Proving Positive Lyapunov Exponents: I. Using Independence

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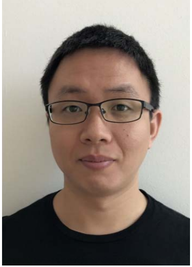
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Lyapunov Exponents of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

Let us fix a compact metric space Ω, a continuous map $T : \Omega \to \Omega$, and an ergodic Borel probability measure μ. The triple (Ω, T, μ) is often referred to as the base dynamical system.

Given a continuous map $A : \Omega \to \mathrm{SL}(2,\mathbb{R})$, we consider the skew product

$$(T, A) : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2,\ (\omega, v) \mapsto (T\omega, A(\omega)v)$$

For each $n \in \mathbb{Z}$, the map $A_n : \Omega \to \mathrm{SL}(2,\mathbb{R})$ is defined by $(T, A)^n) = (T^n, A_n)$.

By Kingman's Subadditive Ergodic Theorem, there is a number $L \geq 0$, called the Lyapunov exponent, such that

$$\begin{aligned} L &= \inf_{n \geq 1} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \lim_{n \to \infty} \frac{1}{n} \int \log \|A_n(\omega)\| \, d\mu(\omega) \\ &= \lim_{n \to \infty} \frac{1}{n} \log \|A_n(\omega)\| \quad \text{for } \mu\text{-a.e. } \omega \end{aligned}$$

Naturally, we are interested in whether $L > 0$ or $L = 0$.

Lyapunov Exponents of One-Parameter Families of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

Example

Consider Schrödinger operators

$$[H\psi](n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$, where the potential $V : \mathbb{Z} \to \mathbb{R}$ is dynamically defined, that is,

$$V(n) = f(T^n\omega)$$

with a base dynamical system (Ω, T, μ) as above and a continuous map $f : \Omega \to \mathbb{R}$. Then the solutions of the generalized eigenvalue equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

can be described via

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A_n^{(E)}(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$

with the Schrödinger cocycle generated by the map

$$A^{(E)}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

Lyapunov Exponents of One-Parameter Families of $\mathrm{SL}(2,\mathbb{R})$-Cocycles

It is natural and customary to denote the Lyapunov exponent of $(T, A^{(E)})$ by $L(E)$. Since $L(E) > 0$ strongly indicates that the generalizes eigenfunctions have exponential behavior, combining this with the existence of polynomially bounded generalized eigenfunctions spectrally almost everywhere, one expects spectral localization (i.e., pure point spectrum with exponentially decaying eigenfunctions) for μ-a.e. $\omega \in \Omega$ when $L(E) > 0$ holds for sufficiently many energies E. Let us denote the exceptional set of energies E by

$$\mathcal{Z} := \{E : L(E) = 0\}$$

Remark

(a) This connection holds almost always, by not always. In particular, there are examples with $\mathcal{Z} = \emptyset$, and yet for all $\omega \in \Omega$, the point spectrum of H is empty.

(b) Since countable sets cannot carry continuous spectral measures, one would want to embark on a proof of spectral localization by showing that $\mathcal{Z}$ is countable.

(c) For technical reasons, one generally desires to show that $\mathcal{Z}$ is discrete.

(d) This is a natural goal as requiring $\mathcal{Z} = \emptyset$ is too restrictive.

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

Let us discuss the goal of proving that $\mathcal{Z}$ is empty or at least small in several settings. We begin with the classical example, which is the simplest of them.

Example (The standard Anderson model)

The potential V is given by a realization of a sequence of independent identically distributed random variables. In our setting, this arises via the choices

- $\Omega = I^{\mathbb{Z}}$, where $I \subset \mathbb{R}$ is a compact interval
- $T : \Omega \to \Omega$ is the shift, $[T\omega](n) = \omega(n+1)$
- $\mu = \nu^{\mathbb{Z}}$, where ν is a probability measure supported by I (and $\#\mathrm{supp}\, \nu \geq 2$)
- $f : \Omega \to \mathbb{R}$, $f(\omega) = \omega(0)$

In this model one can show that $\mathcal{Z} = \emptyset$, and the proof is a straightforward application of Fürstenberg's Theorem about products of i.i.d. $\mathrm{SL}(2, \mathbb{R})$ matrices.

