On the Duffin-Schaeffer Conjecture: 3

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Talk Plan

This talk will give some details about the key technical ideas in the iterative argument.

1. Show that it suffices to get ‘quality increments’
   - Show how the quality of the final graph can be estimated easily
   - Show that quality of the final graph controls our original graph
2. Reduce the question of quality increments to the difficult case
   - Some details about quality increments in the easy case
   - Some details about quality increments in a side case

'Quality'

Recall:

Definition (Quality of a graph)

*We define the **quality** of G with vertex sets V, W and set of primes P as*

$$q(G) = \delta^{10} \cdot \#V \cdot \#W \cdot \prod_{p \in P} p$$

where δ is the edge density.

P is the set of all primes where we have chosen $(V_p, W_{\widehat{p}})$ or $(V_{\widehat{p}}, W_p)$ in earlier iterations.

The iterative argument

$$q(G) = \delta^{10} \cdot \#V \cdot \#W \cdot \prod_{p \in P} p$$

Lemma (Quality controls our original graph)

Let G_{start} have edge set E_{start}. Then

$$\#E_{start} \ll \frac{q(G_{start})}{q(G_{end})} x^{2c}$$

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If $\max(\#V_{\widehat{p}}/\#V, \#W_{\widehat{p}}/\#W) \geq 10^{40}/p$ *then we can choose a subgraph G' from $V_p, V_{\widehat{p}}, W_p, W_{\widehat{p}}$ with*

$$q(G') \geq q(G).$$

(And a version of this could be done in the hard case when $\#V_p \approx \#V$ and $\#W_p \approx \#W$.)

Quality controls our original graph

Let's begin by focusing on the first lemma.

Lemma (Quality controls our original graph)

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- There are a, b such that $a|v$ for all $v \in V_{end}$, $b|w$ for all $w \in W_{end}$, and $(v, w) \in E_{end} \Rightarrow \gcd(v, w) = \gcd(a, b) \geq x^{1-c}$.

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Recall our definition of quality:

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We calculate

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But trivially $\#E_{end} \ll x^2/(ab)$, $\delta_{end} \leq 1$, so since $\gcd(a, b) \geq x^{1-c}$

$$q(G_{end}) \ll x^{2c}.$$

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This gives the lemma!

This argument loses essentially nothing. It is vital that we have a sharp bound for $q(G_{end})$.

Quality controls our original graph III

Moreover, this can be adapted to take into account 'most numbers have few prime factors'. Recall:

$$E_{start} = E_t = \Big\{ v, w \in \mathcal{S} : \sum_{\substack{p \mid vw/\gcd(vw)^2 \\ p \geq t}} \frac{1}{p} \geq 10 \Big\}.$$

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Lemma (Anatomy of integers)

$$\#\Big\{ n \leq x : \sum_{\substack{p \mid n \\ p \geq t}} \frac{1}{p} \geq 1 \Big\} \leq e^{-t} x.$$

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Lemma (Anatomy of integers)

$$\#\Big\{ n \leq x : \sum_{\substack{p|n \\ p \geq t}} \frac{1}{p} \geq 1 \Big\} \leq e^{-t} x.$$

Using this, provided $\sum_{p|ab/\gcd(a,b)^2, p \geq t} 1/p \leq 5$ we find that

$$\#E_{end} \leq \#\Big\{ v, w \leq 2x : \sum_{\substack{p | vw/\gcd(vw)^2 \\ p \geq t}} \frac{1}{p} \geq 10,\ a|v,\ b|w \Big\} \ll \frac{x^2}{ab} e^{-t}$$

Thus we win an extra factor of e^{-t}. (If $ab/\gcd(a,b)^2$ have a lot of small prime factors then we will get a big quality increment.)

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Let G_{start} have edge set $\mathcal{E}_t$. Then either

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or $\sum_{p \in P_{end}, p \geq t} 1/p \geq 5$ and

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In the the first case we want to show $q(G_{start}) \geq q(G_{end})$ and in the second case we want to show $q(G_{end}) \geq e^t q(G_{start})$.

So regardless $\#\mathcal{E}_t \ll x^{2c} e^{-t}$, which is what we need.

Quality increments

Now lets focus on the second lemma.

