On the Duffin-Schaeffer Conjecture: I

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Talk Plan

I'll give four talks on my recent proof with D. Koukoulopoulos of the Duffin-Schaeffer conjecture.

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This first talk will be a colloquium-style overview.

1. Introduction/motivation
2. Statement and consequences
3. High-level overview of key steps

Introduction

Question

How well can you approximate a real number by rationals?

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Theorem (Dirichlet)

Let $\alpha \in \mathbb{R}$. Then there exists infinitely many $a, q \in \mathbb{Z}$ such that

$$\left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2}.$$

Question

Can we do better than this? What about $1/q^3$? $1/q^4$?

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What if we only allow denominators from some subset?

Improved approximations

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Lemma (Golden ratio is badly approximable)

Let $\alpha = (1+\sqrt{5})/2$. For every $a, q \in \mathbb{Z}$ we have

$$\left|\alpha - \frac{a}{q}\right| \geq \frac{1}{3q^2}.$$

Therefore Dirichlet's theorem is essentially best possible!

However, this is specific to a small class of *badly approximable numbers*.

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For **individual** $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, how well we can approximate is usually very difficult.

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Are there infinitely many $a, q \in \mathbb{Z}$ *such that*

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Lemma

Let $\mathcal{S}$ be the set of α such that there are infinitely many a, q with

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^3}.$$

Then $\mathcal{S}$ has measure 0.

Proof: Union bound. Set of α with some approxmation with denominator at least B has size $\le \sum_{q \ge B} 2/q^2 \le 3/B$.

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Theorem (Matomaki)

Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Then there are infinitely many pairs (a, p) with $a \in \mathbb{Z}$ and p prime such that

$$\left|\alpha - \frac{a}{p}\right| \le \frac{1}{p^{4/3-\epsilon}}$$

We expect to improve $4/3$ to 2, but this seems very difficult!

Theorem (Duffin-Schaeffer)

For ***almost all*** $\alpha \in [0, 1]$*, there are infinitely many solutions to*

$$\left|\alpha - \frac{a}{p}\right| < \frac{1}{p^{2-\epsilon}}.$$

Metric Diophantine approximation

- If you want to understand results for **every** α individually, this is often impossibly hard.
- If you allow for a tiny exceptional set, then sometimes you can say much stronger statements.

Principle (Metric Diophantine approximation)

If you are willing to allow an exceptional set of measure 0, you get a much cleaner and more robust theory.

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Question (Main Question)

Let $\Delta : \mathbb{Z} \to \mathbb{R}_{>0}$. Can we understand the set

$$\mathcal{L} := \left\{ \alpha \in [0,1] : \exists \text{ infinitely many } (a,q) \text{ s.t. } \left| \alpha - \frac{a}{q} \right| < \Delta(q) \right\}$$

apart from an exceptional set of measure 0?

Khinchin's Theorem

$$\mathcal{L} := \left\{\alpha \in [0,1] : \ \exists \text{ infinitely many } (a,q) \text{ s.t. } \left|\alpha - \frac{a}{q}\right| < \Delta(q)\right\}$$

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Theorem (Khinchin's Theorem)

Assume that $q^2\Delta(q)$ is ***decreasing****. Then*

$$\text{meas}(\mathcal{L}) = \begin{cases} 1, & \text{if } \sum_q q\Delta(q) = \infty, \\ 0, & \text{if } \sum_q q\Delta(q) < \infty. \end{cases}$$

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This gives an 'almost-all' extension of Dirichlet's theorem!

Corollary

For almost all $\alpha \in [0,1]$, we have infinitely many solutions to

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2 \log q}.$$

For almost no $\alpha \in [0,1]$ do we have infinitely many solutions to

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2 (\log q)^{1+\epsilon}}.$$

0-1 Laws

- 0-1 laws are the reason metric number theory is nice!
- Khinchin's condition that $q^2\Delta(q)$ is decreasing is restrictive.

Question

What happens for ***general*** $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$*?*
Do we have an analogue of Khinchin's Theorem?

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Theorem (Cassels)

*For **any** $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$, we have*

$$\mathrm{meas}(\mathcal{L}) = 0 \quad \textit{or} \quad \mathrm{meas}(\mathcal{L}) = 1.$$

Question

When does $\mathrm{meas}(\mathcal{L}) = 1$, and when does $\mathrm{meas}(\mathcal{L}) = 0$?