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

Example (The continuum Anderson model)

Recognizing that standard Schrödinger operators are continuum operators, and their discrete analogs are studied to investigate phenomena of interest in a technically easier framework, one may ask what can be said about the continuum Anderson model

$$H = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} q_n(\omega) f(x-n)$$

in $L^2(\mathbb{R})$, where $\mathrm{supp}(f) \subseteq [0,1]$ (and $f \not\equiv 0$) and the q_n's are i.i.d. random variables as before:

- $\Omega = I^{\mathbb{Z}}$, where $I \subset \mathbb{R}$ is a compact interval
- $T : \Omega \to \Omega$ is the shift, $[T\omega](n) = \omega(n+1)$
- $\mu = \nu^{\mathbb{Z}}$, where ν is a probability measure supported by I (and $\# \mathrm{supp}\, \nu \geq 2$)
- $q_n(\omega) = \omega(n)$

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

The map $A^{(E)} : \Omega \to \mathrm{SL}(2, \mathbb{R})$ is now given by

$$A^{(E)}(\omega) = \begin{pmatrix} u_D'(1) & u_N'(1) \\ u_D(1) & u_N(1) \end{pmatrix}$$

where u_D, u_N solve

$$-u'' + \omega(0) f u = E u$$

on $[0, 1]$ with

$$\begin{pmatrix} u_D'(0) & u_N'(0) \\ u_D(0) & u_N(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In this model one can show that $\mathcal{Z}$ is discrete, and the proof is a non-straightforward application of Fürstenberg's Theorem about products of i.i.d. $\mathrm{SL}(2, \mathbb{R})$ matrices [D.-Sims-Stolz 2002].

Positive Lyapunov Exponents for Schrödinger Cocycles: Examples

Example (The doubling map model)

The doubling map model is generated by

- $\Omega = \mathbb{T} = \mathbb{R}/\mathbb{Z}$
- $T : \Omega \to \Omega$, $T\omega = 2\omega$
- $\mu = \mathrm{Leb}$
- $f : \Omega \to \mathbb{R}$ suitable, sometimes taken to be $\lambda \cos(2\pi\omega)$

Strictly speaking, the map is non-invertible and hence the resulting operators are defined on $\ell^2(\mathbb{Z}_+)$, but this point is not essential to our discussion.

In this model it had been open whether $\mathcal{Z}$ is discrete, despite several attempts at proving this. See, e.g., [Chulaevsky-Spencer 1995], [D.-Killip 2005], [Sadel-Schulz-Baldes 2008], [Bourgain-Bourgain-Chang 2015], and [Bjerklöv 2020] for partial results.

The binary expansion of $\omega \in \mathbb{T}$ shows that this model may be viewed as a full shift with a non-local sampling function. Thus, there is some underlying independence (as the measure is then a product measure), but the resulting cocycles are not products of i.i.d. $\mathrm{SL}(2, \mathbb{R})$ matrices and hence Fürstenberg's Theorem does not apply.

Positive Lyapunov Exponents for Schrödinger Cocycles: Goals

Our goal is to explain new approaches to proving that $\mathcal{Z}$ is discrete. This will in particular

- provide a straightforward proof of this fact in the second example (continuum random Schrödinger operators)
- provide the first general proof of this fact in the third example (Schrödinger operators defined by uniformly hyperbolic transformations such as the doubling map)

Thus we will

- develop a novel way of applying a variant of Fürstenberg's Theorem, and hence of exploiting independence (remainder of this lecture)
- develop a way to deal with the absence of independence by exploiting the uniform hyperbolicity of the base transformation (tomorrow's lecture)

The Fürstenberg Condition

Definition

We say that a closed subgroup G of $\mathrm{SL}(2,\mathbb{R})$ satisfies the Fürstenberg condition if it is non-compact and there does not exist $\Lambda \subseteq \mathbb{RP}^1$ of cardinality one or two such that $g\Lambda = \Lambda$ for all $g \in G$.

Proposition (Fürstenberg 1963)

Let $E \in \mathbb{R}$ and denote by G_E the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ that contains

$$\{A^{(E)}(\omega) : \omega \in \operatorname{supp}\mu\}$$

If G_E satisfies the Fürstenberg condition, then $L(E) > 0$.

Remark

(a) This is a (very) special case of a more general theorem.

(b) Even in the $\mathrm{SL}(2,\mathbb{R})$ case, it would be more accurate to think of this as a statement about a probability measure ν on $\mathrm{SL}(2,\mathbb{R})$ and the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ containing its topological support.

