Multidimensional Continued Fraction Algorithms, Rauzy Fractals, and Zero Measure Spectrum for Multi-Frequency Schrödinger Operators

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Multi-Frequency Schrödinger Operators

Fix a dimension $d \in \mathbb{N}$ and consider $\alpha \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ that is such that the translation $R_\alpha : \mathbb{T}^d \to \mathbb{T}^d$, $\omega \mapsto \omega + \alpha$ is minimal.

If $g : \mathbb{T}^d \to \mathbb{R}$ is bounded and measurable, we can consider, for each $\omega \in \mathbb{T}^d$, the discrete Schrödinger operator

$$[H_{\alpha,g,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + g(\omega + n\alpha)\psi(n)$$

in $\ell^2(\mathbb{Z})$. We call such an operator a generalized quasi-periodic Schrödinger operator.

By standard arguments involving the ergodicity of Lebesgue measure with respect to R_α, there is a compact set $\Sigma_{\alpha,g}$ such that for Lebesgue almost every $\omega \in \mathbb{T}^d$, the spectrum of $H_{\alpha,g,\omega}$ is equal to $\Sigma_{\alpha,g}$.

The Spectrum of Multi-Frequency Schrödinger Operators

The almost sure spectrum $\Sigma_{\alpha,g}$ can have various topological and measure-theoretic properties. It can be a Cantor (i.e., perfect and nowhere dense) set, but it can also be a finite union of non-degenerate compact intervals. The Cantor spectra that occur can have both positive and zero Lebesgue measure. Among those that have zero Lebesgue measure, examples are known with small, and even zero, Hausdorff dimension.

Roughly speaking, when $d = 1$, it is well known how to produce examples with zero Lebesgue measure and even zero Hausdorff dimension. On the other hand, when $d > 1$, examples are known where the spectrum is a finite union of intervals, and it is (essentially) open how to produce spectra of zero Lebesgue measure.

Definition

A function $g : \mathbb{T}^d \to \mathbb{R}$ is called elementary if it is measurable and takes finitely many values. The set of elementary functions $g : \mathbb{T}^d \to \mathbb{R}$ is denoted by $\mathcal{E}(\mathbb{T}^d)$. A subset of $\mathcal{E}(\mathbb{T}^d)$ is called ample if its $\| \cdot \|_\infty$-closure in $L^\infty(\mathbb{T}^d)$ contains $C(\mathbb{T}^d)$.

Zero Measure Cantor Spectrum

Theorem (Chaika-D.-Fillman-Gohlke)

Let $d = 2$. Then, for Lebesgue almost every $\alpha \in \mathbb{T}^d$, the set

$$\mathcal{Z}_\alpha = \{g \in \mathcal{E}(\mathbb{T}^d) : \Sigma_{\alpha,g} \textit{ is a Cantor set of zero Lebesgue measure}\}$$

is ample.

Remark

(a) In the case $d = 1$, this is a 2006 result of D.-Lenz, and the full measure set of $\alpha \in \mathbb{T}$ is explicit: $\mathbb{T} \setminus \mathbb{Q}$. For $d = 2$, the full measure set is not explicit.
(b) The fact that the result can be extended to a value of d that is greater than one is not obvious, and indeed surprising, since the straightforward extension of the proof for $d = 1$ is known to fail.
(c) To the best of our knowledge, there is no known example of a quasi-periodic multi-frequency potential (i.e., $d > 1$ and $g \in C(\mathbb{T}^d)$) so that the associated Schrödinger operator has zero-measure spectrum. It is unclear whether such an example exists. The fact that arbitrarily small $\|\cdot\|_\infty$ perturbations of an arbitrary $g \in C(\mathbb{T}^d)$ can produce this effect is therefore interesting.
(d) We regard it as an interesting open problem to explore whether this result can be extended to some larger values of d. Several components of our proof indeed do extend to values of d greater than 2.

