Poncelet's Theorem, Paraorthogonal Polynomials and the Numerical Range of Compressed GGT matrices

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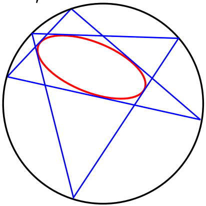
Poncelet's Theorem

In 1813, Jean–Victor Poncelet, while a prisoner of war, proved a remarkable theorem that says if K is an ellipse inside a circle so that there is a triangle circumscribed about K inscribed in the circle,



then there are infinitely many such triangles, indeed, so many that their vertices fill the outer circle and their tangent points of all them fill K.

There has been a huge literature motivated by this gem of projective geometry, even a recent book. In this talk, I will consider three different related developments.

Theme #1: Siebeck's Theorem

In 1864, Jörg Siebeck proved a theorem later popularized in Marden's 1948 book, *Geometry of Polynomials*:

Theorem Let $\{w_j\}_{j=1}^p$ be the vertices of a convex polygon in $\mathbb{C}$ ordered clockwise. Let $m_j \in \mathbb{R}$, $w_{p+1} = w_1$ and let

$$M(z) = \sum_{j=1}^{p} \frac{m_j}{z - w_j}$$

Then the zero's of M are the foci of a curve of class $p-1$ which touches the line segments $w_j w_{j+1}$; $j = 1, \dots, p$ at the point dividing the line in ratio m_j/m_{j+1}.

I am not going to try to give you the rather complicated definitions of the foci of a curve nor of class nor discuss them. I raise this to emphasize there is a n–gon version, that M and its zeros play a special role and that the ratios m_j/m_{j+1} occur.

Theme #2: Finite Blaschke Products

Starting in 2002, Ulrich Daepp, Pamela Gorkin (husband and wife) and collaborators considered finite Blaschke products of the form for $\{z_j\}_{j=1}^n \subset \mathbb{D}$ (maybe not different)

$$B(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \bar{z}_j z}$$

These are precisely Schur functions (analytic maps of $\mathbb{D}$ to itself) which are analytic in a neighborhood of $\overline{\mathbb{D}}$ of magnitude 1 on $\partial\mathbb{D}$ with n zeros (they actually consider zB and sometimes divide their basic function by z; we prefer to take this B and sometime multiply it by z). Since $|zB(z)| < 1$ on $\mathbb{D}$, the map $e^{i\theta} \mapsto \arg\left[e^{i\theta} B(e^{i\theta})\right]$ is strictly increasing in θ and by the argument principle is $n+1$ to 1 so, $\forall \lambda \in \partial\mathbb{D}$, $\exists n+1$ solutions w_j; $j = 1, \ldots, n+1$ of $wB(w) = \bar{\lambda}$ (they just take λ; it will be clear later why we like $\bar{\lambda}$ as our label).

Theme #2: Finite Blaschke Products

The main result in this approach is

Theorem For any $\{z_j\}_{j=1}^n \subset \mathbb{D}$ and any $\lambda \in \partial\mathbb{D}$, there exist $m_j(\lambda) > 0$ with $\sum_{j=1}^{n+1} m_j(\lambda) = 1$ so that

$$\sum_{j=1}^{n+1} \frac{m_j(\lambda)}{z - w_j} = \frac{B(z)}{zB(z) - \bar{\lambda}}$$

The right side of this expression is a rational function of z which is $z^{-1} + \mathrm{O}(|z|^{-2})$ at infinity and with poles exactly at the w_j so the left side is just a partial fraction expansion and $\sum_{j=1}^{n+1} m_j(\lambda) = 1$ follows from the asymptotics at infinity. The main issue is the proof that $m_j > 0$ and they proved this by finding an explicit formula for the m_j in terms of the z's and w's. It is left unmentioned that there is a probability measure and what its significance is.

Theme #2: Finite Blaschke Products

The following theorem is natural to state in this $B(z)$ language

Theorem Fix $\lambda \neq \mu$ both in $\partial\mathbb{D}$ and let $\{w_j\}_{j=1}^{n+1}$ (resp. $\{u_j\}_{j=1}^{n+1}$) be the solutions of $zB(z) = \bar{\lambda}$ (resp. $zB(z) = \bar{\mu}$). Then the $w's$ and $u's$ interlace. Conversely, if one is given interlacing sets, there is a unique n fold Blaschke product so that the w's and u's are the solutions of a $zB(z)$ equation.