(c) The latter formulation would then be applied to the probability measure ν_E on $\mathrm{SL}(2,\mathbb{R})$ given by the push-forward of the single-site probability measure on $\mathbb{R}$ under the map

$$V \mapsto \begin{pmatrix} E - V & -1 \\ 1 & 0 \end{pmatrix}$$

The Fürstenberg-Ishii Condition

Definition

We say that a closed subgroup G of $\mathrm{SL}(2,\mathbb{R})$ satisfies the Fürstenberg-Ishii condition if it contains two elements A, B such that

$$\operatorname{tr} A \neq 0, \quad \operatorname{tr} B \neq 0, \quad \det(AB - BA) \neq 0$$

Proposition (Bucaj-D.-Fillman-Gerbuz-VandenBoom-Wang-Zhang)

Let $E \in \mathbb{R}$ and denote by G_E the smallest closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ that contains

$$\{A^{(E)}(\omega) : \omega \in \operatorname{supp} \mu\}$$

If G_E satisfies the Fürstenberg-Ishii condition, then $L(E) > 0$.

Remark

(a) As before, this is really a statement about a probability measure ν on $\mathrm{SL}(2,\mathbb{R})$.

(b) We were inspired by an assertion in [Ishii 1973] that $\det(AB - BA) \neq 0$ alone implies that the Lyapunov exponent is positive.

(c) However, this is not correct, as shown by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \nu = \frac{1}{2}(\delta_A + \delta_B)$$

The Fürstenberg-Ishii Condition

Proof. We show that the Fürstenberg-Ishii condition implies the Fürstenberg condition.

Recall that the Fürstenberg-Ishii condition guarantees the existence of two elements A, B of the group such that

$$\mathrm{tr}\, A \neq 0, \quad \mathrm{tr}\, B \neq 0, \quad \det(AB - BA) \neq 0$$

Since $AB - BA \neq 0$, this implies that G contains a non-elliptic element, h, and hence is non-compact (use $g_n = h^n$ to see this).

Since $\det(AB - BA) \neq 0$, it follows that A and B have no common eigenvectors. In particular, there cannot be a set $\Lambda \subseteq \mathbb{RP}^1$ of cardinality one with $A\Lambda = B\Lambda = \Lambda$.

Suppose there exists $\Lambda \subseteq \mathbb{RP}^1$ of cardinality two such that $A\Lambda = B\Lambda = \Lambda$, and denote $\Lambda = \{\bar{u}_1, \bar{u}_2\}$. Since $\mathrm{tr}\, A \neq 0$, one cannot have $A\bar{u}_1 = \bar{u}_2$ and $A\bar{u}_2 = \bar{u}_1$, which forces $A\bar{u}_j = \bar{u}_j$ for $j = 1, 2$.

Similarly, $\mathrm{tr}\, B \neq 0$ forces $B\bar{u}_j = \bar{u}_j$, contradicting again the fact that A and B have no common eigenvectors. $\square$

The Fürstenberg-Ishii Condition: the standard Anderson model

Since the single-site distribution is non-degenerate, there are at least two values $a, b \in \mathbb{R}$, $a \neq b$ in its support. Thus, at energy $E \in \mathbb{R}$ we have the two admissible transfer matrices

$$A^{(E)} = \begin{pmatrix} E-a & -1 \\ 1 & 0 \end{pmatrix}, \quad B^{(E)} = \begin{pmatrix} E-b & -1 \\ 1 & 0 \end{pmatrix}$$

Note that for $E \notin \{a, b\}$, we have $\operatorname{tr} A^{(E)} \neq 0$, $\operatorname{tr} B^{(E)} \neq 0$. Moreover,

$$\begin{aligned}
&\det(A^{(E)}B^{(E)} - B^{(E)}A^{(E)}) \\
&= \det\left[\begin{pmatrix} (E-a)(E-b)-1 & a-E \\ E-b & -1 \end{pmatrix} - \begin{pmatrix} (E-b)(E-a)-1 & b-E \\ E-a & -1 \end{pmatrix}\right] \\
&= \det\begin{pmatrix} 0 & a-b \\ a-b & 0 \end{pmatrix} \\
&= -(a-b)^2 \\
&\neq 0
\end{aligned}$$

By the Fürstenberg-Ishii criterion, it follows via this straightforward calculation that $\mathcal{Z} \subseteq \{a, b\}$. Moreover, verifying the Fürstenberg criterion instead, one finds $\mathcal{Z} = \emptyset$.

The Fürstenberg-Ishii Condition: the continuum Anderson model

Recall that the continuum Anderson model is the random operator

$$H = -\frac{d^2}{dx^2} + \sum_{n \in \mathbb{Z}} q_n(\omega) f(x-n)$$

in $L^2(\mathbb{R})$, where $\mathrm{supp}(f) \subseteq [0,1]$ (and $f \not\equiv 0$) and the q_n's are i.i.d. random variables.