Key Components of the Proof

The proof of the previous theorem is based on several ingredients:

- a principle that derives Cantor spectrum of zero Lebesgue measure for Schrödinger operators with potentials taking finitely many values from a dynamical property of the subshift generated by the potentials
- a multi-dimensional continued fraction algorithm, together with an ergodic measure that is equivalent to Lebesgue measure
- an S-adic description of the orbit of a point under the iteration of the multi-dimensional continued fraction algorithm
- a principle showing that for almost every point this induces a natural coding of a suitable torus translation
- a verification that the resulting S-adic subshift obeys the sufficient condition for zero-measure Schrödinger spectrum for almost every point

Zero-Measure Spectrum via the Boshernitzan Criterion

Definition
Given a finite set $\mathcal{A}$, called the alphabet, give the full shift $\mathcal{A}^{\mathbb{Z}}$ the product topology inherited from placing the discrete topology on each factor, and define the shift map

$$S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}, \quad [Sx](n) = x(n+1)$$

A subshift over $\mathcal{A}$ is a closed (hence compact) S-invariant subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$. The language of a subshift X is

$$L(X) := \{x_n \dots x_{n+k-1} : x \in X, n \in \mathbb{Z}, k \in \mathbb{N}\}$$

A subshift X is minimal if each of its S-orbits is dense.

Definition
Let (X, S) be a minimal subshift. We say that (X, S) satisfies the Boshernitzan criterion if there exist an S-invariant probability measure μ, a constant $C > 0$, and a sequence $n_1, n_2, \dots \to \infty$ so that for all $w = w_1 \cdots w_{n_i} \in L(X)$,

$$\mu(\{x \in X : x_1 \cdots x_{n_i} = w\}) > \frac{C}{n_i}$$

Zero-Measure Spectrum via the Boshernitzan Criterion

Given a finite alphabet $\mathcal{A}$ and a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, one can define Schrödinger operators in $\ell^2(\mathbb{Z})$ by generating potentials which are obtained through real-valued sampling along the S-orbits of X. That is, if $f : X \to \mathbb{R}$ is given, we associate with each $x \in X$ the potential $V_x : \mathbb{Z} \to \mathbb{R}$ given by

$$V_x(n) = f(S^n x), \quad n \in \mathbb{Z}$$

The Schrödinger operator H_x in $\ell^2(\mathbb{Z})$ is then given by

$$[H_x \psi](n) = \psi(n+1) + \psi(n-1) + V_x(n)\psi(n)$$

One typically restricts attention to locally constant functions f, that is, functions that depend on only finitely many entries of the input sequence x. If X is minimal and f is locally constant, then a simple strong approximation argument shows that there is a compact set $\Sigma_{X,f} \subset \mathbb{R}$ such that

$$\sigma(H_x) = \Sigma_{X,f} \quad \text{for every } x \in X$$

Zero-Measure Spectrum via the Boshernitzan Criterion

Obviously, a minimal subshift X is finite if and only if every V_x is periodic (in the non-degenerate case of non-constant f), and in this case $\Sigma_{X,f}$ is well known to be a union of finitely many non-degenerate compact intervals. Similarly, if f is constant, the same conclusions hold.

Ruling out these degenerate cases, it is an interesting question whether $\Sigma_{X,f}$ must have zero Lebesgue measure. In fact, Simon had conjectured that this must be the case in complete generality, but this conjecture has been disproved [Avila-D.-Zhang].

Theorem (D.-Lenz)

If the minimal subshift X satisfies the Boshernitzan criterion and f is locally constant, then either all V_x are periodic or the set $\Sigma_{X,f}$ is a Cantor set of zero Lebesgue measure.

The Tribonacci Substitution and the Classical Rauzy Fractal

With the alphabet $\mathcal{A}_3 = \{1, 2, 3\}$, consider the Tribonacci substitution

$$S_T : \mathcal{A}_3 \to \mathcal{A}_3^*, \quad 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

This substitution is primitive, that is, there is a power that sends each symbol to all other symbols. Concretely,

$$S_T^3 : 1 \mapsto 1213121,\ 2 \mapsto 12131,\ 3 \mapsto 1213$$

The associated Tribonacci substitution sequence is the element $u_T = S_T^\infty(1)$ of $\mathcal{A}_3^{\mathbb{Z}_+}$, which is fixed by S_T since $S_T(1)$ begins with 1. We have

$$\begin{aligned}
S_T^0(1) &= 1\\
S_T^1(1) &= 12\\
S_T^2(1) &= 1213\\
S_T^3(1) &= 1213121\\
S_T^4(1) &= 1213121121312\\
S_T^5(1) &= 121312112131212131211213
\end{aligned}$$

and hence $u_T = 121312112131212131211213\ldots$.

