Improving Estimates: Discrete Sum

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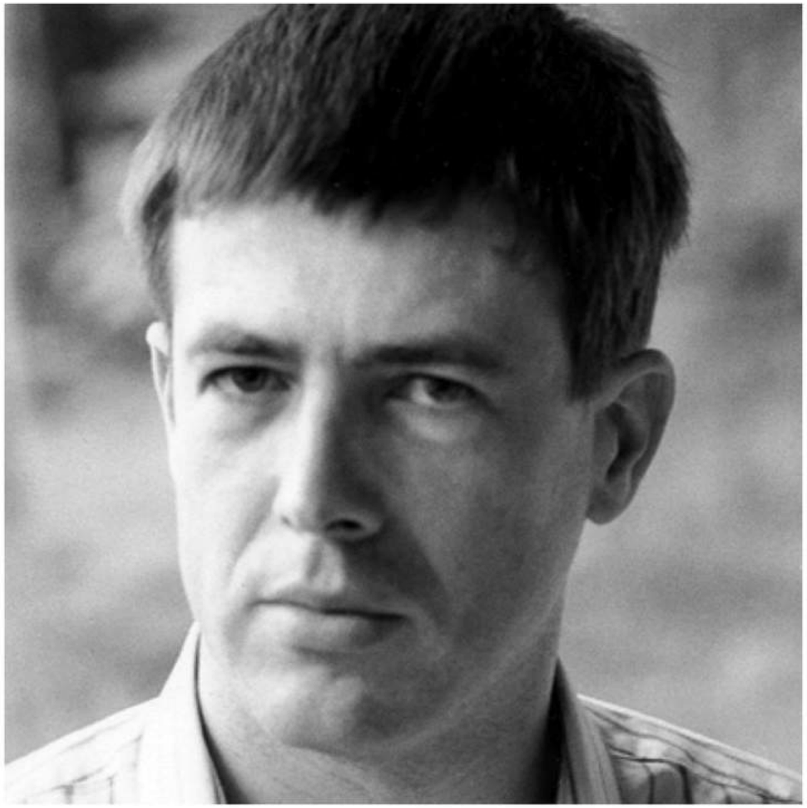
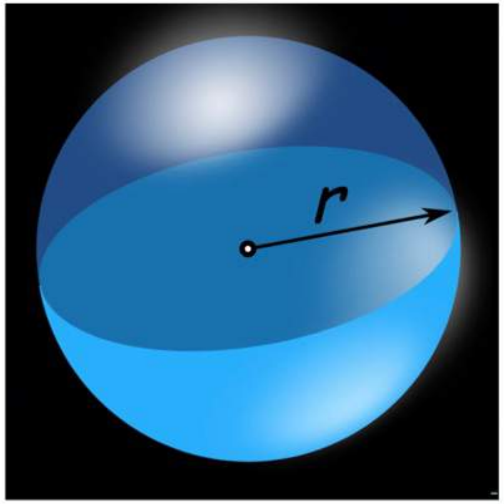