This result was first proven by Gao and Wu in the S_n framework below and the w's and u's enter as vertices of Poncelet $(n+1)$–gons. Their proof is long and involves lots of manipulations of determinants. The later, much shorter, proof of Daepp, Gorkin and Voss constructs some rational Herglotz functions with given interlacing zeros and poles. We have a simple third proof. For reasons that will become obvious later, for now, I'll call this Wendroff's Theorem for Blaschke products. Parameter counting is a little subtle.

Theme #3: Completely Non–Unitary Contractions With Defect Index 1

An operator on a Hilbert space is called a *contraction* if its norm is at most 1. It is called *completely non–unitary* if it has no invariant subspace on which it is unitary. In the finite dimensional case, this is equivalent to be there being no eigenvector with eigenvalue λ obeying $|\lambda| = 1$.

In the finite dimensional case, the defect index of a contraction, A, is defined to be the dimension of the range of $\mathbf{1} - A^*A$. The space S_n is the set of Completely Non–Unitary Contractions on $\mathbb{C}^n$ with defect index 1. They were studied in a series of independent papers by Boris Mirman and by Hwa-Long Gau and Pei Yuan Wu, both series starting in 1998. One important theorem is

Theorem For any $\{z_j\}_{j=1}^n \subset \mathbb{D}$ (maybe not different), there is an operator $A \in S_n$ whose eigenvalues (counting algebraic multiplicity) are the z_j. Any two elements in S_n are unitarily equivalent $\iff$ they have the same eigenvalues.

Theme #3: Numerical Range

Recall that if A is an operator on a Hilbert space, $\mathcal{H}$, then $N(A)$, the *numerical range* of an operator, A, on $\mathcal{H}$ is the set of values $\langle\varphi, A\varphi\rangle$ where we run through all $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ (not ≤ 1!). It is a subtle fact that $N(A)$ is a convex subset of $\mathbb{C}$ and an easy fact that it is compact when $\mathcal{H}$ is finite dimensional.

It is not hard to show that if A is normal, then (the closure of) $N(A)$ is the closed convex hull of the spectrum, so, in the finite dimensional normal case, $N(A)$ is the convex hull of the eigenvalues and so a convex polygon. In particular, if A is a k dimensional unitary operator with simple spectrum, then $N(A)$ is a convex k–gon inscribed in $\partial\mathbb{D}$.

Theme #3: Unitary Dilations

Let $\mathcal{H} \subset \mathcal{K}$ two Hilbert spaces and P the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. If $A \in \mathcal{L}(\mathcal{H})$, $B \in \mathcal{L}(\mathcal{K})$, we are interested in the relation $A = PBP \restriction \mathcal{H}$. If that holds we say that A is a *compression* of B and that B is a *dilation* of A. In case $\dim(\mathcal{K}) < \infty$, we call $\dim(\mathcal{K}) - \dim(\mathcal{H})$ the *rank* of the dilation.

Given a contraction, A on $\mathcal{H}$, one is interested in finding $\mathcal{K}$ and $B \in \mathcal{L}(\mathcal{K})$ so that B is a unitary dilation of A. It is easy to construct such a dilation on $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, so if $\dim(\mathcal{H}) = n$, a rank n unitary dilation, but one can show there is a one parameter family of rank one unitary dilations, $\{B_\lambda\}_{\lambda \in \partial\mathbb{D}}$ of any $A \in S_n$. For different λ, the eigenvalues are different, every point on the circle is an eigenvalue of some B_λ.

Theme #3: Unitary Dilations

The big theorem in these papers is

Theorem Let $A \in S_n$ and $\{B_\lambda\}_{\lambda \in \partial\mathbb{D}}$ its rank one unitary dilations. For each fixed λ, $N(B_\lambda)$ is a solid $n+1$–gon with vertices on $\partial\mathbb{D}$. Each edge of this polygon is tangent to $N(A)$ at a single point and as λ moves through all of $\partial\mathbb{D}$, these tangent points trace out the entire boundary of $N(A)$. Moreover

$$N(A) = \bigcap_{\lambda \in \partial\mathbb{D}} N(B_\lambda)$$

If, for a fixed λ one forms

$$M(z) = \sum_{j=1}^{p} \frac{m_j}{z - w_j}$$

for suitable m_j, then the zeros of M are precisely the eigenvalues of A.

