{D}$ whose Verblunsky coefficients satisfy

$$\sum_{n=0}^{\infty} n^{\gamma} |\alpha_n|^2 < \infty$$

for $\gamma \in (0,1)$. Then there is a set $S \subset \partial\mathbb{D}$ of Hausdorff dimension at most $1-\gamma$ so that for $z \in \partial\mathbb{D} \setminus S$, the associated Szegő matrices $T_n(z)$ are bounded:

$$\sup_{n \geq 0} \| T_n(z) \| < \infty.$$

In particular, μ_{sing} is supported by a set of dimension at most $1-\gamma$.

Prüfer Variables

Let $\{\alpha_n\}_{n\geq 0}$ be the Verblunsky coefficients of a nontrivial probability measure $d\mu$ on $\partial\mathbb{D}$. As mentioned above, the α_n's give rise to a sequence $\{\Phi_n(z)\}_{n\geq 0}$ of monic polynomials (via the Szegő recurrence) that are orthogonal with respect to $d\mu$.

For $\beta \in [0, 2\pi)$, we also consider the monic polynomials $\{\Phi_n(z, \beta)\}_{n\geq 0}$ that are associated in the same way with the Verblunsky coefficients $\{e^{i\beta}\alpha_n\}_{n\geq 0}$. The parameter β corresponds to a variation of the initial condition for the Szegő recursion. In particular, the orthogonal polynomials of both the first and second kind arise for suitable choices of β, and hence we can bound the Szegő matrices once we have bounds for these two relevant values of β.

Let $\eta \in [0, 2\pi)$. Define the Prüfer variables R_n, θ_n by

$$\Phi_n(e^{i\eta}, \beta) = R_n(\eta, \beta) \exp\left[i(n\eta + \theta_n(\eta, \beta))\right]$$

where $R_n > 0$, $\theta_n \in [0, 2\pi)$, and $|\theta_{n+1} - \theta_n| < \pi$. Thus, the desired boundedness statement for the β-dependent orthogonal polynomials can be established by bounding the β-dependent Prüfer radius $R_n(\eta, \beta)$ as $n \to \infty$ outside a set of η's that has sufficiently small Hausdorff dimension.

Prüfer Equations

The Prüfer variables obey the following pair of equations:

$$\frac{R_{n+1}^2(\eta,\beta)}{R_n^2(\eta,\beta)} = 1 + |\alpha_n|^2 - 2\,\mathrm{Re}\left(\alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}\right),$$

$$e^{-i(\theta_{n+1}(\eta,\beta)-\theta_n(\eta,\beta))} = \frac{1-\alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}}{\left[1+|\alpha_n|^2 - 2\,\mathrm{Re}\left(\alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}\right)\right]^{1/2}}.$$

Since the Szegő matrices $T_n(z)$ are expressed via the normalized polynomials, we note that when $\{\alpha_n\} \in \ell^2$, we have for $r_n(\eta,\beta) = |\varphi_n(\eta,\beta)|$ the following two-sided estimates:

$$r_n(\eta,\beta) \sim R_n(\eta,\beta) \sim \exp\left(-\sum_{j=0}^{n-1} \mathrm{Re}(\alpha_j e^{i[(j+1)\eta+\beta+2\theta_j(\eta,\beta)]})\right)$$

Uniformly Hölder Continuous Measures and A.E. Boundedness

Theorem

Assume that $\sum n^{\gamma}|\alpha_n|^2 < \infty$ *holds for some* $\gamma \in (0,1)$*. Suppose that* ν *is a finite Borel measure on* $(0, 2\pi)$ *with the following two properties:*

(i) *There is a* $\delta > 0$ *such that* ν *is supported by* $(\delta, 2\pi - \delta)$*.*

(ii) *There is a* $D \in (1-\gamma, 1)$ *such that* ν *is uniformly D-Hölder continuous, that is,* $\nu(I) \lesssim |I|^D$ *for every interval* $I \subseteq (0, 2\pi)$*.*

Then

$$\sup\{R_n(\eta, \beta) : \beta \in [0, 2\pi), n \in \mathbb{Z}_+\} < \infty$$

for ν*-almost every* η*.*

Assuming this theorem, we can give the proof of the main result:

Deriving the Main Result

Proof of the Hausdorff Dimension Conjecture. We have to show that

$$S = \{\eta \in [0, 2\pi) : \sup_{n \geq 0} \| T_n(e^{i\eta}) \| = \infty\}$$

has Hausdorff dimension at most $1 - \gamma$.
Assuming this fails, and hence $\mathbf{dim}_{\mathrm{H}}(S) > 1 - \gamma$, it follows from standard results in measure theory that there is a finite Borel measure ν with the following two properties:

(i) There is a $\delta > 0$ such that ν is supported by $S \cap (\delta, 2\pi - \delta)$.

(ii) There is a $D \in (1 - \gamma, 1)$ such that ν is uniformly D-Hölder continuous, that is, $\nu(I) \lesssim |I|^D$ for every interval $I \subseteq (0, 2\pi)$.