Jean Bourgain

Elias Stein

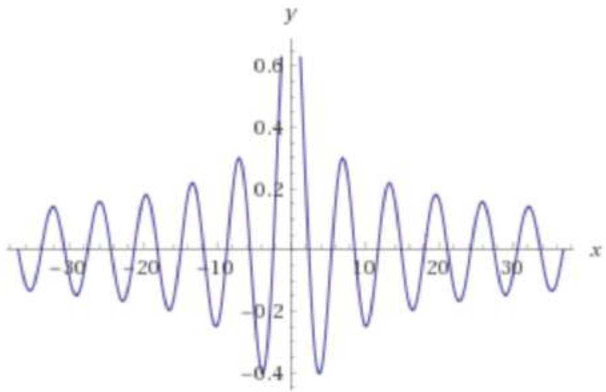
Continuous Case



$$A_r f(x) = \int_{\mathbb{S}^{d-1}} f(x - ry)\, d\sigma(y)$$

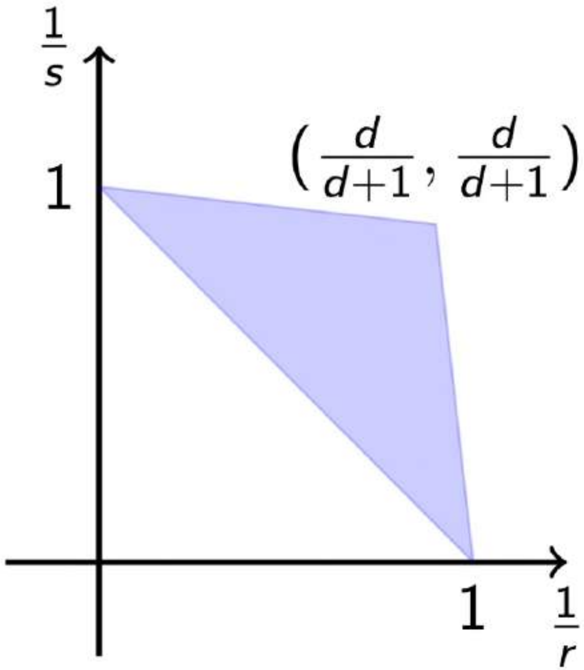
Fourier Transform of $d\sigma$

$$\widehat{d\sigma}(\sigma) \simeq |\xi|^{-\frac{d-1}{2}} e^{\pm i|\xi|}$$



1. The rate of decay: Gain of $\frac{d-1}{2}$ derivatives
2. The oscillation at unit speed in the Fourier transform

L^p Improving: Littman & Strichartz



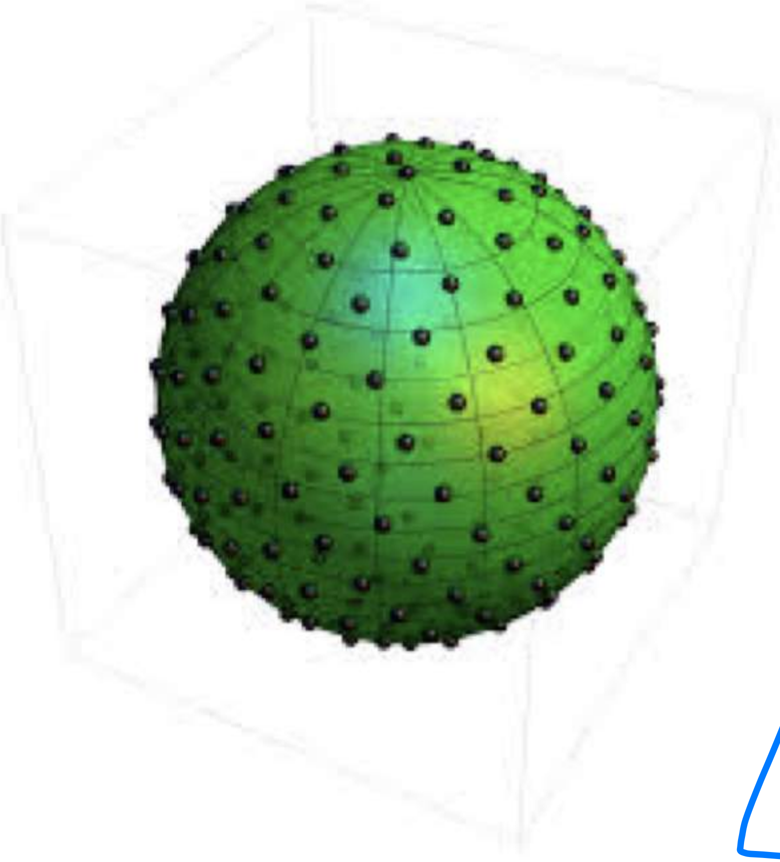
$$\boxed{\langle A_1 f, g\rangle \lesssim \|f\|_r \|g\|_s}$$

$$L^{\frac{d+1}{d}} \to L^{d+1}$$

<u>More generally</u>:

- Good understanding for "non-flat" curves and codimension 1 surfaces
- Interplay between geometric/combinatorial methods, with a desire to suppress Fourier Transform.

Discrete Spherical Averages



$$\boxed{A_\lambda f = \frac{1}{\lambda^{d-2}} \sum_{n\,:\,|n|=\lambda} f(x-n)}$$

discrete sphere has
codimension 2

Why λ^{d-2}? In $\{x :: \lambda \leq |x| \leq 2\lambda\}$ there are about λ^d lattice points, and about λ^2 radii which are the square root of an integers. Finally, $d \geq 5$, so the lattice points are equitably distributed among the spheres.

Not all radii are the same: Interested in 'scale-free' bounds. For a cube E of side length λ, and operator M which is 'local', we seek bounds of the form

$$|E|^{-1}\langle Mf, g\rangle \lesssim \langle f\rangle_{E,r}\langle g\rangle_{E,s},$$

where f, g are supported on E, and

$$\langle f\rangle_{E,r}^{r} = |E|^{-1}\sum_{n\in E}|f(n)|^{r}.$$

<u>Lagrange:</u> Every integer is sum of 4 $\square$'s.

dim = 4 is critical. $d \geq 5$ below.

Kevin Hughes, Kessler-Lacey 2017 $d \geq 5$

$$\langle A_\lambda^{\mathbb{Z}^d} f \rangle_{E,p'} \lesssim C_{p,\omega(\lambda^2)} \langle f \rangle_{E,p}$$

$\ell(E) = \lambda$ $\qquad \frac{d+1}{d-1} < p < 2$

- $\omega(\lambda^2)$ = # of distinct prime factors of λ

- Can you go to $p > \frac{d+2}{d}$?

- For $\frac{d}{d-2} < p < 2$ $\qquad C_{p,\omega(\lambda^2)} = C_p$

* Schlag - Sogge type results for Stein maximal function (Kesler, L., Mena)

* Sharp result for averages over squares (Han, L. – Yang)

* Prime integers. Endpoint type results (L. Hamed Mousavi, Jacob Rahmi)

* Sharp result for paraboloid (Dasu, Demeter Langowski)
* a good result for arbitrary polynomials (several people)
* Averages w.r.t. divisor function Christina Giannitsi

* Polynomial curves in high dim.
  Dendrinos, Hughes, Vitturi

* Many results hold for sparse bounds
  Weighted Theory follows.