This classification is much harder than showing $\mathrm{meas}(\mathcal{L}) \in \{0, 1\}$!

0-1 laws II

0-1 laws remind me of a result from probability.

Lemma (Borel-Cantelli)

Let $E_1, E_2, \dots$ be random events.

1. *If $\sum_j \mathbb{P}(E_j) < \infty$, then almost surely only finitely many E_j occur.*
2. *If $\sum_j \mathbb{P}(E_j) = \infty$ and the E_j are **independent**, then almost surely infinitely many E_j occur.*

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Choose $\alpha \in [0, 1]$ uniformly at random, let E_q the event that α is in

$$\bigcup_{a \pmod q} \left[\frac{a}{q} - \Delta(q), \frac{a}{q} + \Delta(q)\right]$$

First Borel-Cantelli shows that measure 0 part of Khinchin holds for **all** Δ!

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Guess

If $\sum_q q\Delta(q) = \infty$ *then* $\text{meas}(\mathcal{L}) = 1$*.*

- Would remove the decreasing condition in Khinchin's theorem.
- This is saying the E_q are 'quasi-independent' events.

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Proposition (Duffin-Schaeffer)

This guess is false! There exists $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ *such that*

$$\sum_q q\Delta(q) = \infty \quad \textit{but} \quad \mathrm{meas}(\mathcal{L}) = 0.$$

Morally due to overlaps with $a_1/q_1 = a_2/q_2$.

Idea: restrict attention to reduced fractions a/q with $\gcd(a, q) = 1$.

Duffin-Schaeffer conjecture

$$\mathcal{L}^* := \left\{\alpha \in [0,1] : \exists \text{ infinitely many coprime } (a,q) \text{ s.t. } \left|\alpha - \frac{a}{q}\right| < \Delta(q)\right\}$$

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Conjecture (Duffin-Schaeffer)

For ***any*** $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ *we have*

$$\text{meas}(\mathcal{L}^*) = \begin{cases} 1, & \text{if } \sum_q \phi(q)\Delta(q) = \infty, \\ 0, & \text{if } \sum_q \phi(q)\Delta(q) < \infty. \end{cases}$$

Large amount of partial progress dealing with important cases, thanks to *Duffin, Schaeffer, Erdős, Vaaler, Pollington, Vaughan, Harman, Haynes, Beresnevich, Velani, Aistleitner, . . .*

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Theorem (Koukoulopoulos-M.)

The Duffin-Schaeffer conjecture is true.

Duffin-Schaeffer conjecture II

This can be translated back into our original classification problem.

Corollary (Catlin's conjecture)

Let $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{>0}$ *and* $\widetilde{\Delta}(q) := \sup_{q|n} \Delta(n)$. *Then*

$$\text{meas}(\mathcal{L}) = \begin{cases} 1, & \text{if } \sum_q \phi(q)\widetilde{\Delta}(q) = \infty, \\ 0, & \text{if } \sum_q \phi(q)\widetilde{\Delta}(q) < \infty. \end{cases}$$

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Using a result of Beresnevich-Velani, can also determine the Hausdorff measure of $\mathcal{L}$ or $\mathcal{L}^*$ by convergence/divergence criteria.

Corollary

Let $\Delta : \mathbb{Z}_{>0} \to [0, 1/2]$ *and*

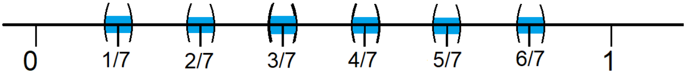
$$s := \inf\{\beta \in \mathbb{R}_{\geq 0} : \sum_q \phi(q)\Delta(q)^\beta < \infty\}.$$

Then $\dim_{\mathcal{H}}(\mathcal{L}^*) = \min(s, 1)$.

Proof is a fun blend of ideas from number theory, combinatorics, ergodic theory,...

Let

$$E_q^* := \bigcup_{(a,q)=1} \left[\frac{a}{q} - \Delta(q), \frac{a}{q} + \Delta(q)\right].$$



We want to show that if $\sum_q \phi(q)\Delta(q) = \infty$, then almost all α lie in infinitely many E_q^*.

Step 1: Quasi-independence

Let

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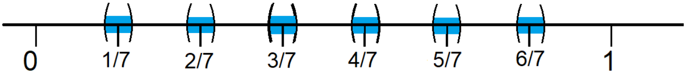

We want to show that if $\sum_q \phi(q)\Delta(q) = \infty$, then almost all α lie in infinitely many E_q^*.