The Bottom Line

The point of our work is the claim that while the authors didn't realize it, they were secretly studying OPUC (orthogonal polynomials on the unit circle). This allows us to find proofs and some extensions of the results and to illuminate the meaning of some formulae. For example,

$$\sum_{j=1}^{n+1} \frac{m_j(\lambda)}{z - w_j} = \frac{B(z)}{zB(z) - \bar{\lambda}}$$

is a formula that can be interpreted as a matrix resolvent on the left and a Cramer's rule ratio of determinants on the right. Moreover, as I'll explain, the above is essentially a very special case of a result called *Khrushchev's formula*, a result published in 2001, remarkably the year before Daepp, Gorkin and Mortini proved the above!

We'll also be able to find new results in OPUC theory motivated by the above theories.

Szegő Recursion and Verblunsky Coefficients

We'll give a lightning review of OPUC to fix notation, state our main results and, if time allows, say something about the proofs.

The standard reference (he said modestly) for OPUC is my pairs of books which total 1044 pages. In response to a request from a friend, I wrote a 30 page summary of the high points which I called "OPUC on one foot" after a story in the Talmud. Here I need to remind/tell you of a few of the high points so I think of the next few slides as "OPUC on one toe".

OPUC involves the study of probability measures $d\mu$ on $\partial\mathbb{D}$ which are non–trivial, i.e. not finite point measures. Given such a measure, one can look at the inner product in $L^2(\partial\mathbb{D}, d\mu)$ and form the monic orthogonal polynomials, $\Phi_n(z)$, and orthonormal polynomials, $\varphi_n(z)$.

Szegő Recursion and Verblunsky Coefficients

Let $\mathcal{P}_n$ be the $n+1$–dimensional space of polynomials in z of degree at most n and define for $P_n \in \mathcal{P}_n$

$$P_n^*(z) = z^n \overline{P_n\left(\frac{1}{\bar{z}}\right)}$$

Alas, it is standard to use this symbol even though it is n dependent; one hopes n will be clear from the context! Szegő recursion says that

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n\Phi_n^*(z)$$

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - \alpha_n z\Phi_n(z)$$

The second equation is what you get by applying * to the first, so often, only the first is written. $\alpha_n \equiv -\overline{\Phi_{n+1}^*(0)}$ are complex numbers, called *Verblunsky coefficients*; they lie in $\mathbb{D}$. There is also inverse Szegő recursion. Note that Φ_n determines $\{\alpha_j\}_{j=0}^{n-1}$.

The Norm of Φ_n

One useful calculation is to note if one writes $z\Phi_n(z) = \Phi_{n+1}(z) + \bar{\alpha}_n \Phi_n^*(z)$, the vectors on the right side are orthogonal, so

$$\|z\Phi_n\|^2 = \|\Phi_{n+1}\|^2 + |\alpha_n|^2 \|\Phi_n^*\|^2 \Rightarrow \|\Phi_{n+1}\| = \rho_n \|\Phi_n\|$$

where $\rho_j = \sqrt{1 - |\alpha_j|^2}$. This is where $|\alpha_n| < 1$ from. We also see that, by induction that

$$\|\Phi_n\| = \prod_{j=0}^{n-1} \rho_j$$

This implies that $\|\Phi_{n+1}\| \leq \|\Phi_n\|$ which should not be surprising, since, by the relation between minimization and orthogonality, one knows that Φ_n is the monic polynomial of degree n that minimizes the norm of all such norms.