What are Proofs like

- Circle Method
- Use High Pass, Low Pass method (Bourgain, Ionescu)
- See continuous and $\mathbb{Z}/q\mathbb{Z}$ versions of the averages

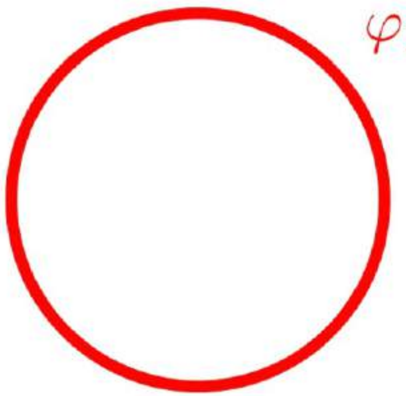
Proof of Littman, Strichartz Estimate

$|\widehat{d\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}$

Auxilary Question: For $f = \mathbf{1}_F$, $g = \mathbf{1}_G$ supported on the unit cube, and integer N, prove that $A_1 f \leq \text{HI} + \text{LO}$, where

$$\langle \text{HI}, g\rangle \lesssim N^{-\frac{d-1}{2}}[|F|\cdot|G|]^{1/2}, \qquad \langle \text{LO}, g\rangle \lesssim N|F|\cdot|G|.$$

Optimal $N \simeq [|F|\cdot|G|]^{-\frac{1}{d+1}}$, which proves $A_1 : L^{\frac{d+1}{d},1} \mapsto L^{d+1,\infty}$.



$\varphi_{1/N} * A_1$

Set $\text{LO} = \varphi_{1/N} * A_1 f$, so that $\text{LO} \lesssim N|F|$.

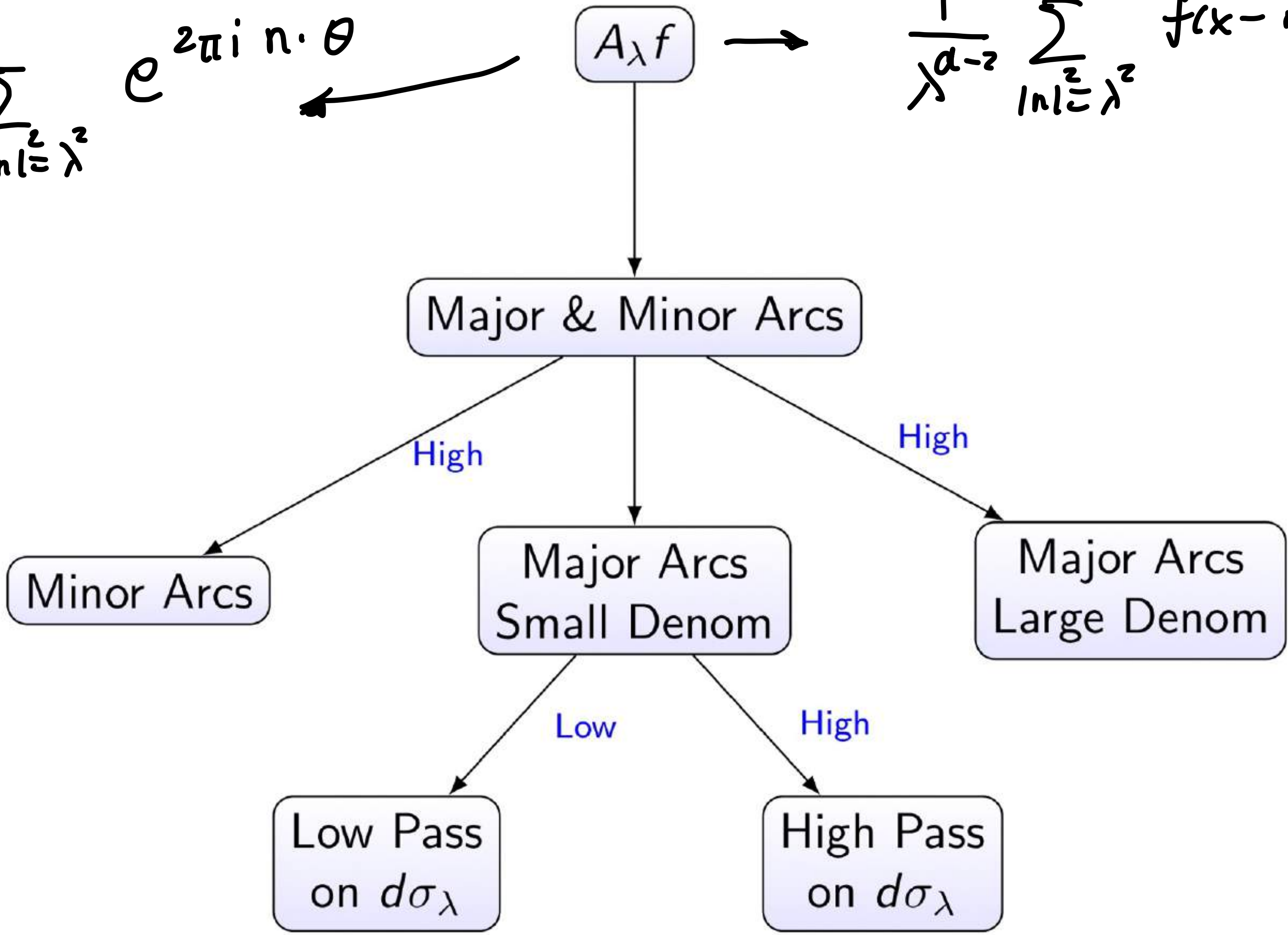
Use stationary decay estimate on HI.