Lemma (Quality increment in the easy case)

If $\max(\#V_{\widehat{p}}/\#V, \#W_{\widehat{p}}/\#W) \geq 10^{40}/p$ *then we can choose a subgraph* G' *from* $V_p, V_{\widehat{p}}, W_p, W_{\widehat{p}}$ *with*

$$q(G') \geq q(G).$$

(And a version of this could be done in the hard case when $\#V_p \approx \#V$ and $\#W_p \approx \#W$.)

Quality increments: notation

Recall $V_p = \{v \in V : p|v\}$, $V_{\widehat{p}} = \{v \in V : p \nmid v\}$.

- Let $\alpha_p, \beta_p \in [0, 1]$ be defined by:
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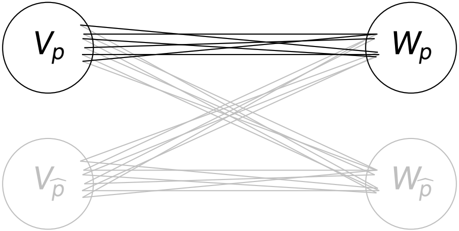
$G_{p,p}$:



$\delta_{p,p}\alpha_p\beta_p\#V\#W$ edges

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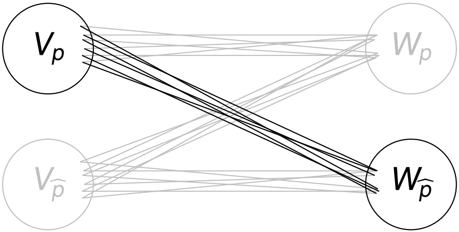

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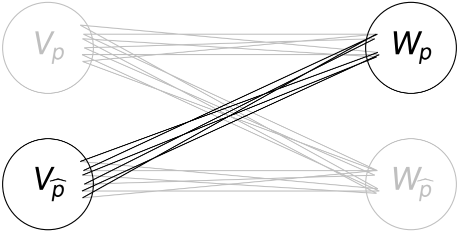

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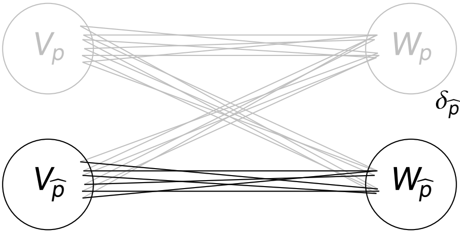

$\delta_{\widehat{p},\widehat{p}}(1-\alpha_p)(1-\beta_p)\#V\#W$ edges

Calculation of qualities

With this notation, it is easy to calculate the qualities of each subgraph:

$$\frac{q(G_{p,p})}{q(G)} = \Big(\frac{\delta_{p,p}}{\delta}\Big)^{10}\alpha_p\beta_p, \qquad \frac{q(G_{\widehat{p},\widehat{p}})}{q(G)} = \Big(\frac{\delta_{\widehat{p},\widehat{p}}}{\delta}\Big)^{10}(1-\alpha_p)(1-\beta_p),$$

$$\frac{q(G_{\widehat{p},p})}{q(G)} = p\Big(\frac{\delta_{\widehat{p},p}}{\delta}\Big)^{10}(1-\alpha_p)\beta_p, \qquad \frac{q(G_{p,\widehat{p}})}{q(G)} = p\Big(\frac{\delta_{p,\widehat{p}}}{\delta}\Big)^{10}\alpha_p(1-\beta_p).$$

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Since $\#E = \#E_{p,p} + \#E_{p,\widehat{p}} + \#E_{\widehat{p},p} + \#E_{\widehat{p},\widehat{p}}$ we find that

$$\delta = \delta_{p,p}\alpha_p\beta_p + \delta_{p,\widehat{p}}\alpha_p(1-\beta_p) + \delta_{\widehat{p},p}(1-\alpha_p)\beta_p + \delta_{\widehat{p},\widehat{p}}(1-\alpha_p)(1-\beta_p).$$

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Question

With this constraint, for which values of α_p, β_p must one of the subgraphs have increased quality?

This is an elementary problem in real analysis.