The Tribonacci Substitution and the Classical Rauzy Fractal

Note that

$$\begin{aligned} S_T^{k+3}(1) &= S_T^{k+2}(12) \\ &= S_T^{k+2}(1)S_T^{k+2}(2) \\ &= S_T^{k+2}(1)S_T^{k+1}(13) \\ &= S_T^{k+2}(1)S_T^{k+1}(1)S_T^{k+1}(3) \\ &= S_T^{k+2}(1)S_T^{k+1}(1)S_T^{k}(1) \end{aligned}$$

In particular, for $T_k := |S_T^k(1)|$, we have

$$T_{k+3} = T_{k+2} + T_{k+1} + T_k$$

which explains the terminology.

In particular, we must have

$$T_k = \lambda_T^k(1 + o(1))$$

where $\lambda_T > 1$ solves

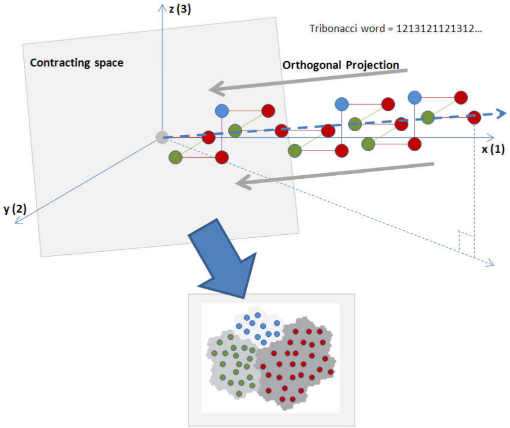
$$\lambda^3 - \lambda^2 - \lambda - 1 = 0$$

The Tribonacci Substitution and the Classical Rauzy Fractal

The classical Rauzy fractal is constructed as follows:

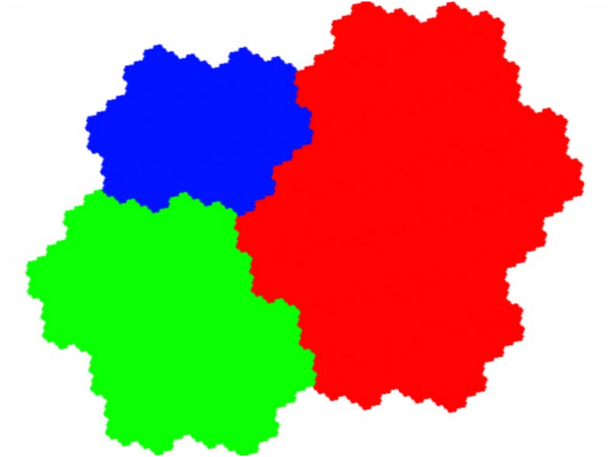
- Consider $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ and associate $x \leftrightarrow 1$, $y \leftrightarrow 2$, $z \leftrightarrow 3$.
- Scan u_T from left to right and build a "staircase" by starting at $(0, 0, 0)$ and increasing that component by one which corresponds to the symbol currently being scanned.
- Since $u_T = 1213121121312121311213\ldots$, the sequence of points so generated begins with $(1, 0, 0)$, $(1, 1, 0)$, $(2, 1, 0)$, $(2, 1, 1)$, $(3, 1, 1)$, $(3, 2, 1)$, $(4, 2, 1)$, etc.
- Note that these points cluster along a line L_T. We'll return to this point momentarily.
- Project the points in the direction of this line to the orthogonal complement P_T of L_T.
- The closure of the image in the plane P_T is the classical Rauzy fractal.
- If we color the points corresponding to the three different symbols in three different colors, the Rauzy fractal partitions into three subsets, which happen to be similar to itself. This is a manifestation of the self-similarity of the Tribonacci substitution sequence: $S_T(u_T) = u_T$.