Volume of annulus is about $1/N$

Fixed Radius Case

$$\frac{1}{\lambda^{d-2}} \sum_{|n|^2=\lambda^2} e^{2\pi i n\cdot\theta}$$

$$\frac{1}{\lambda^{d-2}} \sum_{|n|^2=\lambda^2} f(x-n)$$



$$A_\lambda f(x) = \int_{\mathbb{T}^d} a_\lambda(\xi) \widehat{f}(\xi)\, d\xi \tag{1}$$

$$a_\lambda = c_\lambda + r_\lambda \tag{2}$$

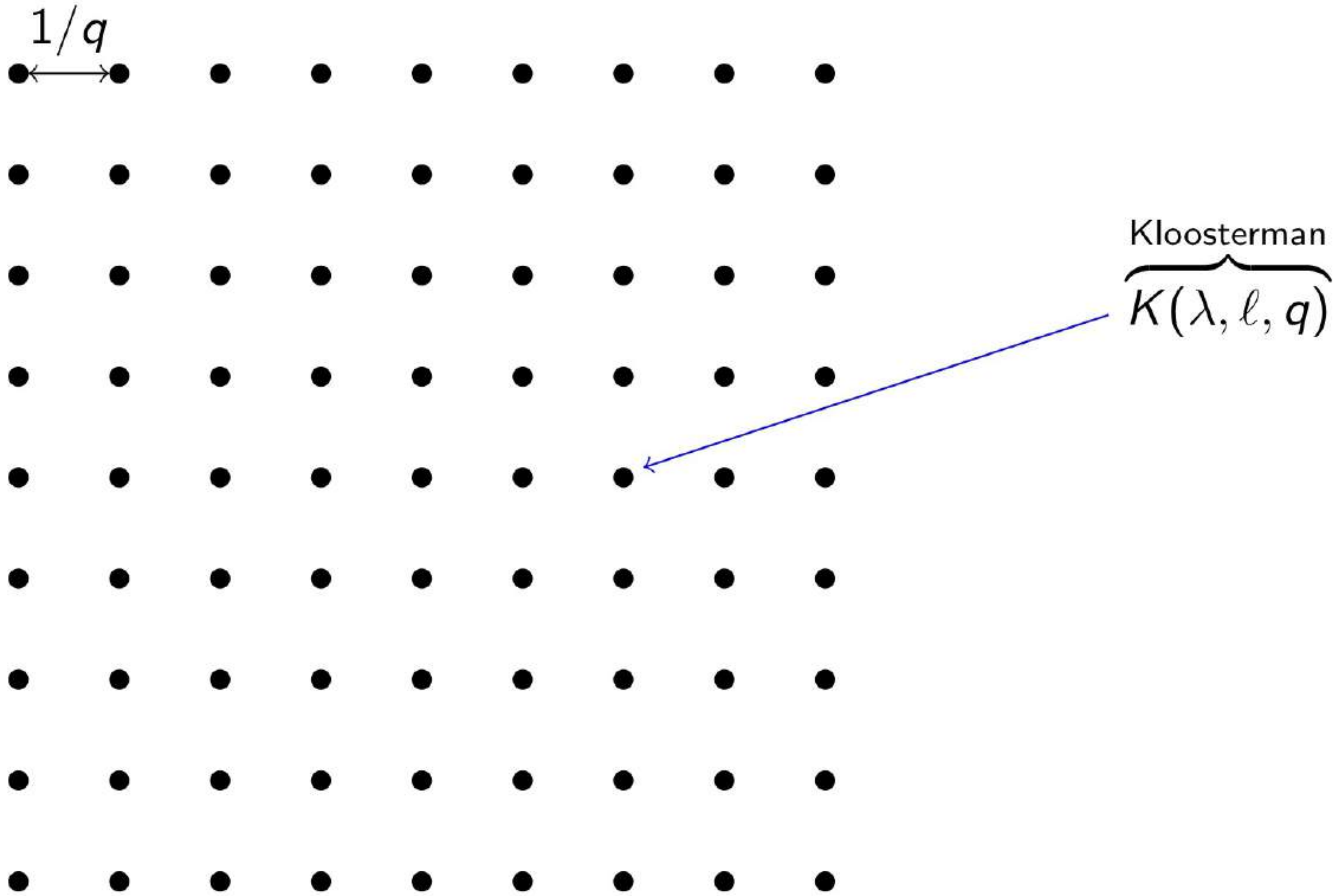
$$c_\lambda(\xi) = \sum_{1 \leq q \leq \lambda} c_{\lambda,q}(\xi), \tag{3}$$