- By Borel-Cantelli, we would be done if these E_q^* behaved as if the were independent. Following the proof, it suffices to show

$$\text{meas}(E_q^* \cap E_r^*) = (1 + o(1))\,\text{meas}(E_q^*)\,\text{meas}(E_r^*) \quad \text{for all } q, r.$$

...but this is too much to hope for.

Step 2: Gallagher's 0-1 Law

Building on Cassels' result, with ergodic theory Gallagher showed

Theorem (Gallagher)

$\text{meas}(\mathcal{L}^*) = 0$ *or* $\text{meas}(\mathcal{L}^*) = 1$.

So $\text{meas}(\mathcal{L}^*) > 0$ implies $\text{meas}(\mathcal{L}^*) = 1$!

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- Using Gallagher's theorem, we only require much weaker quasi-independence.
- It suffices to show

$$\sum_{q,r} \mathrm{meas}(E_q^* \cap E_r^*) \leq 1000000 \sum_q \mathrm{meas}(E_q^*) \sum_r \mathrm{meas}(E_r^*)$$

(an upper bound on average).

This is much more realistic!

Step 3: Sieve bound for measures

Pollington-Vaughan used sieve methods to get an essentially sharp upper bound for $\mathrm{meas}(E_q^* \cap E_r^*)$.

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unless both of the following hold:

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Thus we want to show that on average, it cannot be the case that q, r have a large GCD and have lots of small prime factors.

Step 4: Anatomy of integers

We first concentrate on the 'lots of small prime factors' bit.

The Pollington-Vaughan bound implies

$$\sum_{\substack{p|q \text{ or } r\\ p\geq t}} \frac{1}{p} \leq 10 \quad \Rightarrow \quad \operatorname{meas}(E_q^* \cap E_r^*) \leq (\log t) \operatorname{meas}(E_q^*) \operatorname{meas}(E_r^*).$$

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Lemma (Most numbers don't have lots of prime factors)

$$\frac{1}{x}\#\Big\{n \leq x : \sum_{\substack{p|n \\ p\geq t}} \frac{1}{p} \geq 8\Big\} \leq e^{-t^2}.$$

Thus the rarity of numbers with lots of prime factors outweighs the fact that $\text{meas}(E_q^* \cap E_r^*)$ can be a bit larger **if they occur in the support of Δ with the normal frequency**.

Erdős-Vaaler used this to establish the Duffin-Schaeffer conjecture when $\Delta_q = O(1/q^2)$.

Step 5: Arithmetic Combinatorics

We still need to understand the ‘large GCD’ bit.

Question

If I have a set of integers with lots of pairs of elements having a large GCD, what must that set look like?

There is one easy way of constructing a large set where *all* elements have a large GCD: Take all multiples of a large number.

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Theorem (Approximate structure of GCD sets)

Let $\mathcal{A}$ be a set with many pairs having a large GCD. Then one of the follow holds:

1. *$\mathcal{A}$ is ‘small’.*
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Thus the trivial construction is essentially the only way to make a **large** set.

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Thus the trivial construction is essentially the only way to make a **large** set. **WARNING:** I’m lying/oversimplifying quite a lot here.

Putting it all together

- Using Gallagher+weak Borel-Cantelli, it suffices to show on average

$$\mathrm{meas}(E_q^* \cap E_r^*) \leq 10^6 \,\mathrm{meas}(E_q^*)\,\mathrm{meas}(E_r^*)$$

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- If B $\mathscr{A}$ essentially has a fixed divisor d, then dividing by d reduces to the Erdős-Vaaler setup.
  Since few numbers satisfy 2 we can handle this case.
- Therefore done in either case!

Summary

Using standard ideas:

- **Ergodic theory:** Gallagher's 0-1 law
- **Probability:** Weak Borel-Cantelli
- **Analytic number theory:** Pollington-Vaughan bound

we reduce to a problem of 'quasi-independence'.

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- **Anatomy of integers:** Few integers with many prime factors

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- **Anatomy of integers:** Few integers with many prime factors

we reduce to a structure theorem. Using

- **Graph theory:** Reframe sets as (weighted) dense graphs
- **Combinatorics:** 'Compression' argument

we prove the structure theorem.

Questions

Thanks for listening!