Problem: On the convexity of zeta functions

Cooperation workshop (online)

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FWF Der Wissenschaftsfonds.

SFB QMC

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Notation

We consider the following Dirichlet series:

$$\psi(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for which we assume that the coefficients $a_n \geq 0$ for $n \geq 1$ and the series is uniformly convergent for $s > 1$.
In this class we can find **the Riemann zeta-function**

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and **the Lambda function**

$$\lambda(s) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}.$$

Notation: Convexity and Concavity of f

Definition

- A function f is **convex** if

$$f(\alpha x+\beta y)\le \alpha f(x)+\beta f(y)$$

  holds for all x, y in the domain of f and for all non-negative α, β with $\alpha+\beta=1$.

- The function f is **concave** if

$$f(\alpha x+\beta y)\ge \alpha f(x)+\beta f(y).$$

- Also f is **strictly convex (or strictly concave)** if equality is possible only when $\alpha=0, \beta=0$ or $x=y$.

Concavity of the function $1/\psi(s)$

Cerone and Dragomir (2009)

Find Dirichlet series ψ with non-negative coefficients which has the property that the function $1/\psi$ is concave where it is defined.

Cerone and Dragomir (2009)

Consider the Dirichlet series $\psi(s) = \sum_{n\geq 1} a_n/n^s$, the following statements are equivalent:

- the function $1/\psi$ is a concave on $(1, \infty)$,
- for any $s_1, s_2 > 1$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we have

$$\psi(\alpha s_1 + \beta s_2) \leq \frac{\psi(s_1)\psi(s_2)}{\alpha\psi(s_1) + \beta\psi(s_2)}.$$

- for any $s > 1$, we have

$$\psi^{''}(s)\psi(s) \geq 2(\psi^{'}(s))^2.$$

Concavity of the function $1/\zeta(s)$

Cerone and Dragomir (2009)

The function $1/\zeta(s)$ is concave on the interval $[\zeta^{-1}(e), \infty)$, where ζ^{-1} denotes the inverse function of ζ and

$$\zeta^{-1}(e) \simeq 1.474464287.$$

Conjecture: Cerone and Dragomir (2009)

The function $1/\zeta(s)$ is concave on the whole interval $(1, \infty)$,

Concavity of the function $1/\zeta(s)$

Alkan (2019)

The function $1/\zeta(s)$ is strictly concave on the interval $(1, \infty)$.

Alkan (2020)

The function $1/\zeta(s)$ is strictly concave for $s \geq 0.22$.

Remark

- The **strict concavity** of $1/\zeta$ is seen to be consequence of

$$\zeta^{''}(s)\zeta(s) - 2(\zeta^{'}(s))^2 > 0, \quad \text{on the interval } (1, \infty).$$

- The function $1/\zeta$ is **strict concave** if and only if $\frac{d^2}{ds^2}\left(\frac{1}{\zeta(s)}\right) < 0$.
- We have

$$\zeta^{''}(s)\zeta(s) - 2(\zeta^{'}(s))^2 = \frac{2\gamma}{(s-1)^3} + O\left(\frac{1}{(s-1)^2}\right)$$

Our suggestion

Our suggestion

Find the best value of c such that the function $1/\zeta$ is strictly concave for $s \geq c$.

References:

1. E. Alkan, Biased behavior of weighted Mertens sums, *Int. J. Number Theory* **16** (2020) pp. 547–577.
2. E. Alkan, Resolution of a conjecture on the convexity of zeta functions, *J. Math. Anal. Appl.* **472** (2019) pp. 1987–2016.
3. P. Cerone and S. S. Dragomir, Some convexity properties of Dirichlet series with positive terms, *Math. Nachr.* **282** (2009) pp. 964–975.