$$c_{\lambda,q}(\xi) = \sum_{\ell \in \mathbb{Z}_q^d} K(\lambda, q, \ell)\, \underbrace{\widetilde{\Phi}_q(\xi - \ell/q)}_{\text{cut off fn}} \widetilde{d\sigma_\lambda}(\xi - \ell/q), \tag{4}$$

$$\underbrace{K(\lambda, q, \ell)}_{\text{Kloosterman sum}} = q^{-d} \sum_{a \in \mathbb{Z}_q^\times} e_q\left(-a\lambda^2\right) \underbrace{\sum_{n \in \mathbb{Z}_q^d} e_q\left(|n|^2 a + n \cdot \ell\right)}_{\text{Gauss sum}} \tag{5}$$

$$e_q(\theta) = e^{2\pi i \theta / q}. \tag{6}$$

The Mulitiplier $C_{\lambda,q}$



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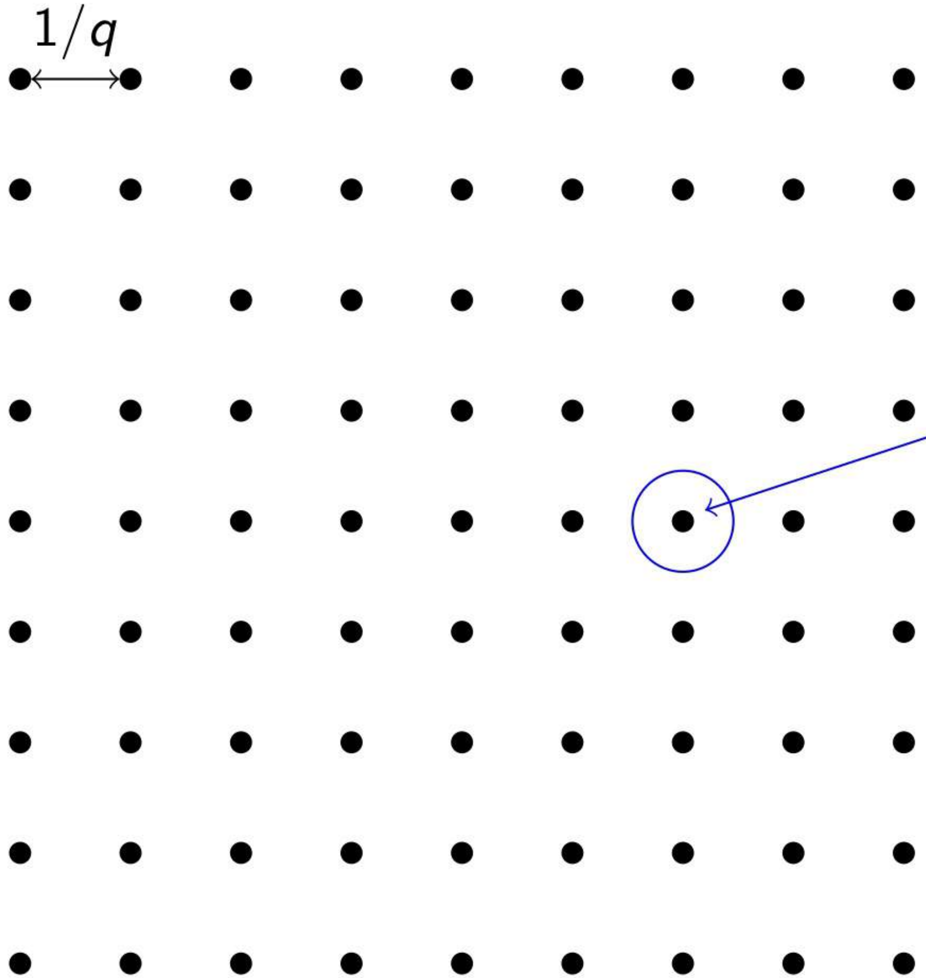

$$\overbrace{K(\lambda,\ell,q)}^{\text{Kloosterman}}\ \underbrace{\Phi_q(\xi-\ell/q)}_{\text{Cutoff}}$$

