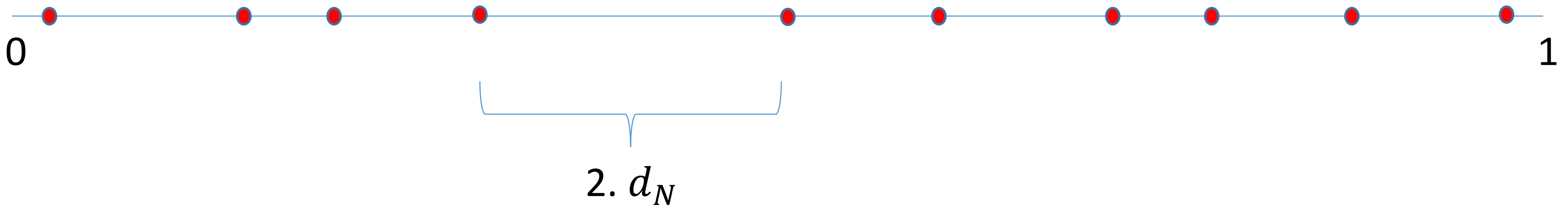
Finite sequence $x_1, x_2, \ldots, x_N$ in $[0,1)$



dispersion $d_N := \sup_{x} \min_{i=1,2,\ldots N} |x - x_i|$

(essentially) $\inf_{x_1, x_2, \ldots, x_N} d_N \approx 0{,}5 * \frac{1}{N}$

Infinite sequence $x_1, x_2, \dots, x_N, \dots$ in [0 ,1)

dispersion $d_N \coloneqq \sup_x \min_{i=1,2,\dots N} |x - x_i|$

$d_N \geq 0{,}5 * \frac{1}{N}$ always (every sequence, every N)

e.g.: van der Corput sequence in base 2 ?

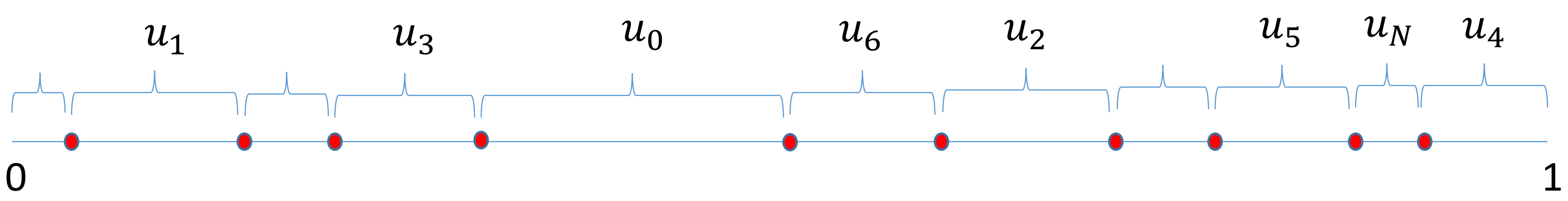
$d_N \leq 1 * \frac{1}{N}$ for every N

c := $\inf_{x_1, x_2, \dots} \sup_N N * d_N$ ⇒ $0.5 \leq c \leq 1$

Indeed: $\mathbf{c} = \frac{\mathbf{1}}{\mathbf{\log 4}} = \mathbf{0,721}$

Proof: Assume $d_N \leq c * \frac{1}{N}$ for every N

Consider first N elements $x_1, x_2, \ldots, x_N$ in [0 ,1)



$d_N = \frac{u_0}{2}$ $\quad d_{N+1} \geq \frac{u_1}{2}$ $\quad d_{N+2} \geq \frac{u_2}{2}$ $\quad \ldots \quad d_{2N} \geq \frac{u_N}{2}$ hence

$$\frac{1}{2} = \frac{u_0}{2} + \frac{u_1}{2} + \ldots + \frac{u_N}{2} \leq d_N + d_{N+1} + \ldots + d_{2N} \leq c * \left(\frac{1}{N} + \frac{1}{N+1} + \ldots + \frac{1}{2N} \right) \approx c * \log 2$$

Can achieve equality ... if choose $x_n := \left\{ \frac{\log(2n-3)}{\log 2} \right\}$

dimension 2: finite sequence $x_1, x_2, \dots, x_N$ in $[0,1)^2$

dispersion with respect to maximum norm $d_N \coloneqq \sup_x \min_{i=1,2,\dots N} \left\| x - x_i \right\|_\infty$

(essentially) $\inf_{x_1,x_2,\dots,x_N} d_N \approx 0{,}5 * \frac{1}{\sqrt{N}}$

Infinite sequence $x_1, x_2, \dots, x_N, \dots$ in $[0,1)$

$$c := \inf_{x_1,x_2,\dots} \sup_N \sqrt{N} * d_N$$

Easy: $0.5 \le c \le 1$

Best we know: $0.5493 = \frac{1}{2}\sqrt{\frac{1}{2(\sqrt{2}-1)}} \le c \le \frac{1}{\log 4} = 0{,}721$

Problem: improve estimates for dimension 2,

higher dimensions ? (especially: does c tend to 0.5 for growing dimension ?)

References:

H.Niederreiter: *Random Number Generation and Quasi-Monte Carlo Methods*. SIAM books 1992 (Chapter 6).

G.Larcher and H. Niederreiter: *A lower bound for the dispersion of multidimensional sequences*. In: K.Nagasaka, Ed., Analytic Number Theory and Related Topics. World Scientific, 1991.