On the Duffin-Schaeffer Conjecture: 4

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Talk Plan

This talk will give some details about the key technical ideas in the iterative argument.

1. Handle quality increments in the difficult case
2. Show these quality increments are suitable for the proof
3. Put everything together to finish proof
4. Reflect on the argument
5. Further problems

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- Thus it just requires us to get suitable quality increments when $\alpha_p, \beta_p \approx 1$.

Recall: if $\alpha_p, \beta_p \approx 1$ we **cannot** obtain a quality increment in general (with our current setup). We've reduced to the situation of our counterexample!

Need to use extra structure specific to the DS problem.

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Show for every set $\mathcal{S} \subset [x, 2x]$ *and every* $t \geq 1$

$$\sum_{\substack{q,r\in\mathcal{E}_t \\ \gcd(q,r)\geq x^{1-c}/t}} \underbrace{\frac{\phi(q)}{q}\frac{\phi(r)}{r}}_{\textit{weights} \approx 1} = O\Big(\frac{x^{2c}}{t}\Big),$$

where

$$\sum_{q\in\mathcal{S}} \underbrace{\frac{\phi(q)}{q}}_{\textit{weights} \approx 1} = O(x^c), \qquad \mathcal{E}_t := \Big\{(q,r)\in\mathcal{S}^2 : \sum_{\substack{p|qr/\gcd(q,r)^2 \\ p\geq t}} \frac{1}{p} \geq 10\Big\}.$$

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Since $\phi(q)/q = \prod_{p|q}(1-1/p)$, we gain an additional factor of $(1-1/p)$ whenever we choose to restrict to V_p or W_p.

So we can afford to lose a few factors of $1 - 1/p$ in our quality

To handle the difficult case we therefore need to use an argument which is sensitive to the specific weights in the DS problem.

Weights in the DS problem II

To handle the difficult case we therefore need to use an argument which is sensitive to the specific weights in the DS problem.

Aim:

1. Show that we can obtain suitable increments if we allow for a loss of $(1 - 1/p)$ factors
2. Show that we can still get an adequate result if we have the $\phi(q)/q$ weights.

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Let's first show that we can get suitable increments if these losses are acceptable.

Difficult case increments

Lemma (Almost-quality increment in difficult case)

Let G be a GCD graph, p a prime and $\alpha_p, \beta_p \geq 1 - 10^{10}/p$. Then one of the following holds:

1. $q(G_{p,p}) \geq \Big(1 - \frac{1}{p}\Big)^2 \Big(1 - \frac{1}{p^{3/2}}\Big) q(G)$,
2. *There is a $G' \in \{G_{p,\widehat{p}}, G_{\widehat{p},p}\}$ such that $q(G') \geq q(G)$.*

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In the second case we get a quality increment. In the first case

- The weights $\phi(q)/q$ will balance out the factor $(1 - 1/p)^2$.
- The total loss from $(1 - 1/p^{3/2})$ over all iterations is bounded since $\prod_p (1 - 1/p^{3/2})$ converges.

Proof for difficult case

Imagine for a contradiction $q(G_{p,p}) \leq (1 - 1/p)^2(1 - 1/p^{3/2})q(G)$ and $q(G_{p,\widehat{p}}), q(G_{\widehat{p},p}) \leq q(G)$.

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$$\delta_{p,p} \leq \delta(1-1/p)^{2/10}(1-1/p^{3/2})^{1/10}\alpha_p^{-1/10}\beta_p^{-1/10}$$
$$\delta_{p,\widehat{p}} \leq \delta p^{1/10}\alpha_p^{-1/10}(1-\beta_p)^{-1/10}.$$

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Substituting this all into our constraint

$$\delta = \delta_{p,p}\alpha_p\beta_p + \delta_{p,\widehat{p}}\alpha_p(1 - \beta_p) + \delta_{\widehat{p},p}(1 - \alpha_p)\beta_p + \delta_{\widehat{p},\widehat{p}}(1 - \alpha_p)(1 - \beta_p)$$

gives

$$1 \leq \Big(1 - \frac{1}{p}\Big)^{2/10}\Big(1 - \frac{A}{p}\Big)^{9/10}\Big(1 - \frac{B}{p}\Big)^{9/10}\Big(1 - \frac{1}{p^{3/2}}\Big)^{1/10} + \frac{B^{9/10} + A^{9/10}}{p} + O\Big(\frac{1}{p^{9/5}}\Big).$$

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$$=1-\frac{(2+9A+9B)}{10p}+\frac{B^{9/10}+A^{9/10}}{p}-\frac{1}{p^{3/2}}+O\Big(\frac{1}{p^{9/5}}\Big).$$

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Contradiction if p is large!