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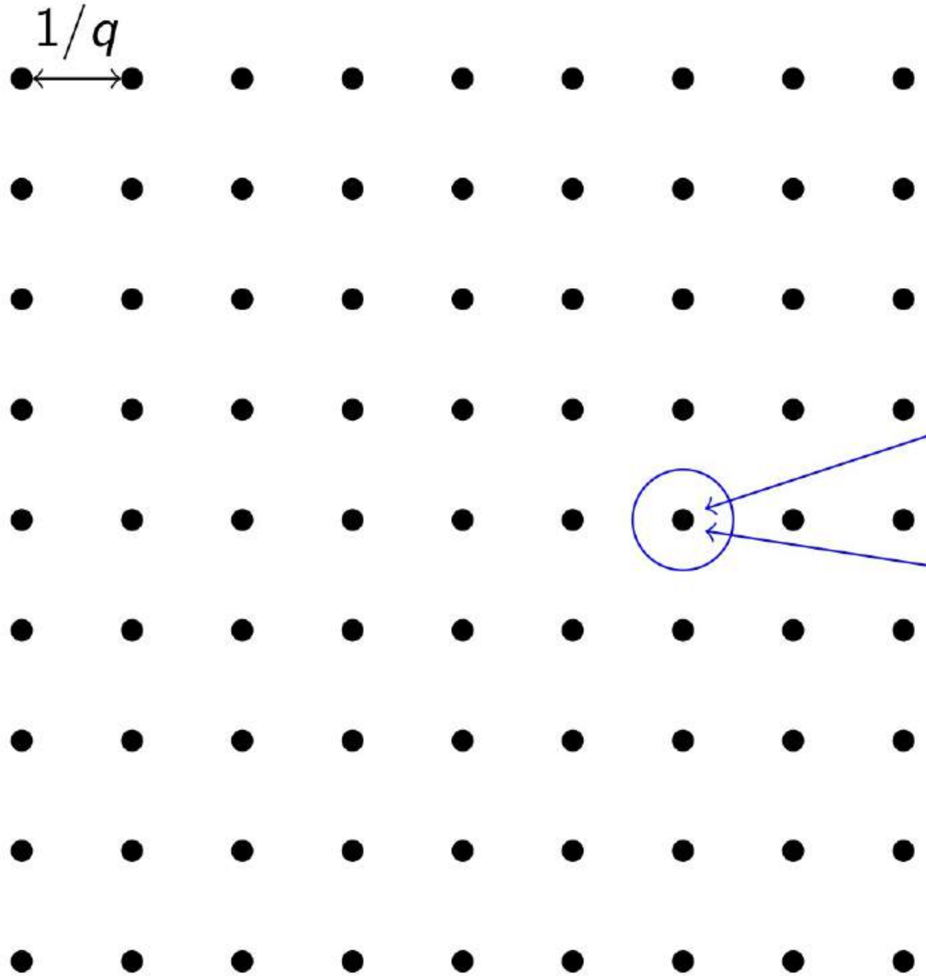
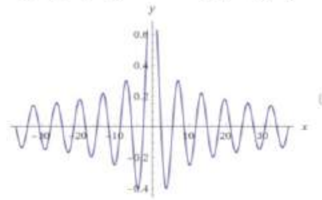


$$\overbrace{K(\lambda, \ell, q)}^{\text{Kloosterman}} \underbrace{\Phi_q(\xi - \ell/q)}_{\text{Cutoff}}$$

$$\widehat{d\sigma}(\xi - \ell/q)$$



Key Estimates

eventually gives $\omega(\lambda^2)$ term.

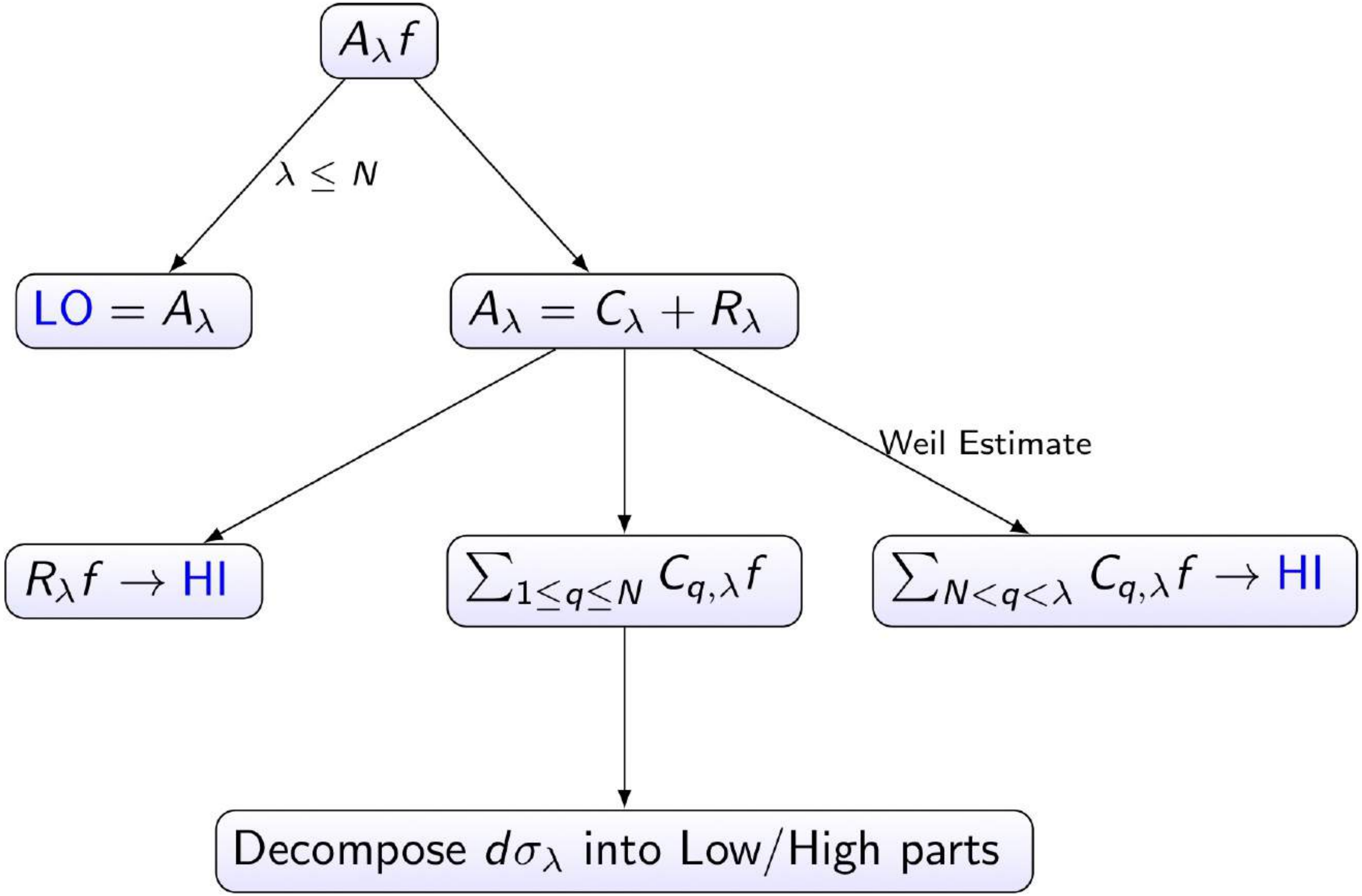
Andre Weil:

$$|K(\lambda, q, \ell)| \lesssim \lambda^{\epsilon + \frac{1-d}{2}} \rho(\lambda^2, q)$$

Write $q = 2^r st$ with st odd and $(t, \lambda^2) = 1$, then $\rho(\lambda^2, q) = \sqrt{2^r s}$. This is a 'square root gain' over the Gauss sum bound.

Akos Magyar:

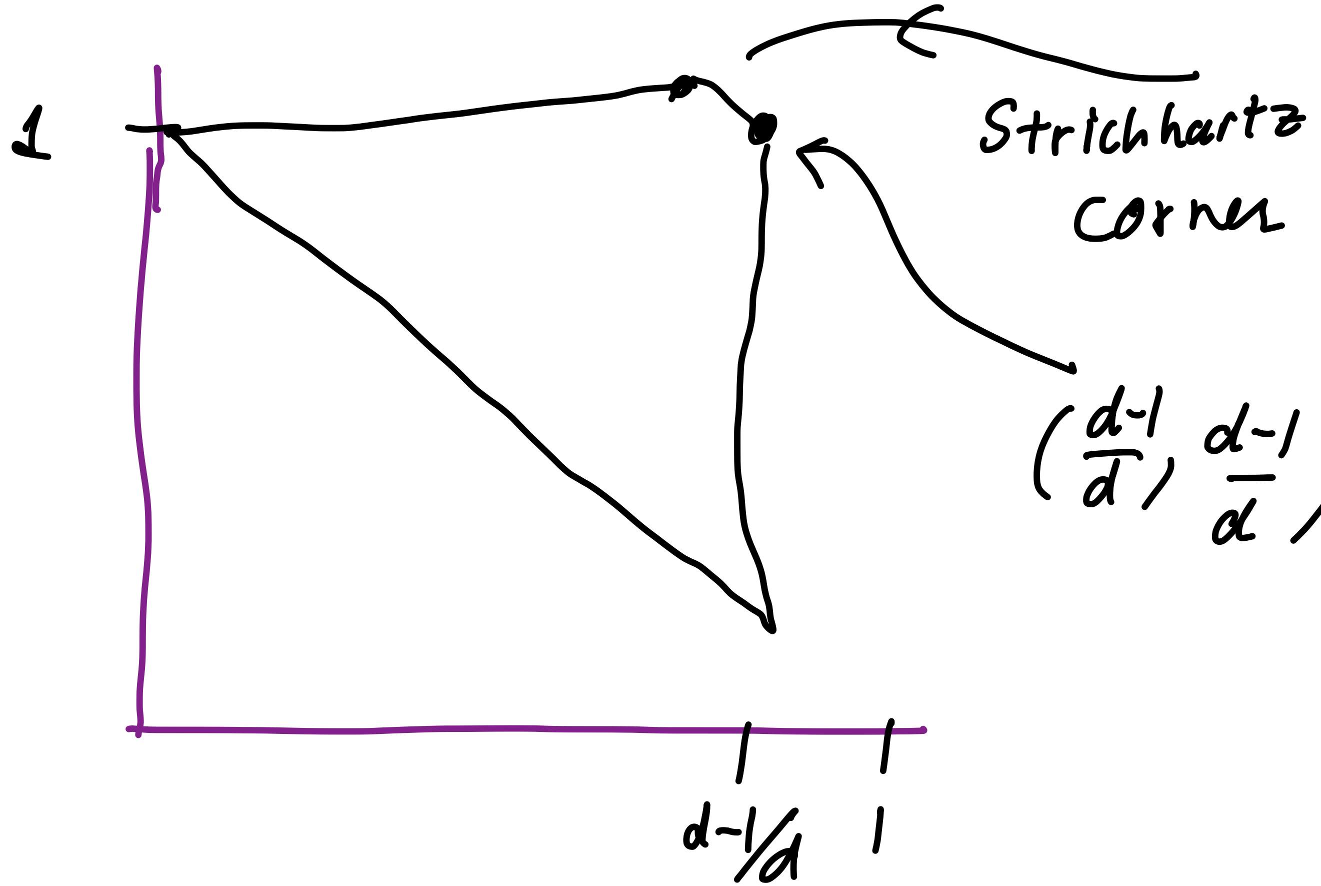
$$\|r_\lambda\|_\infty \lesssim \lambda^{\epsilon + \frac{3-d}{2}}$$