Carathéodory Functions and All That

Given a probability measure, $d\mu$, on $\partial\mathbb{D}$, we define two associated functions on $\mathbb{D}$:

$$F(z) = \int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta); \qquad F(z) = \frac{1+zf(z)}{1-zf(z)}$$

called the *Carathéodory function* and *Schur function* of $d\mu$ after their defining properties. They obey $\mathrm{Re}(F(z)) > 0$; $F(0) = 1$ and $|f(z)| < 1$. Schur associated a set of parameters to a Schur function via $f_0 \equiv f$

$$\gamma_n(f) = f_n(0); \qquad f_n(z) = \frac{\gamma_n + zf_{n+1}(z)}{1+\bar{\gamma}_n z f_{n+1}(z)}$$

If f is a finite degree m Blaschke product, then $\gamma_m \in \partial\mathbb{D}$ and the process terminates. If not (in which case we call f a *non–trivial Schur function*), we can define the *Schur iterates*, f_n, and *Schur parameters*, $\gamma_n(f) \in \mathbb{D}$, for all n.

The Geronimus and Verblunsky Theorems

Schur's Theorem (1917) There is a one–one correspondence between non–trivial Schur functions and sequences $\{\gamma_n\}_{n=0}^{\infty}$ in $\mathbb{D}$ given by the map from f to its Schur parameters.

Verblunsky's Theorem (1935) There is a one–one correspondence between non–trivial probability measures on $\partial\mathbb{D}$ and sequences $\{\alpha_n\}_{n=0}^{\infty}$ in $\mathbb{D}$ given by the map from a measure to its OPUC and the Verblunsky coefficients defined via Szegő recursion.

Geronimus' Theorem (1944) Let $d\mu$ be a non–trivial probability measures on $\partial\mathbb{D}$ and f its Schur function. Then

$$\alpha_n(d\mu) = \gamma_n(f)$$

This theorem explains why one writes Szegő recursion with the complex conjugate and minus sign on α.

Khrushchev's Formula

In 2002, Khrushchev found a new approach to Rakhmanoff's Theorem with lots of other interesting stuff. A basic result he needed was the following

Theorem (Khrushchev's formula) Let f be the Schur function of some non–trivial probability measure, $d\mu$, on the unit circle and let f_n be its n^{th} Schur iterate. Let $B_n(z) = \Phi_n(z)/\Phi_n^*(z)$. Then the Schur function of the probability measure $|\varphi_n(e^{i\theta})|^2 d\mu$ is $f_n(z)B_n(z)$.

This formula is an OPUC analog of the fact that the Green's function for a whole line Schrödinger operator is the product of two suitably normalized Weyl solutions.

Wendroff's Theorems for OPRL and OPUC

Wendroff's Theorem for OPRL (Wendroff 1961) For OPRL, the zeros of P_n and P_{n+1} lie in $\mathbb{R}$ and strictly interlace. Conversely, given any $2n+1$ interlacing points in $\mathbb{R}$, there exist a set of OPRL so those are the zeros of P_n and P_{n+1}. Moreover all such measures have the same Jacobi parameters $\{a_j\}_{j=1}^{n-1}$ and $\{b_j\}_{j=1}^{n}$ and so the same $\{p_j\}_{j=1}^{n+1}$.

Wendroff's Theorem for OPUC (Geronimus 1946 in Annals of Math!) For OPUC, the zeros of Φ_n lie in $\mathbb{D}$. Conversely, given any n points in $\mathbb{D}$ (allowing multiplicity), there exist a set of OPUC so those are the zeros of Φ_n. Moreover all such measures have the same Verblunsky coefficients $\{\alpha_j\}_{j=0}^{n-1}$ and so the same $\{\varphi_j\}_{j=0}^{n}$.

Of course, the results on the zeros long predate these theorems. Uniqueness of the α's in the OPUC theorem comes from inverse Szegő recursion. One way (Erdélyi, Nevai, Zhang and Geronimo) of getting existence is to use $N^{-1}d\theta/|\Phi_n(e^{i\theta})|^2$ which turns out to have Φ_n as an OPUC and remaining $\alpha_j = 0$ for $j \geq n$.

Compressed Multiplication Operators

Let $\mathcal{P}_j$ be the space of polynomials of degree at most j and let P_n be the projection onto $\mathcal{P}_{n-1}$. We call it P_n since $\dim \operatorname{ran} P_n$ is n. Given a non-trivial measure, let M_z be multiplication on by z on $L^2(\partial\mathbb{D}, d\mu)$. By a *compressed multiplication operator*, I mean the compression of the unitary M_z to polynomials of degree at most $n-1$, i.e. $A = P_n M_z P_n$ restricted to $\mathcal{P}_{n-1}$.