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Therefore either a quality increment, or $G_{p,p}$ with a controlled loss in quality

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Lemma (Quality controls our DS graph)

Let G_{start} have edge set $\mathcal{E}_t$. If $\sum_{p\in P_{end}, p\geq t} 1/p \leq 5$ then

$$\#\mathcal{E}_t \ll \frac{q(G_{start})}{q(G_{end})} x^{2c} e^{-t}.$$

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Refined version:

Lemma (Quality controls our DS graph with weighting)

Let G_{start} have edge set $\mathcal{E}_t$. If $\sum_{p\in P_{end}, p\geq t} 1/p \leq 5$ then

$$\sum_{q,r\in\mathcal{E}_t} \frac{\phi(q)}{q}\frac{\phi(r)}{r} \ll \frac{q(G_{start})}{q(G_{end})} x^{2c} e^{-t} \prod_{p\in P_{bad}} \Big(1-\frac{1}{p}\Big)^2,$$

where P_{bad} is the set of primes where we choose $G_{p,p}$ in the difficult case.

Quality controls our DS graph with weighting

Before, proof used

$$\#E_{end} \leq \#\Big\{v, w \leq 2x : \sum_{\substack{p | vw/\gcd(vw)^2 \\ p \geq t}} \frac{1}{p} \geq 10,\ a|v,\ b|w\Big\} \ll \frac{x^2}{ab} e^{-t}.$$

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Instead, we use

$$\sum_{(q,r)\in E_{end}} \frac{\phi(q)}{q}\frac{\phi(r)}{r} \leq \sum_{\substack{v,w\leq 2x \\ a|v,\, b|w \\ \sum_{p|vw/\gcd(vw)^2, p\geq t} 1/p \geq 10}} \frac{\phi(v)}{v}\frac{\phi(w)}{w}$$

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$$\leq \frac{\phi(a)}{a}\frac{\phi(b)}{b}\sum_{\substack{v'\leq 2x/a,\, w'\leq 2x/b\\ \sum_{p|v'w'/\gcd(v'w')^2, p\geq t} 1/p\geq 5}} 1$$

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$$\ll \frac{\phi(a)}{a}\frac{\phi(b)}{b}\frac{x^2}{ab}e^{-t}$$

$$\ll \frac{x^2}{ab}e^{-t}\prod_{p\in P_{bad}}\Big(1-\frac{1}{p}\Big)^2.$$

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into our (weighted) quality gives

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Thus

$$\sum_{q,r\in\mathcal{E}_t} \frac{\phi(q)}{q}\frac{\phi(r)}{r} \ll q(G_{start}) \ll \frac{q(G_{start})}{q(G_{end})}x^{2c}e^{-t}\prod_{p\in P_{bad}}\Big(1-\frac{1}{p}\Big)^2.$$

(An analogous argument works for when $\sum_{p|ab/\gcd(a,b)^2} 1/p \geq 5$)

Summary for quality increments

For our iteration, we consider three cases: if $\alpha_p \approx \beta_p \approx 0$, if $\alpha_p \approx \beta_p \approx 1$ or if neither holds.

- Easy case: We can always obtain a quality increment unless $\alpha_p \approx \beta_p \approx 1$ or $\alpha_p \approx \beta_p \approx 0$.

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- If $\alpha_p \approx \beta_p \approx 1$, then either we have a quality increment, or choose $G_{p,p}$ and a loss of at most a factor $(1 - 1/p)^2(1 - 1/p^{3/2})$.
- This is (just!) acceptable for our final bounds.

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4. For the choice of Δ we are considering, this says

$$\sum_{q,r \in \mathcal{E}_t} \frac{\phi(q)}{q} \frac{\phi(r)}{r} \ll e^{-t} x^{2c}.$$

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9. If $\alpha_p \approx \beta_p \approx 1$, we find a subgraph with $q(G_{i+1}) \geq q(G_i)$ or $G_{i+1} = G_{p,p}$ with $q(G_{i+1}) \geq q(G_i)(1-1/p)^2(1-1/p^{3/2})$.

