On the Duffin-Schaeffer Conjecture: 2

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Talk Plan

This talk will give an introduction to the key new technical ideas in the proof.

1. Reduce to addive combinatorial problem about sets with large GCDs
2. Reformulate as a combinatorial problem about graphs
3. Overview of the iterative argument for graphs

Main Aim

Recall that we are given a function $\Delta : \mathbb{Z}_{>0} \to \mathbb{R}_{\geq 0}$.

The aim is to show 'quasi independence' on average

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$$\sum_{q,r} \text{meas}(E_q^* \cap E_r^*) \leq 10^6 \sum_{q,r} \text{meas}(E_q^*)\,\text{meas}(E_r^*)$$

(for some suitable range of summation for q, r)

Here

$$E_q := \bigcup_{\substack{1 \leq a \leq q \\ \gcd(a,q)=1}} \left[\frac{a}{q} - \Delta(q), \frac{a}{q} + \Delta(q)\right].$$

Special case

Let's focus on a special case to emphasize the main ideas.

- $\Delta(q) \in \{q^{-1-c}, 0\}$ for some constant $c \in [0,1]$.
- $\Delta(q) = 0$ if q has a repeated prime factor or a factor $< 10^{10^{10}}$.
- $\Delta(q)$ is supported on a union of dyadic intervals $[x_1, 2x_1] \cup [x_2, 2x_2] \cup \dots$ with $x_{i+1} > x_i^2$ and

$$\sum_{q \in [x_i, 2x_i]} \phi(q)\Delta(q) \in [1,2].$$

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$$\sum_{q\in[x_i,2x_i]} \phi(q)\Delta(q) \in [1,2].$$

In this setup, we have that

$$\mathrm{meas}(E_q^*) \ll \frac{\phi(q)}{q} q^{-c}$$

($A \ll B$ means $A = O(B)$ means $|A| \leq cB$ for a constant $c > 0$)

Aim

Let $\mathcal{S} = \{q \in [x, 2x] : \Delta(q) \neq 0\}$. Then

$$\sum_{q,r\in\mathcal{S}} \mathrm{meas}(E_q^* \cap E_r^*) = O(1).$$

Simplification II

Lemma (Pollington-Vaughan bound)

$$\sum_{\substack{p|q \text{ or } r\\ p\geq t}} \frac{1}{p} \leq 11 \quad \Rightarrow \quad \mathrm{meas}(E_q^*\cap E_r^*) \leq 100(\log t)\,\mathrm{meas}(E_q^*)\,\mathrm{meas}(E_r^*).$$

Moreover, if $\gcd(q,r) \leq x^{1-c}/t$ *then*

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This reduces to showing for every set $\mathcal{S} \subset [x, 2x]$ and every $t \geq 1$

$$\sum_{\substack{q,r\in\mathcal{E}_t\\ \gcd(q,r)\geq x^{1-c}/t}} \frac{\phi(q)}{q}\frac{\phi(r)}{r} = O\Big(\frac{x^{2c}}{t}\Big),$$

where

$$\sum_{q\in\mathcal{S}} \frac{\phi(q)}{q} = O(x^c), \qquad \mathcal{E}_t := \Big\{(q,r)\in\mathcal{S}^2 : \sum_{\substack{p|qr/\gcd(q,r)^2\\ p\geq t}} \frac{1}{p} \geq 10\Big\}.$$

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$$\sum_{\substack{q,r \in \mathcal{E}_t \\ \underbrace{\gcd(q,r) \geq x^{1-c}/t}_{\text{GCD condition}}}} \underbrace{\frac{\phi(q)}{q}\frac{\phi(r)}{r}}_{\text{weights} \approx 1} = O\Big(\frac{x^{2c}}{t}\Big),$$

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Model problem

If we ignore the $\phi(q)/q$ weights, this simplifies to: If $\#\mathcal{S} = x^c$ and $\mathcal{S} \subseteq [x, 2x]$, is it the case that

$$\#\Big\{(q,r) \in \mathcal{E}_t : \gcd(q,r) \geq x^{1-c}/t\Big\} = O(x^{2c}/t)?$$

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Problem (Model problem)

What sets $\mathcal{S} \subseteq [x, 2x]$ have $\#\mathcal{S} \approx x^c$ and 1% of pairs $(s_1, s_2) \in \mathcal{S}^2$ with $\gcd(s_1, s_2) \geq x^{1-c}$?