Let us for simplicity consider the case where $f = \chi_{[0,1)}$ and the q_n are Bernoulli random variables taking the values $\lambda > 0$ and 0 with probabilities p and $1-p$, respectively.

In this case we have exactly two basic admissible transfer matrices. Taking $E > \lambda$, they have the form

$$A^{(E)} = \begin{pmatrix} \cos\sqrt{E-\lambda} & \frac{1}{\sqrt{E-\lambda}} \sin\sqrt{E-\lambda} \\ -\sqrt{E-\lambda}\sin\sqrt{E-\lambda} & \cos\sqrt{E-\lambda} \end{pmatrix}$$

$$B^{(E)} = \begin{pmatrix} \cos\sqrt{E} & \frac{1}{\sqrt{E}} \sin\sqrt{E} \\ -\sqrt{E}\sin\sqrt{E} & \cos\sqrt{E} \end{pmatrix}$$

The Fürstenberg-Ishii Condition: the continuum Anderson model

$$A^{(E)} = \begin{pmatrix} \cos\sqrt{E-\lambda} & \frac{1}{\sqrt{E-\lambda}}\sin\sqrt{E-\lambda} \\ -\sqrt{E-\lambda}\sin\sqrt{E-\lambda} & \cos\sqrt{E-\lambda} \end{pmatrix}$$

$$B^{(E)} = \begin{pmatrix} \cos\sqrt{E} & \frac{1}{\sqrt{E}}\sin\sqrt{E} \\ -\sqrt{E}\sin\sqrt{E} & \cos\sqrt{E} \end{pmatrix}$$

Note first that for $E > \lambda$ of the form $E = (k\pi)^2$ or $E = (k\pi)^2 + \lambda$, we immediately get $L(E) = 0$ as one of $A^{(E)}$, $B^{(E)}$ is the identity, and the other has powers whose norms cannot grow exponentially.

In particular, $\mathcal{Z}$ contains an infinite discrete set.

On the other hand,

$$E \mapsto \operatorname{tr} A^{(E)}, \quad \operatorname{tr} B^{(E)}, \quad \det(A^{(E)}B^{(E)} - B^{(E)}A^{(E)})$$

are analytic functions that do not vanish identically. It follows from the Fürstenberg-Ishii criterion that $\mathcal{Z}$ is discrete.

A Basic Inverse Spectral Theory Result

Lemma

If $V_1, V_2 \in L^2[0,T)$ and the transfer matrices mapping solution data from 0 to T obey

$$A^{(E)}(V_1) = A^{(E)}(V_2) \text{ for every } E \in \mathbb{C}$$

then $V_1 = V_2$ Lebesgue almost everywhere on $[0,T)$.

Proof.

Recall the definition of the Weyl-Titchmarsh-function m_j associated with V_j: taking β large enough, for every $E \in \mathbb{C} \setminus [-\beta, \infty)$, there is a unique (modulo an overall multiplicative constant) solution $u_j = u_j(\cdot, E)$ of $-u_j'' + V_j u_j = E u_j$ that satisfies a Dirichlet boundary condition at T, and one then defines

$$m_j(E) = \frac{u_j'(0,E)}{u_j(0,E)}.$$

By the equality of the transfer matrices, we have $m_1 = m_2$, and hence $V_1 \equiv V_2$ (a.e.) by a fundamental inverse spectral theory principle; e.g., [Borg 1952], [Marchenko 1952], [Simon 1999]. $\square$

Setting and Assumptions

Fix two parameters $0 < \delta \leq m$, and define

$$\mathcal{W} = \bigcup_{\delta \leq s \leq m} L^2[0, s).$$

To distinguish the fibers, let us denote the length of the domain by $s = \ell(f)$ whenever $f \in L^2[0, s)$. We specify a continuum Anderson model by choosing a probability measure $\widetilde{\mu}$ on $\mathcal{W}$ subject to the uniform boundedness assumption

$$\nu\text{-ess}\sup \|f\|_{L^2} < \infty.$$

We naturally obtain the full shift

$$\Omega = \mathcal{W}^{\mathbb{Z}}, \quad \mu = \widetilde{\mu}^{\mathbb{Z}}, \quad [T\omega]_n = \omega_{n+1}$$

Then, for each $\omega \in \Omega$, we obtain a potential V_ω by concatenating $\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots$, and an associated Schrödinger operator $H_\omega = -\partial_x^2 + V_\omega$.