The Tribonacci Substitution and the Classical Rauzy Fractal



The construction of the classical Rauzy fractal
(Source: Wikipedia)

The Tribonacci Substitution and the Classical Rauzy Fractal



The classical Rauzy fractal
(Source: Milton Minervino)

The Tribonacci Substitution and the Classical Rauzy Fractal

The Tribonacci substitution matrix is defined by $M_T = (m_{ij})_{1\le i,j\le 3}$, where m_{ij} is the number of times the symbol i occurs in $S_T(j)$. Thus,

$$M_T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

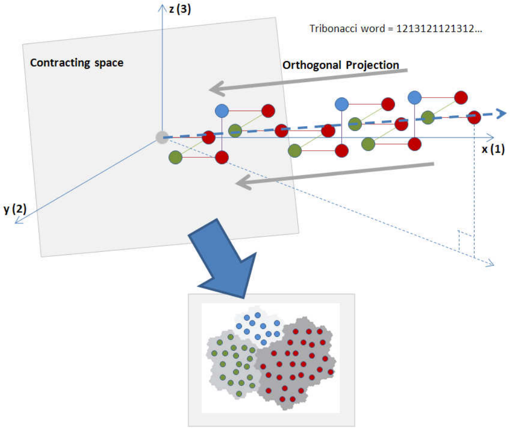
The primitivity of S_T of course reflects the primitivity of M_T, seen via

$$M_T^3 = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

It implies by Perron-Frobenius that there is a simple leading eigenvalue $\lambda_T > 1$, and the corresponding eigenvector v_T, which can be chosen so that all its components are strictly positive, determines the line L_T mentioned earlier.

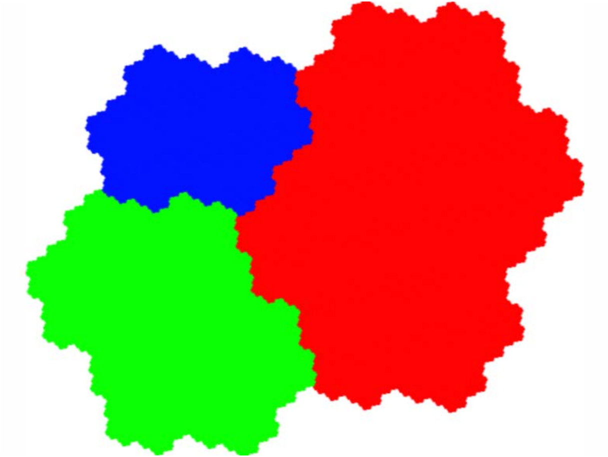
In fact, $\det(\lambda I - M) = \lambda^3 - \lambda^2 - \lambda - 1$, and the roots are $\lambda_T, \mu_T, \bar{\mu}_T$ with $|\mu_T| < 1$; and hence L_T is the expanding subspace and P_T is the contracting subspace associated with M.

The Tribonacci Substitution and the Classical Rauzy Fractal



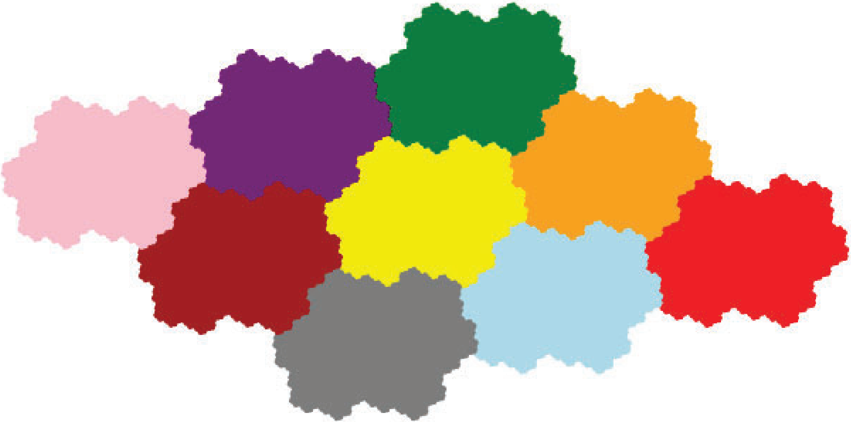
The construction of the classical Rauzy fractal
(Source: Wikipedia)