We hope to say that any such set must be very structured, and so can then be analysed explicitly.

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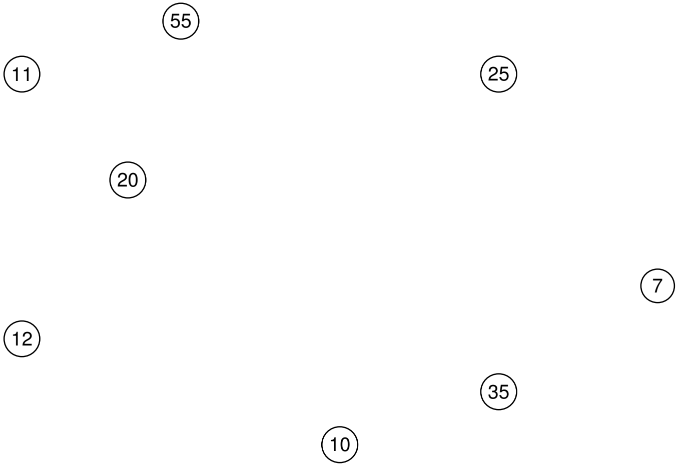

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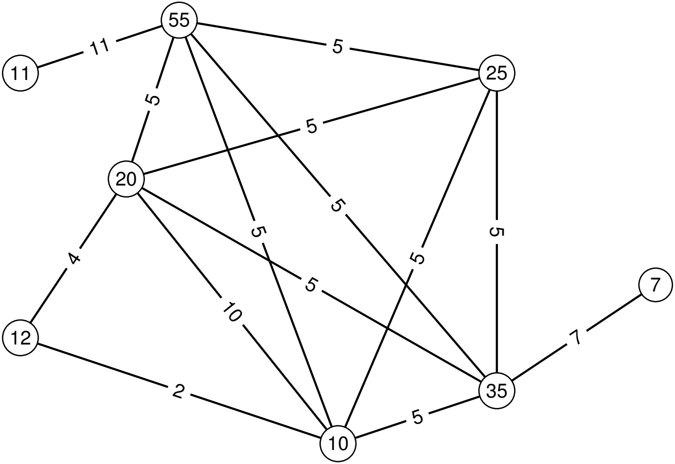

A positive proportion of pairs have a large GCD, so a **dense graph**.

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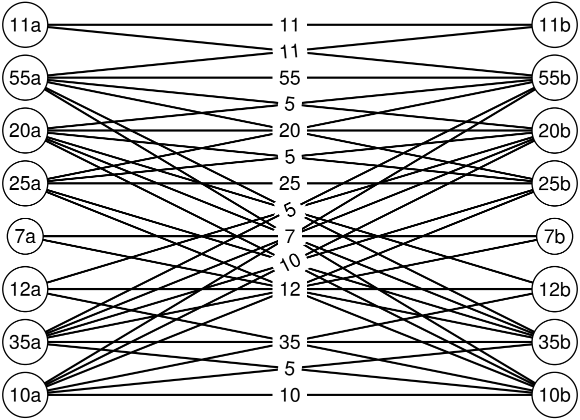

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- We don't try to control the (finite) number of iterations.
- Since the final graph cannot be iterated further, we hope that it will be simple to explicitly analyse.
  (e.g. all vertices are a multiple of d, so everything connected.)
- If we increase structure at each step, we hope that we can understand our original graph from the final one.

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$$V_p := \{v \in V : p|v\}, \qquad V_{\widehat{p}} := \{v \in V : p \nmid v\},$$
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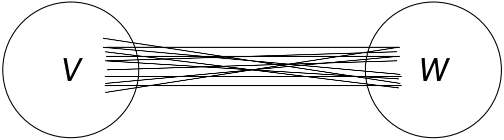

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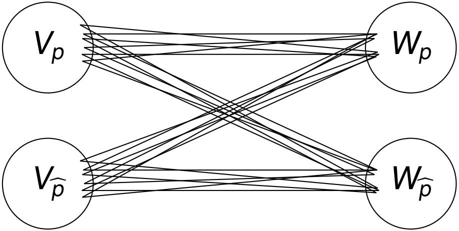