Further calculation

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$$\delta_{p,p} \leq \frac{\delta}{(\alpha_p\beta_p)^{1/10}}, \qquad \delta_{\widehat{p},p} \leq \frac{\delta}{(p(1-\alpha_p)\beta_p)^{1/10}}, \qquad \text{etc}$$

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$$1 \leq (\alpha_p\beta_p)^{9/10} + (1-\alpha_p)^{9/10}(1-\beta_p)^{9/10} + \frac{\alpha_p^{9/10}(1-\beta_p)^{9/10} + (1-\alpha_p)^{9/10}\beta_p^{9/10}}{p^{1/10}}.$$

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For any $x, y \in [0,1]$ with $xy \geq (1-x)(1-y)$

$$\begin{aligned}
(xy)^{9/10} + (1-x)^{9/10}(1-y)^{9/10} &\leq (xy)^{2/5}\Big((xy)^{1/2} + (1-x)^{1/2}(1-y)^{1/2}\Big) \\
&\leq (xy)^{2/5}\Big(\frac{x+y}{2} + \frac{(1-x)+(1-y)}{2}\Big) \\
&= (xy)^{2/5}.
\end{aligned}$$

Quality increment

- If $q(G_{p,p}), q(G_{\widehat{p},\widehat{p}}), q(G_{p,\widehat{p}}), q(G_{\widehat{p},p}) \leq q(G)$ then

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- If $xy \geq (1-x)(1-y)$ then $(xy)^{9/10} + (1-x)^{9/10}(1-y)^{9/10} \leq (xy)^{2/5}$.
- By symmetry we may assume $\alpha_p\beta_p \geq (1-\alpha_p)(1-\beta_p)$ and $\alpha_p \geq \beta_p$ (so $\alpha_p(1-\beta_p) \geq \beta_p(1-\alpha_p)$).

Quality increment

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- If $xy \ge (1-x)(1-y)$ then $(xy)^{9/10} + (1-x)^{9/10}(1-y)^{9/10} \le (xy)^{2/5}$.
- By symmetry we may assume $\alpha_p\beta_p \ge (1-\alpha_p)(1-\beta_p)$ and $\alpha_p \ge \beta_p$ (so $\alpha_p(1-\beta_p) \ge \beta_p(1-\alpha_p)$).

Thus

$$1 \le \alpha_p^{2/5}\beta_p^{2/5} + \frac{\alpha_p^{2/5}(1-\beta_p)^{2/5}}{p^{1/10}} \le \beta_p^{2/5} + \frac{(1-\beta_p)^{2/5}}{p^{1/10}}.$$

But this only holds if $\beta_p = 1 - O(p^{-1/6})$. By being slightly more careful, or substituting this into the constraint again, this can be refined to $\beta_p \ge 1 - 10^{10}p^{-1}$.

Quality increment

- If $q(G_{p,p}), q(G_{\widehat{p},\widehat{p}}), q(G_{p,\widehat{p}}), q(G_{\widehat{p},p}) \le q(G)$ then

$$1 \le (\alpha_p\beta_p)^{9/10}+(1-\alpha_p)^{9/10}(1-\beta_p)^{9/10}+\frac{\alpha_p^{9/10}(1-\beta_p)^{9/10}+(1-\alpha_p)^{9/10}\beta_p^{9/10}}{p^{1/10}}.$$

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But this only holds if $\beta_p = 1 - O(p^{-1/6})$. By being slightly more careful, or substituting this into the constraint again, this can be refined to $\beta_p \ge 1 - 10^{10}p^{-1}$.

So there is a quality increment unless $\alpha_p \approx \beta_p \approx 1$ or $\alpha_p \approx \beta_p \approx 0$.

Quality increments: Summary

Lemma (Quality increment in non-extremal cases)

Assume $\min(\alpha_p, \beta_p) \leq 1 - 10^{40}/p$ *and* $\max(\alpha_p, \beta_p) \geq 10^{40}/p$.

Then there is a $G' \in \{G_{p,p}, G_{p,\widehat{p}}, G_{\widehat{p},p}, G_{\widehat{p},\widehat{p}}\}$ *with*

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Useful technical point: The same argument actually shows on of the following holds

$$\max\Big(q(G_{p,p}), q(G_{\widehat{p},\widehat{p}})\Big) \geq q(G),$$
$$\max\Big(q(G_{p,\widehat{p}}), q(G_{\widehat{p},p})\Big) \geq 100q(G).$$

(This is useful for dealing with the case $\sup_{p \in P_{end}, p \geq t} 1/p > 5$)

Quality increments in extreme cases

We have an easy argument which gives a good iteration in most cases.