The Tribonacci Substitution and the Classical Rauzy Fractal



The classical Rauzy fractal
(Source: Milton Minervino)

The Tribonacci Substitution and the Classical Rauzy Fractal



Periodic tiling of the plane by copies of the Rauzy fractal
(Source: Milton Minervino)

The Tribonacci Substitution and the Classical Rauzy Fractal

Let us list some highlights of the foundational paper [Rauzy 1982]:

- One can tile the plane periodically by copies of the Rauzy fractal.
- In other words, the decomposition of the Rauzy fractal into the three colored similar pieces induces a partition of $\mathbb{T}^2$.
- Moreover, one can use this partition of $\mathbb{T}^2$ to code an orbit of the translation
$$T_\alpha : \mathbb{T}^2 \to \mathbb{T}^2, \quad \omega \mapsto \omega + \alpha$$
with a suitable $\alpha \in \mathbb{T}^2$ by undoing the association of the three colors and the three symbols in $\mathcal{A}_3$.
- This recovers the Tribonacci substitution sequence (or rather the subshift it generates).
- In other words, this particular primitive substitution subshift (which is well known to satisfy the Boshernitzan criterion) arises from a coding of a torus translation.

The key point underlying our result is that this can be turned around!

S-Adic Systems and Subshifts

An S-adic system over $\mathcal{A}$ is defined by a choice of a directive sequence $\boldsymbol{\tau} = (\tau_n)_{n=0}^{\infty}$ of substitutions on $\mathcal{A}$.

For $0 \leq m < n$, we consider compositions of the form $\tau_{[m,n]} = \tau_m \cdots \tau_n$. For $a \in \mathcal{A}$, we write $w_n(a) = \tau_{[0,n]}(a)$, and for the substitution matrices, we write $M_I = M_{\tau_I}$ for an interval I. Clearly, for $I = [m, n]$, one has

$$M_{[m,n]} = M_{\tau_m} M_{\tau_{m+1}} \cdots M_{\tau_n}$$

The language associated to $\boldsymbol{\tau}$ is

$$L(\boldsymbol{\tau}) := \{w \in \mathcal{A}^* : w \triangleleft w_n(a) \text{ for some } a \in \mathcal{A} \text{ and } n \in \mathbb{N}_0\}$$

It is easy to check that

$$X = X(\boldsymbol{\tau}) := \{x \in \mathcal{A}^{\mathbb{Z}} : L(x) \subseteq L(\boldsymbol{\tau})\}$$

is a non-empty subshift, provided that

$$\lim_{n\to\infty} \max_{a\in\mathcal{A}} |w_n(a)| = \infty$$

In this case, we call $X(\boldsymbol{\tau})$ the S-adic subshift generated by $\boldsymbol{\tau}$.

The Cassaigne-Selmer Algorithm

Denote $\mathbb{R}_+ = [0, \infty)$ and let

$$\Delta = \Delta_3 = \{(x_1, x_2, x_3) \in \mathbb{R}_+^3 : x_1 + x_2 + x_3 = 1\}$$

The Cassaigne-Selmer algorithm is given by

$$T : \Delta \to \Delta, \quad (x_1, x_2, x_3) \mapsto \begin{cases} \left(\frac{x_1 - x_3}{x_1 + x_2}, \frac{x_3}{x_1 + x_2}, \frac{x_2}{x_1 + x_2}\right) & \text{if } x_1 \geq x_3 \\ \left(\frac{x_2}{x_2 + x_3}, \frac{x_1}{x_2 + x_3}, \frac{x_3 - x_1}{x_2 + x_3}\right) & \text{if } x_3 > x_1 \end{cases}$$

There is an ergodic T-invariant probability measure ν on Δ which is equivalent to Lebesgue measure.

The Associated S-Adic Subshift

The Cassaigne-Selmer algorithm is of the form

$$T\colon \Delta \to \Delta, \quad \mathbf{x} \mapsto \frac{A(\mathbf{x})^{-1}\mathbf{x}}{\|A(\mathbf{x})^{-1}\mathbf{x}\|_1}$$

for some locally constant matrix valued function $A\colon \Delta \to \mathrm{GL}(3,\mathbb{Z})$.