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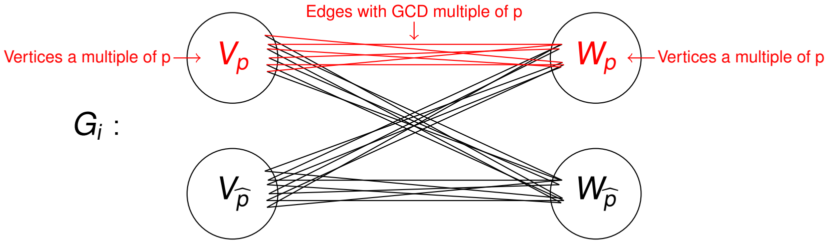

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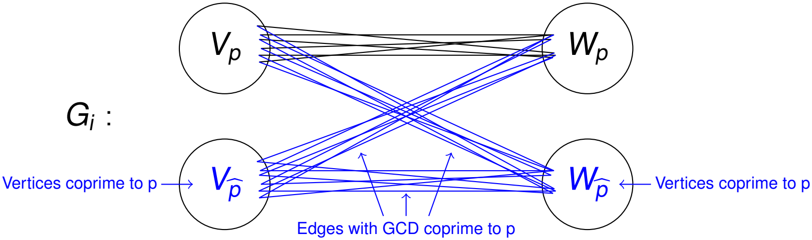

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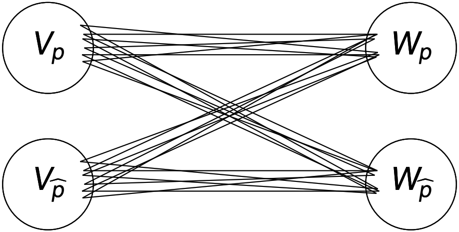

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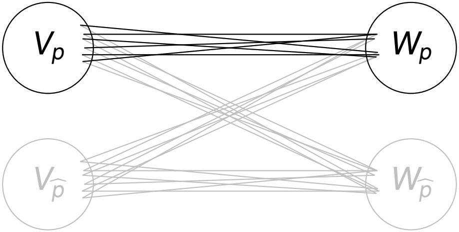

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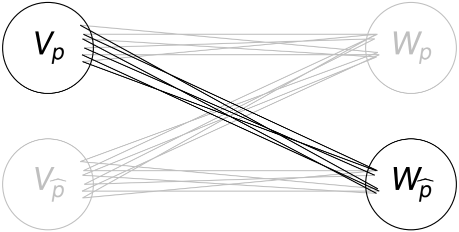

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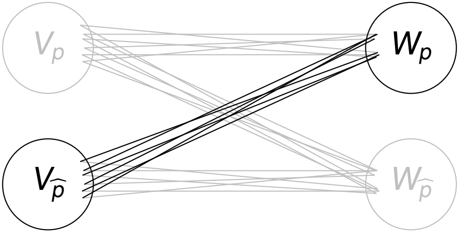

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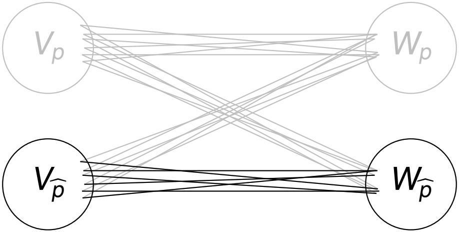

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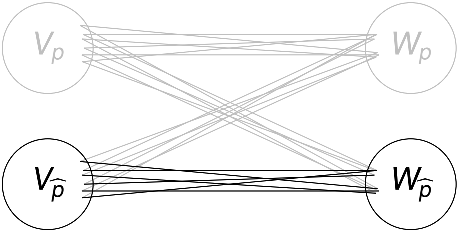

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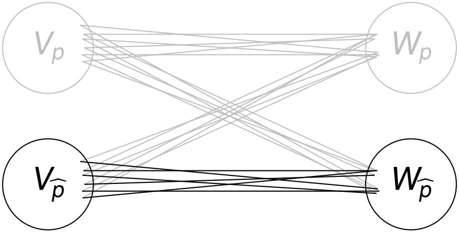

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We need to keep track of previous iterations, so consider a bipartite graph G with a set P of primes.

Definition (Quality of a graph)

*We define the **quality** of G with vertex sets V, W and set of primes P as*

$$q(G) = \delta^{10} \cdot \#V \cdot \#W \cdot \prod_{p \in P} p$$

where δ is the edge density.

P is the set of all primes where we have chosen $(V_p, W_{\widehat{p}})$ or $(V_{\widehat{p}}, W_p)$ in earlier iterations.