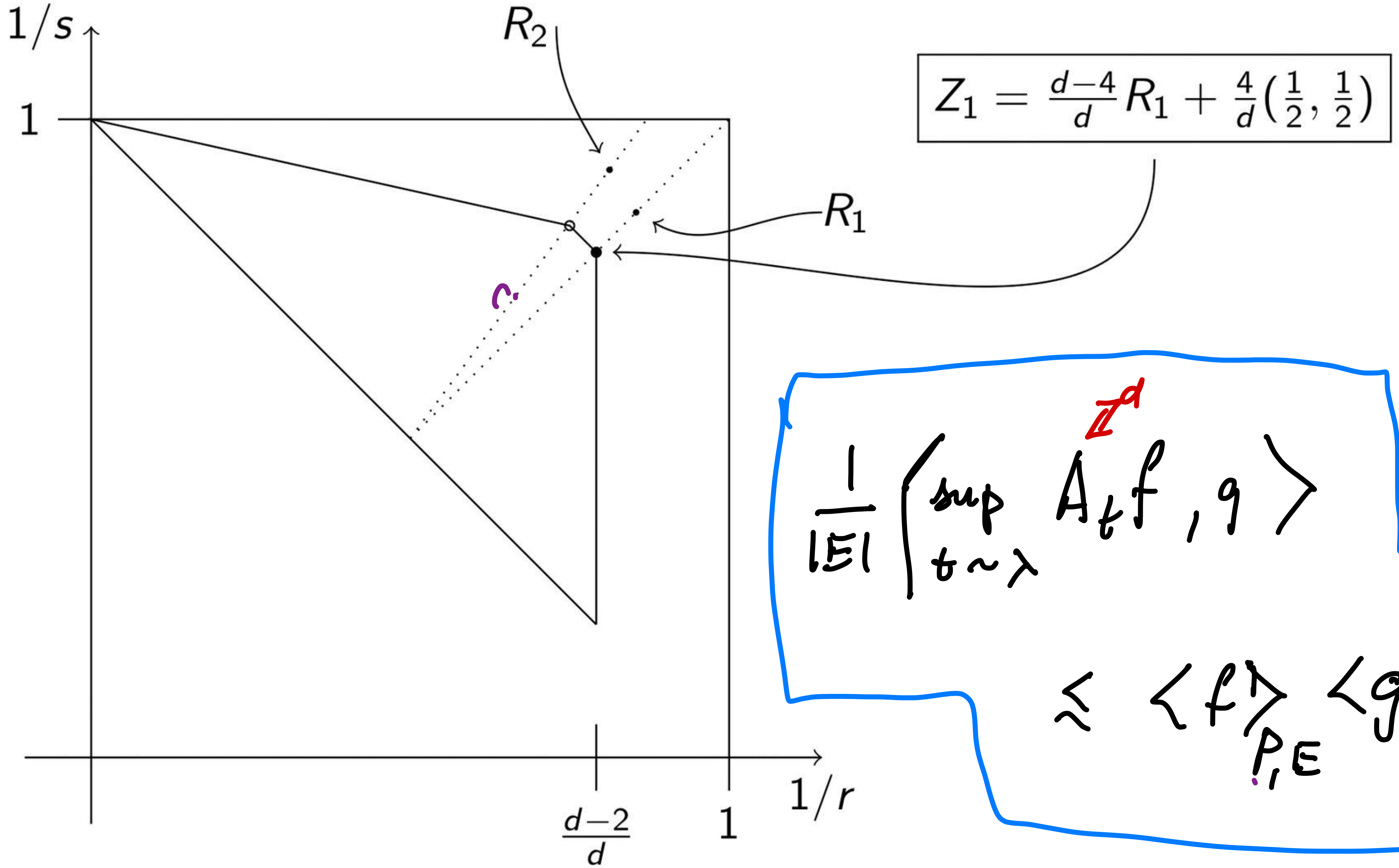
Variant of continuous approach.

Schlag - Sogge Inequalities

$$\left\langle \sup_{t\sim 1} | A_t^{\mathbb{R}^d} f |, g \right\rangle \lesssim \|f\|_p \, \|g\|_q$$



Schlag Type Exponents : Kesler, L., Mena Arias



$\mathbb{Z}^d$

$$\frac{1}{|E|}\left\langle \sup_{t\sim\lambda} A_t f , g \right\rangle$$

$$\lesssim \langle f \rangle_{p,E} \langle g \rangle_{p',E}$$

Recent breakthrough result of Krause Mirek, Tao $\forall$ poly. $p: \mathbb{Z} \to \mathbb{Z}$

$$\lim \frac{1}{N} \sum_{1}^{N} f T^n g T^{p(n)} \text{ exists.}$$

- depends upon ℓ^2 improving estimates.

Extension of KMT would require bilinear improving estimate.