We select for each $\mathbf{x} \in \Delta$ a substitution $\varphi(\mathbf{x})$ on the alphabet $\mathcal{A}_3 = \{1,2,3\}$ such that $A(\mathbf{x})$ coincides with the substitution matrix $M_{\varphi(\mathbf{x})}$:

$$\varphi(\mathbf{x}) = \begin{cases} \gamma_1 & \text{if } x_1 \geq x_3 \\ \gamma_2 & \text{if } x_3 > x_1 \end{cases}$$

with the Cassaigne-Selmer substitutions

$$\gamma_1\colon \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 13 \\ 3 \mapsto 2 \end{cases} \qquad \gamma_2\colon \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 13 \\ 3 \mapsto 3 \end{cases}$$

The Associated S-Adic Subshift

The orbit of a point $\mathbf{x} \in \Delta$ under the action of T defines an S-adic system, called a substitutive realization of (Δ, T, A), given by the directive sequence

$$\phi(\mathbf{x}) = (\varphi(T^n\mathbf{x}))_{n=0}^{\infty}$$

The corresponding subshift is given by $(X(\phi(\mathbf{x})), S)$.

On the other hand, we relate to each point $\mathbf{x}$ in the 3-dimensional simplex Δ a point on the torus $\mathbb{T}^2$ by the map $\pi : \Delta \to \mathbb{T}^2$, which denotes the projection to the first 2 coordinates.

Note that π is not a surjective map but for

$$\mathbb{T}^2_\Delta = \{t \in \mathbb{T}^2 : t_1 + t_2 \leq 1\}$$

the map $\pi\colon \Delta \to \mathbb{T}^2_\Delta$, $\mathbf{x} \mapsto \pi(\mathbf{x})$ is a bijection.

Below we will use the following fact: if $\alpha \in \mathbb{T}^2 \setminus \mathbb{T}^2_\Delta$, then $-\alpha \in \mathbb{T}^2_\Delta$.

Natural Codings of Torus Translations

For $\alpha \in \mathbb{T}^2$, let as before

$$R_\alpha : \mathbb{T}^2 \to \mathbb{T}^2, \quad R_\alpha(\omega) = \omega + \alpha$$

denote the associated torus translation.

Definition

A collection $\mathfrak{F} = \{\mathcal{F}_1, \ldots, \mathcal{F}_h\}$ is called a natural measurable partition of $\mathbb{T}^2$ if

- $\bigcup_{i=1}^h \mathcal{F}_i = \mathbb{T}^2$
- $\mathcal{F}_j \cap \mathcal{F}_k$ has zero measure for each $j \neq k$
- each $\mathcal{F}_i$ is measurable with dense interior and zero measure boundary

Given the map R_α, the language associated with $\mathfrak{F}$, denoted $L(\mathfrak{F})$, is the set of finite words $w = w_0 \cdots w_n \in \{1, \ldots, h\}^*$ such that $\bigcap_{k=0}^n R_\alpha^{-k} \mathring{\mathcal{F}}_{w_k} \neq \emptyset$, where $\mathring{A}$ denotes the interior of A.

Natural Codings of Torus Translations

Definition

A subshift (X, S) is called a natural coding of $(\mathbb{T}^2, R_\alpha)$ if its language coincides with the language of a natural measurable partition $\{\mathcal{F}_1, \dots, \mathcal{F}_h\}$ and

$$\bigcap_{n\in\mathbb{N}} \overline{\bigcap_{k=0}^{n} R_\alpha^{-k} \mathring{\mathcal{F}}_{x_k}}$$

consists of a single point for every $x = (x_n)_{n\in\mathbb{Z}} \in X$.

Theorem (Berthé-Steiner-Thuswaldner, Fogg-Noûs)

Let ϕ be the substitutive realization of the Cassaigne-Selmer algorithm. For ν-almost every $\mathbf{x} \in \Delta$, the subshift $(X(\phi(\mathbf{x})), S)$ is a natural coding of $(\mathbb{T}^2, R_{\pi(\mathbf{x})})$.