In particular, the previous theorem is applicable to this measure and it ensures for ν-almost every η the boundedness of the Prüfer radius as $n \to \infty$ for all initial phases β, and in particular those two that correspond to the entries of the Szegő matrices. In particular, for ν-almost every η, the Szegő matrices $T_n(e^{i\eta})$ remain bounded as $n \to \infty$, in contradiction with the definition of S and the fact that S supports the measure ν. This completes the proof. $\square$

Estimates for the WKB Transform

Let us write

$$\psi(k,\eta,\beta) = (k+1)\eta + \beta + 2\theta_n(\eta,\beta)$$
$$\omega(s,\eta,\beta) = (s+1)\eta + \beta + \frac{1}{\eta}\sum_{k=0}^{s-1} |\alpha_k|^2$$

Theorem

Assume that $\sum n^{\gamma}|\alpha_n|^2 < \infty$ holds for some $\gamma \in (0,1)$. Suppose that ν is a finite Borel measure on $(0,2\pi)$ with the following two properties:

(i) *There is a $\delta > 0$ such that ν is supported by $(\delta, 2\pi - \delta)$.*

(ii) *There is a $D \in (1-\gamma, 1)$ such that ν is uniformly D-Hölder continuous, that is, $\nu(I) \lesssim |I|^D$ for every interval $I \subseteq (0,2\pi)$.*

Then

$$\sup_{\beta} \sum_{s=0}^{L} \left| \int f(\eta) e^{i\omega(s,\eta,\beta)}\, d\nu(\eta) \right|^2 \lesssim (L+1)^{1-D} \int |f(\eta)|^2\, d\nu(\eta),$$

for all $f \in L^2((0,2\pi), d\nu)$ and $L \in \mathbb{Z}_+$.

Remling's Divide-and-Conquer Strategy

Lemma

Given a sequence of Verblunsky coefficients $\{\alpha_n\}_{n\in\mathbb{Z}_+} \subseteq \mathbb{D}$ *that is not finitely supported, define the strictly increasing sequence* $\{x_n\}_{n\in\mathbb{Z}_+} \subseteq \mathbb{Z}_+$ *by*

- $x_0 = 0$,
- *for every* $n \in \mathbb{Z}_+$, x_{n+1} *is the smallest power of* 2 *so that* $x_{n+1} > x_n$ *and* $\alpha_j \neq 0$ *for at least one* $j \in [x_n, x_{n+1})$.

If $\sum n^\gamma |\alpha_n|^2 < \infty$ *holds for some* $\gamma \in (0,1)$, *then for every*

$$D \in (1-\gamma, 1)$$

and every integer

$$N \in \left(\frac{2-\gamma-D}{D+\gamma-1}, \infty\right)$$

we have

$$\sum_{n=1}^{\infty} |x_n - x_{n-1}|^{\frac{1-D}{2}} \|\alpha\chi_{[x_{n-1},x_n)}\|_2 < \infty$$

and

$$\sup_n \|\alpha\chi_{[x_{n-1},x_n)}\|_1 \| |x_n - x_{n-1}|^{\frac{1-D}{2}} \alpha\chi_{[x_{n-1},x_n)}\|_2^N < \infty$$

Remling's Divide-and-Conquer Strategy

With

$$N_n := \max\left\{1, \left\lfloor \frac{1}{\sqrt{\sum_{m=x_{n-1}}^{x_n-1} (x_n - x_{n-1})^{1-D}|\alpha_m|^2}} \right\rfloor\right\}$$

we subdivide the interval $[x_{n-1}, x_n)$ into N_n subintervals and then define the next generations of this iteration scheme inductively.

Lemma

In order for

$$\sum_{n=1}^{\infty} \sum_{l_1,l_2,\dots,l_{j-1}=1}^{N_n} \left| \sum_{k=y_{j-1}(l_{j-1}-1)}^{y_{j-1}(l_{j-1})-1} \alpha_k e^{i\omega(k,\eta)} e^{is\tau(k,\eta)} \sum_{m=y_{j-1}(l_{j-1}-1)}^{k} \alpha_m e^{i\omega(m,\eta)} \right|$$

to converge for ν-almost every η and $s \in \mathbb{N}$, it suffices to show that

$$\sum_{n=1}^{\infty} \sum_{l_1,l_2,\dots,l_j=1}^{N_n} \left| \sum_{k=y_j(l_j-1)}^{y_j(l_j)-1} \alpha_k e^{i\omega(k,\eta)} e^{is\tau(k,\eta)} \sum_{m=y_j(l_j-1)}^{k} \alpha_m e^{i\omega(m,\eta)} \right|$$

converges for ν-almost every η and $s \in \mathbb{N}$.

Remling's Divide-and-Conquer Strategy

Proposition

Assume that $\sum n^\gamma |\alpha_n|^2 < \infty$ *holds for some* $\gamma \in (0,1)$*, and let* $\{x_n\}$ *be chosen as in the previous lemma. Suppose that* ν *is a finite Borel measure on* $(0, 2\pi)$ *with the following two properties:*

(i) *There is a* $\delta > 0$ *such that* ν *is supported by* $(\delta, 2\pi - \delta)$*.*

(ii) *There is a* $D \in (1-\gamma, 1)$ *such that* ν *is uniformly D-Hölder continuous, that is,* $\nu(I) \lesssim |I|^D$ *for every interval* $I \subseteq (0, 2\pi)$*.*

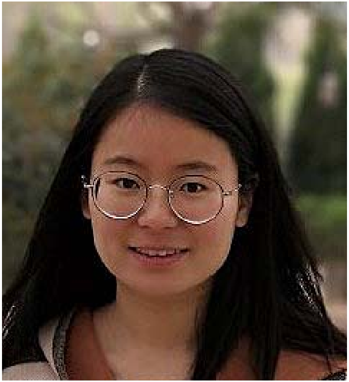
Then

$$\sup\{R_{x_n}(\eta, \beta) : \beta \in [0, 2\pi), n \in \mathbb{Z}_+\} < \infty$$

for ν*-almost every* η*.*

Lastly, interpolate between the x_n's via Kiselev's 1999 maximal function estimate!

Thank you!



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