Characteristic Polynomials of Compressed Multiplication Operators

Theorem A depends only on $\{\alpha_j\}_{j=0}^{n-1}$, i.e. two A's with the same such α's are unitarily equivalent. The eigenvalues of A are precisely the zeros of Φ_n (up to algebraic multiplicity) so

$$\det(z - A) = \Phi_n(z)$$

This implies that if two A's are unitarily equivalent, they have the same Φ_n, and, so, by inverse Szegő recursion, the same $\{\alpha_j\}_{j=0}^{n-1}$. To prove the guts of theorem, let q be a polynomial in $\mathcal{P}_{n-1}$ with $(A - \zeta)q = 0$. Then $(z - \zeta)q(z)$ must lie in the kernel of P_n but any polynomial of degree at most n in the kernel is a multiple of Φ_n. To say that $q \neq 0$ and $(z - \zeta)q(z)$ is a multiple of Φ_n precisely says that $\Phi_n(\zeta) = 0$.

Trivial Measures and POPUC

Suppose now that $d\mu$ is a trivial measure on $\partial\mathbb{D}$, say with $n+1$ pure points, $\{w_j\}_{j=1}^{n+1}$. Then $\{z^k\}_{k=0}^{n}$ are still independent, so one can used Gram–Schmidt to form $\{\Phi_j\}_{j=1}^{n}$. As the norm minimizer, one also has that

$$\Phi_{n+1} = \prod_{j=1}^{n+1}(z - w_j)$$

Since this has norm 0, one expects and indeed finds that Φ_{n+1} is given by Szegő recursion but with $|\alpha_n| = 1$. That is trivial measures are described by sets of $n+1$ Verblunsky coefficients, the first n in $\mathbb{D}$ and the last in $\partial\mathbb{D}$. The corresponding multiplication operators are precisely the $n+1$–dimensional unitaries with a cyclic vector.

Trivial Measures and POPUC

This motivates, the following: Suppose, we are given a non–trivial measure with Verblunsky coefficients, $\{\alpha_j\}_{j=0}^{\infty}$, and we consider, $\Phi_n(z)$. Given $\lambda \in \partial\mathbb{D}$, we define the *paraorthogonal polynomial* (POPUC) by

$$\Phi_{n+1}(z;\lambda) = z\Phi_n(z) - \bar{\lambda}\Phi_n^*(z)$$

Theorem Fix $\{\alpha_j\}_{j=0}^{n-1}$ all in $\mathbb{D}$ and let A be the corresponding compressed multiplication operator. The POPUC of degree $n+1$ are in one to one correspondence with the rank one unitary dilations of A. The eigenvalues of the unitary, U_λ, associated to $\Phi_{n+1}(z;\lambda)$ are the zeros of that polynomial so that

$$\det(z - U_\lambda) = \Phi_{n+1}(z;\lambda)$$

GGT Matrices

The operators, A and U_λ act on spaces with natural orthonormal bases, namely $\{\varphi_j\}_{j=0}^{n-1}$ and $\{\varphi_j\}_{j=0}^{n}$. One can compute the matrix elements explicitly and I called these matrices *GGT matrices* and the bases *GGT bases*. The explicit form is important in some of the proofs in our work but since we don't need it below, I won't be explicit here. There is another matrix representation called CMV which is superior for the infinite dimensional case, in part, because the OPs may not be a complete basis in the infinite case. But since the GGT bases are complete for $\mathcal{P}_n$, they are simpler for this study.

S_n

We next summarize our main results:

Theorem 1 Every compressed multiplication operator lies in S_n.

Theorem 2 Every element in S_n is unitarily equivalent to a compressed multiplication operator.

Theorem 3 For any set of n elements (with multiplicity) in $\mathbb{D}$, there is a compressed multiplication operator with those eigenvalues. Two compressed multiplication operators with the same characteristic polynomial are unitarily equivalent.

These theorems imply the main classification result on S_n.