Remark

If $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_h\}$ is a natural measurable partition of $\mathbb{T}^2$, then the language generated by R_α on $\mathcal{F}$ coincides with the language generated by $R_{-\alpha}$ on the natural measurable partition $\{-\mathcal{F}_1, \dots, -\mathcal{F}_h\}$. In particular, if (X, S) is a natural coding of $(\mathbb{T}^2, R_\alpha)$, then it is also a natural coding of $(\mathbb{T}^2, R_{-\alpha})$.

S-Adic Subshifts Satisfying the Boshernitzan Criterion

Let $\phi = (\varphi_k)_{k=0}^{\infty}$ be a directive sequence generating an S-adic system, $(X(\phi), S)$.

Definition

For $a, b \in \mathcal{A}$, we say that a precedes b *at level n* if there are $m \in \mathbb{N}$ and $c \in \mathcal{A}$ such that $ab \lhd \varphi_{[n+1,n+m]}(c)$. For an interval $I = [n+1, n+\ell]$, we say φ_I is a word builder at level n if, whenever a precedes b at level n, there is $c \in \mathcal{A}$ such that $ab \lhd \varphi_I(c)$.

Theorem (Chaika-D.-Fillman-Gohlke)

Suppose there exists a constant $N > 0$ so that, for infinitely many n_0, there exist $n_0 < n_1 < n_2 < n_3$ so that

- $M_{[n_0+1,n_1]}$ *and* $M_{[n_2+1,n_3]}$ *are positive matrices*
- $\varphi_{[n_1+1,n_2]}$ *is a word builder at level* n_1
- $\max\{\|M_{[n_0+1,n_1]}\|, \|M_{[n_1+1,n_2]}\|, \|M_{[n_2+1,n_3]}\|\} \leq N$

Then $(X(\phi), S)$ satisfies Boshernitzan's criterion.

Boshernitzan's Criterion for Codings of Translations

Theorem (Chaika-D.-Fillman-Gohlke)

For Lebesgue almost every $\alpha \in \mathbb{T}^2_\Delta$*, the subshift* $(X(\phi(\pi^{-1}(\alpha))), S)$ *satisfies Boshernitzan's criterion. In particular, for almost every* $\alpha \in \mathbb{T}^2$*, the toral translation* $(\mathbb{T}^2, R_\alpha)$ *admits a natural coding that satisfies Boshernitzan's criterion.*

Sketch of Proof. The main steps are the following:

- when running the Cassaigne-Selmer algorithm T, identify a local situation in Δ that generates a word builder over a finite stretch of the iteration
- show that this local situation has positive measure with respect to ν
- use the Birkhoff ergodic theorem to show that almost every trajectory enters the local situation infinitely often
- conclude that for almost every point, there are are infinitely many word builders

One can then deduce that the subshift $(X(\phi(\mathbf{x})), S)$ satisfies the sufficient condition for the Boshernitzan criterion from the previous slide for ν-almost every $x \in \Delta$. $\square$

Deriving the Main Result

Proof that zero-measure Cantor spectrum is ample in $\mathcal{E}(\mathbb{T}^2)$. Assume that (X, S) is a natural coding of $R_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ associated with the natural measurable partition $\{\mathcal{F}_1, \ldots, \mathcal{F}_h\}$.

Given $w = w_0 \cdots w_n \in L(X)$, let

$$\mathcal{F}_w = \bigcap_{k=0}^{n} R_\alpha^{-k} \mathcal{F}_{w_k}$$

which is nonempty by the definition of $L(X)$. Let χ_w denote the characteristic function of $\mathcal{F}_w$, and let $\mathcal{A}$ denote the algebra generated by $\{\chi_w : w \in L(X)\}$.

It can then be seen that $\mathcal{A}$ is ample as any $f \in C(\mathbb{T}^d)$ is uniformly continuous and $\mathrm{diam}(\mathcal{F}_w)$ can be made as small as desired by taking $|w|$ sufficiently large.

In particular, $\mathcal{A} \setminus \{\text{constants}\}$ is then ample as well.

Now conclude by taking the full measure sets of α's in $\mathbb{T}^2$ that generate a translation $R_\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ that is minimal and admits a natural coding that satisfies the Boshernitzan criterion. $\square$

Thank you!

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