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- Begin with G_{start}, a dense bipartite graph with vertex sets in $[x, 2x]$ and edges formed by joining v, w if $\gcd(v, w) \geq x^{1-c}$.
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Let G_{start} have edge set E_{start}. Then

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Lemma (Quality increment)

If $\max(\#V_{\widehat{p}}/\#V, \#W_{\widehat{p}}/\#W) \geq 10^{40}/p$ then we can choose a subgraph G' from $V_p, V_{\widehat{p}}, W_p, W_{\widehat{p}}$ with

$$q(G') \geq q(G).$$

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If we pretended that we can always find G', then we see that we can ensure $q(G_{end}) \geq q(G_{start})$, so

$$\#E_{start} \ll x^{2c}$$

This is almost what we want! If $\#E_{start} \approx x^{2c}$ then iterations must lose nothing, which is very close to giving our structure theorem.

Application to DS

For the DS problem we only need to save in the upper bound based on the threshold t for small prime factors. In this case

$$E_{start} = \mathcal{E}_t := \Bigg\{(q,r) \in \mathcal{S}^2 : \sum_{\substack{p \mid qr/\gcd(q,r)^2 \\ p \geq t}} \frac{1}{p} \geq 10\Bigg\}.$$

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Lemma (Quality controls our DS graph)

Let G_{start} have edge set $\mathcal{E}_t$. Then either

$$\#\mathcal{E}_t \ll \frac{q(G_{start})}{q(G_{end})} x^{2c} e^{-t}$$

or $q(G_{end}) \geq e^t q(G_{start})$.

Thus in either case $\#\mathcal{E}_t \ll x^{2c} e^{-t}$, which is what we need.

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- Since each prime can only be chosen once, this procedure terminates when there are no primes left.

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- By Lemma, there is a subgraph $G' \in \{G_{p,p}, G_{\widehat{p},p}, G_{p,\widehat{p}}, G_{\widehat{p},\widehat{p}}\}$ with $q(G') \geq q(G)$. If $G' = G_{p,\widehat{p}}$ or $G_{\widehat{p},p}$ then $P_{i+1} := P_i \cup \{p\}$ and otherwise $P_{i+1} := P_i$. We set $G_{i+1} := G'$.
- Since each prime can only be chosen once, this procedure terminates when there are no primes left.
- Since we can't iterate any more, if $p | \gcd(v, w)$ for some $v \in V_{end}, w \in W_{end}$ then $p|v$ for all v and $p|w$ for all w.

Idealised iteration procedure

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- If $p \in P_{end}$ then $p|v$ for all v and $p \nmid w$ for all w, or vice versa.
- Thus there are a, b such that $a|v$ for all $v \in V_{end}$, $b|w$ for all $w \in W_{end}$, and $(v, w) \in E_{end} \Rightarrow \gcd(v, w) = \gcd(a, b) \geq x^{1-c}$.

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Recall:

Lemma (Quality increment)

If $\max(\#V_{\widehat{p}}/\#V, \#W_{\widehat{p}}/\#W) \geq 10^{40}/p$ *then we can choose a subgraph G' from $V_p, V_{\widehat{p}}, W_p, W_{\widehat{p}}$ with*

$$q(G') \geq q(G).$$

This adequately gives quality increments **provided** α_p, β_p **are not both close to 1**.

(The argument generalizes easily to weighted graphs - DS problem has $\phi(q)/q$ weights on vertices.)

Difficult case

Question

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This is because the model problem has a counterexample!

Example (S. Chow)

If $\mathcal{S} = \{n!/j : j \in [n/2, n]\} \subseteq [(n-1)!, 2(n-1)!]$ *then:*

- *All pairs* $s_1, s_2 \in \mathcal{S}$ *have* $\gcd(s_1, s_2) \geq n!/n^2$.
- *There is no* $d \in \mathbb{Z}$ *of size* $\gg (n-1)!/\#\mathcal{S}$ *which divides a positive proportion of elements.*

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- To account for this, we use a different (more technical) definition of quality.
- This change in argument means the iteration needs to be done it two stages.
- This difficult situation is precisely that of the counterexample.

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- This essentially puts us in the Erdős-Vaaler situation and so compatible with bounds from the ‘anatomy of integers’.
- Extra care is needed when almost all elements are a multiple of many small primes p. In this case the $\phi(q)/q$ weights which are natural in the DS problem save us!

Questions

Thank you for listening.