We need to think about what happens in the remaining cases:

1. $\alpha_p \approx \beta_p \approx 0$,
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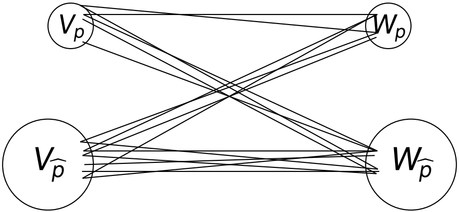
Let's first think about $\alpha_p \approx \beta_p \approx 0$.

Few vertices a multiple of p

If $\alpha_p \approx \beta_p \leq 10^{10}/p$ then V_p, W_p are very small, so there should be virtually no edges between them (a proportion $O(1/p^2)$).



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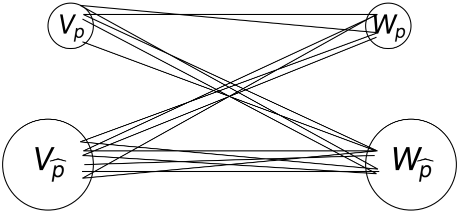

Edges between V_p, W_p are the only edges corresponding to a $\gcd$ being a multiple of p.

If they make a proportion $\leq 1/p^{3/2}$ of edges, we can remove all such edges for all primes p and we will only ever lose at most 1% of our edges/quality in total, since $\prod_p (1 - 1/p^{3/2})$ converges.

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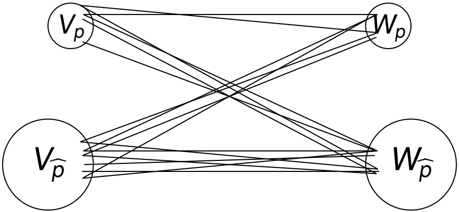

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Good if $\delta_{p,p} \leq \delta p^{3/2} \alpha_p^{-1} \beta_p^{-1}$

Few vertices a multiple of p

On the other hand, if $\alpha_p, \beta_p \le 10^{10}/p$ are small but there are many more edges than expected between V_p, W_p, then $G_{p,p}$ must be of much higher density.

$$\frac{q(G_{p,p})}{q(G)} = \Big(\frac{\delta_{p,p}}{\delta}\Big)^{10}\alpha_p\beta_p.$$

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Thus we're good either way!

Few vertices a multiple of p: Summary

Thus for fairly trivial reasons we don't need to worry about the case $\alpha_p \approx \beta_p \approx 0$.

Lemma (Few vertices a multiple of p gives a tiny quality loss)

If $\alpha_p, \beta_p \le 10^{40}/p$, then for $G' = G_{p,\widehat{p}} \cup G_{\widehat{p},p} \cup G_{\widehat{p},\widehat{p}}$ we have

$$q(G') \ge q(G)\Big(1 - \frac{10}{p^{3/2}}\Big).$$

or

$$q(G_{p,p}) \ge 10^{10} q(G).$$

The loss in the first case is so small that it is OK for us.

Reduced to difficult case

So far:

- We have adequate quality increments **provided α_p, β_p are not both close to 1**.
- The argument is actually very flexible and works for **weighted graphs** (which is actually what comes up in DS problem)
- This is close to a structural result; we can reduce to the situation where for every prime p dividing a GCD p divides most of the elements on both sides.
- Recall: if $\alpha_p, \beta_p \approx 1$ we **cannot** obtain a quality increment in general (with our current setup).

Next time we'll see how to handle quality increments in this case too, relying on extra structure in the DS problem.

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- Thus it suffices to show $q(G_{start}) \le q(G_{end})$.
- If α_p, β_p are not both close to 0 or both close to 1, it is easy to find a quality increment.
- The case $\alpha_p \approx \beta_p \approx 0$ can be handled by looking at it specifically.
- We're left to handle the difficult case of $\alpha_p \approx \beta_p \approx 1$.

Questions

Thank you for listening.