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9. If $\alpha_p \approx \beta_p \approx 1$, we find a subgraph with $q(G_{i+1}) \geq q(G_i)$ or $G_{i+1} = G_{p,p}$ with $q(G_{i+1}) \geq q(G_i)(1 - 1/p)^2(1 - 1/p^{3/2})$.
10. We end up with G_{end} where we can't iterate further and

$$q(G_{end}) \gg q(G_{start}) \prod_{p \in P_{bad}} \left(1 - \frac{1}{p}\right)^2.$$

Putting everything together II

5. We re-interpret this as a (weighted) sum over edges in a complicated graph G_{start}. The sum is bounded by $q(G_{start})$.
6. We repeatedly pass to subgraphs $G_{start} \supseteq G_1 \supseteq G_2 \supseteq \dots$ by choosing a prime p and a subgraph $G_{p,p}$, $G_{p,\widehat{p}}$, $G_{\widehat{p},p}$ or $G_{\widehat{p},\widehat{p}}$.
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$$q(G_{end}) \gg q(G_{start}) \prod_{p\in P_{bad}} \Big(1-\frac{1}{p}\Big)^2.$$
11. Since we can't iterate, G_{end} is very simple, and we calculate
$$q(G_{end}) \ll x^{2c} e^{-t} \prod_{p\in P_{bad}} \Big(1-\frac{1}{p}\Big)^2.$$

Putting everything together III

Thus

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I've sketched how to make the argument work, but I produced the definition of 'quality' out of thin air.

Question

Why this definition of quality?

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Naiive guess: If $V \subseteq [V_1, 2V_1]$ and $W \subseteq [W_1, 2W_1]$, then the number of pairs with gcd at least d should be

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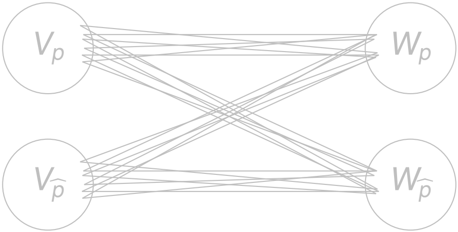

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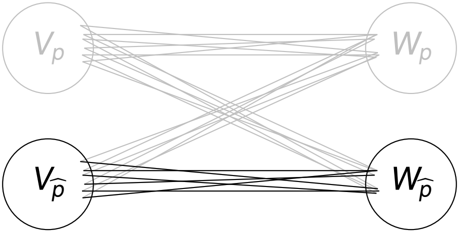

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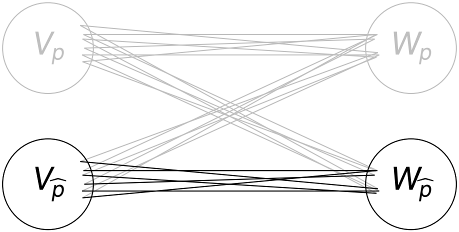

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Gain: Potentially increased the edge density

Need an increase in edge density to outweigh loss in vertices

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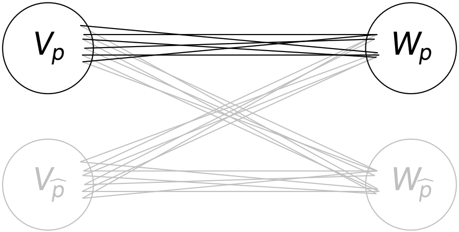

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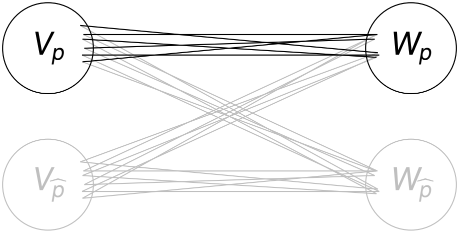

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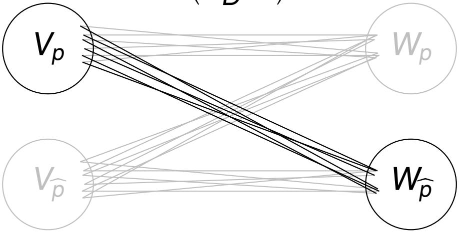

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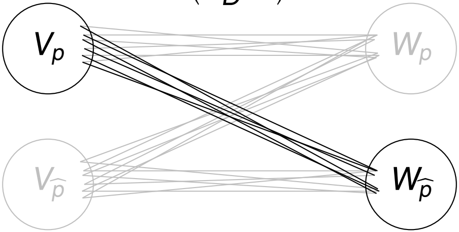

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Loss: Smaller vertex sets, potential loss of edge density

Gain: Naiive guess smaller by a factor of p

Need decrease in edge density and vertices to be outweighed by gain through p-factor

Further avenues

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- *Why* does the proof only just work?

Questions

Thank you for listening.