Numerical Range

For each $\lambda \in \partial\mathbb{D}$, let U_λ be the associated unitary, Φ_{n+1} the associated POPUC, $\{w_j\}_{j=1}^{n+1}$ the zeros of Φ_{n+1}, cyclicly ordered, and η_j the associated normalized eigenvectors, so $\eta_j(z) = N_j^{-1}\Phi_{n+1}(z)/(z-w_j)$. Let $m_j = |\langle \eta_j, \varphi_n\rangle|^2 > 0$ (since $\deg(\eta_j) = n$) so $\sum_{j=1}^{n+1} m_j = 1$. Let A be the dimension n compressed multiplication operator.

Theorem 4 For $j = 1, \ldots, n+1$, the line from w_j to w_{j+1} intersects $N(A)$ in a single point, ζ_j, and $|\zeta_j - w_j|/|\zeta_j - w_{j+1}| = m_j/m_{j+1}$. In particular, $\prod_{j=1}^{n+1} |\zeta_j - w_j| = \prod_{j=1}^{n+1} |\zeta_j - w_{j+1}|$.

Theorem 5 For each λ, we have that $N(U_\lambda)$ is a solid $(n+1)$-gon whose sides are tangent to $N(A)$. $\partial N(A)$ is a strictly convex analytic curve and one has that

$$N(A) = \cap_{\lambda\in\partial\mathbb{D}} N(U_\lambda)$$

Blaschke Products

With the above definition of the m_j, the spectral measure for φ_n is $d\nu = \sum_{j=1}^{n+1} m_j \delta_{w_j}$.

Theorem 6 One has that

$$\int \frac{1}{z - e^{i\theta}} d\nu(\theta) = \frac{\Phi_n(z)}{z\Phi_n(z) - \bar{\lambda}\Phi_n^*(z)}$$

If $\{z_j\}_{j=1}^n$ are the zeros of Φ_n, then

$\Phi_n(z)/\Phi_n^*(z) = \prod_{j=1}^n \frac{z-z_j}{1-\bar{z}_j z}$, so this is the Blaschke product theorem of Daepp et al quoted above with a very different proof of the positivity of the m_j and of the formula.

This result can be proven from Khrushchev's formula by taking limits to extend his formula to trivial measures. It can also be obtained from general formulae I have in my OPUC book for M–functions. We have two simple direct proofs.

Blaschke Products

Theorem 7 If $\{\alpha_j\}_{j=0}^{n-1}$ are the Verblunsky coefficients of the original problem, the Verblunsky coefficients of $d\nu$ are given by

$$\alpha_j(d\nu) = -\lambda\bar{\alpha}_{n-1-j}, \quad j = 0, \dots, n-1; \qquad \alpha_n(d\nu) = \lambda$$

The $\lambda = 1$ case of this result is implicit in a remark in my OPUC book on rank two decoupling of CMV matrices but our proofs are more direct.

Wendroff's Theorem

Theorem 8 (Wendroff's Theorem for (P)OPUC) The zero's of POPUC's for two values of λ interlace. Conversely, given two sets of $n+1$ interlacing points on $\partial\mathbb{D}$, there exist unique $\{\alpha_j\}_{j=0}^{n-1}$ in $\mathbb{D}$ and λ, μ in $\partial\mathbb{D}$ so these are zeros of the associated POPUCs.

This theorem is equivalent to a result of Gao–Wu quoted above. We have a new proof. At about the same time as this independently of Gao-Wu and each other, Golinskii and Cantero–Moral–Velázquez proved the first result. The strict interlacing of the first half of the next theorem is due to Simon. We have a new proof and the full result.

Theorem 9 (Wendroff's Theorem for Second Kind POPUC) Let $\{w_j\}_{j=1}^{n+1}$ be the zeros of a POPUC, Φ_{n+1}, ordered clockwise and $\{y_j\}_{j=1}^{n+1}$ be the zeros of associated second kind POPUC, Ψ_{n+1}, ordered clockwise, so that y_1 is the first zero after w_1 going clockwise. Then the w's and y's strictly interlace and one has that

$$\prod_{j=1}^{n+1} y_j = -\prod_{j=1}^{n+1} w_j$$

Conversely, if $\{w_j\}_{j=1}^{n+1}$ and $\{y_j\}_{j=1}^{n+1}$ are strictly interlacing and obey the above, then there is a unique set of Verblunsky coefficients $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{D}$ and $\lambda \in \partial\mathbb{D}$ so that $\{w_j\}_{j=1}^{n+1}$ is the set of zeros of the associated POPUC and $\{y_j\}_{j=1}^{n+1}$ the zeros of the associated second kind OPUC.