Setting and Assumptions

More specifically, define

$$s_n = s_n(\omega) := \begin{cases} \sum_{j=0}^{n-1} \ell(\omega_j) & n \geq 1 \\ 0 & n = 0 \\ -\sum_{j=n}^{-1} \ell(\omega_j) & n \leq -1 \end{cases}$$

denote $I_n = [s_n, s_{n+1})$, and define

$$V_\omega(x) = \omega_n(x - s_n), \qquad \text{for each } x \in I_n$$

For each $w \in \mathcal{W}$, $E \in \mathbb{C}$, $A^E(w)$ is the unique $\mathrm{SL}(2, \mathbb{C})$ matrix with

$$\begin{bmatrix} \psi(s_1) \\ \psi'(s_1) \end{bmatrix} = A^E(w) \begin{bmatrix} \psi(0) \\ \psi'(0) \end{bmatrix} \tag{1}$$

whenever $H_\omega \psi = E\psi$ with $\omega_0 = w$. The Lyapunov exponent $L(E)$ is then defined as before as the average and typical exponential growth rate of the resulting matrix products.

Assumptions and Main Result

One obvious obstruction is if all elements of the support of ν commute in the free product sense, so that one cannot distinguish permutations of elements of the support after concatenation. When this is the case, all realizations V_ω are periodic, and the model is not actually random.

This is the only obstruction; we formulate the negation of this as our nontriviality assumption. For $f_j \in L^2[0, a_j)$, $j = 1, 2$, we write

$$(f_1 \star f_2)(x) = \begin{cases} f_1(x) & 0 \le x < a_1 \\ f_2(x - a_1) & a_1 \le x < a_1 + a_2. \end{cases}.$$

The nontriviality assumption is the following:

$$\text{There exist } f_j \in \operatorname{supp} \nu \text{ such that } f_1 \star f_2 \neq f_2 \star f_1.$$

Let us note that the equality that is assumed to fail is in L^2, so we really mean that $f_1 \star f_2$ and $f_2 \star f_1$ differ on a set of positive Lebesgue measure.

The Main Result

Theorem (Bucaj-D.-Gerbuz-Fillman-VandenBoom-Wang-Zhang)
If ν satisfies the uniform boundedness and nontriviality assumptions, then

$$\mathcal{Z} = \{E : L(E) = 0\}$$

is discrete.

Proof. With f_j from the nontriviality assumption, denote by $M_j(E) = A^E(f_j)$ the associated transfer matrix across the respective interval, and let

$$Q(E) = M_1(E)M_2(E) - M_2(E)M_1(E)$$

By the Fürstenberg-Ishii criterion,

$$\mathrm{tr}\, M_1(E) \neq 0,\ \mathrm{tr}\, M_2(E) \neq 0,\ \det Q(E) \neq 0 \Rightarrow E \notin \mathcal{Z}$$

The functions $\mathrm{tr}\, M_1(E)$, $\mathrm{tr}\, M_2(E)$, $Q(E)$, and $\det Q(E)$ are analytic, and the first two are known to not vanish identically by Floquet theory. The nontriviality assumption and the inverse spectral theory lemma imply that $Q(E)$ does not vanish identically.

We wish to show that $\det Q(E)$ does not vanish identically, either, as this will conclude the proof.

The Main Result

Suppose for the sake of contradiction that $\det Q(E) = 0$ identically. Fix an interval $I \subseteq \mathbb{R}$ such that $M_1(E)$ is elliptic for all $E \in I$; such an interval exists due to Floquet theory.

Since $\det Q(E) = 0$ for $E \in I$, M_1 and M_2 have a common eigenvector. Furthermore, since M_1 is elliptic, the eigenvector may be chosen of the form

$$v = \begin{pmatrix} 1 \\ w \end{pmatrix}, \quad w \in \mathbb{C} \setminus \mathbb{R}$$

Then, since M_1 and M_2 have real entries, one deduces that they both have

$$\bar{v} = \begin{pmatrix} 1 \\ \bar{w} \end{pmatrix}$$

as an eigenvector, and therefore $Q(E) = 0$ for all $E \in I$. But $Q(E)$ only vanishes on a discrete set; contradiction.

Thus, $\operatorname{tr} M_1(E), \operatorname{tr} M_2(E), \det Q(E)$ are nonzero analytic functions, and therefore the Fürstenberg-Ishii condition, sufficient for $L(E) \notin \mathcal{Z}$, holds away from a discrete set. $\square$

Thank you!

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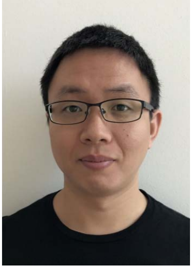

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