Compressed Multiplication Operators

Let $A = P_n M_z P_n$. Then $\|Ap\| = \|p\|$ if $\deg(p) \leq n-2$, while $\|A\varphi_{n-1}\| = |\alpha_{n-1}|$ by Szegő recursion, so $\mathbf{1} - A^*A = \rho_{n-1}^2 \langle \varphi_{n-1}, \cdot \rangle \varphi_{n-1}$. Since the eigenvalues of A are zeros of Φ_n, no eigenvalue has absolute value 1. Hence $A \in S_n$ and we have that

Theorem 1 Every compressed multiplication operator lies in S_n.

Proof of Theorem 2

Recall that Theorem 2 says that "Every element in S_n is unitarily equivalent to a compressed multiplication operator." We have two proofs of this result, neither short.

One proof uses the polar decomposition of an $A \in S_n$. Let x_{n-1} be a unit vector in $\operatorname{ran}(\mathbf{1} - A^*A)$. Complete nonunitarity is equivalent to x_{n-1} being cyclic for the unitary part of the polar decomposition so it is equivalent to a unitary GGT matrix which we turn on its head and then use the full polar decomposition to see that A is equivalent to a compressed GGT matrix.

The other proof uses x_{n-1} and an induction motivated by inverse Szegő recursion to construct a basis $\{x_j\}_{j=0}^{n-1}$ which yield a set of OPUC.

A third proof uses Theorem 3 and the classification theorem for S_n but since we want to use Theorem 2 to prove the classification theorem, we don't use this.

Proof of Theorem 3

Recall that Theorem 3 says that "For any set of n elements (with multiplicity) in $\mathbb{D}$, there is a compressed multiplication operator with those eigenvalues. Two compressed multiplication operators with the same characteristic polynomial are unitarily equivalent."

A little thought shows that this is a restatement of Wendroff's Theorem for OPUC.

Edges of $N(U_\lambda)$

Let $\mathcal{P}_j$ be the space of polynomials of degree at most j so U_λ acts on $\mathcal{P}_n$ and A on $\mathcal{P}_{n-1}$. For any $\psi \in \mathcal{P}_n$, we have that $\langle \psi, U_\lambda \psi \rangle = \sum_{j=1}^{n+1} z_j |\langle \eta_j, \psi \rangle|^2$ so that the only such expectations that lie in the line between z_j and z_{j+1} come from $\psi =$ linear combinations of η_j and η_{j+1}. Since $\psi \in \mathcal{P}_{n-1} \iff \langle \varphi_n, \psi \rangle = 0$, the only unit vector in both $\mathcal{P}_{n-1}$ and that 2D space is (recall $m_j = |\langle \eta_j, \phi_n \rangle|^2$) $\psi = [\langle \varphi_n, \eta_j \rangle \eta_{j+1} - \langle \varphi_n, \eta_{j+1} \rangle \eta_j] / \sqrt{m_j + m_{j+1}}$ so

$$\langle \psi, A\psi \rangle = \frac{m_j z_{j+1} + m_{j+1} z_j}{m_j + m_{j+1}}$$

which by a simple calculation implies that

Theorem 4 For $j = 1, \dots, n+1$, the line from w_j to w_{j+1} intersects $N(A)$ in a single point ζ_j and $|\zeta_j - w_j| / |\zeta_j - w_{j+1}| = m_j / m_{j+1}$. In particular, $\prod_{j=1}^{n+1} |\zeta_j - w_j| = \prod_{j=1}^{n+1} |\zeta_j - w_{j+1}|$.

Proof of Theorem 5

Recall that Theorem 5 says that "For each λ, we have that $N(U_\lambda)$ is a solid $n+1$-gon whose sides are tangent to $N(A)$. $\partial N(A)$ is a strictly convex analytic curve and one has that

$$N(A) = \cap_{\lambda \in \partial\mathbb{D}} N(U_\lambda)''$$

It is easy to see that U_λ is analytic in λ, so its eigenvectors are real analytic in λ and so the tangent points move analytically. Given that $N(A)$ is convex, we see that its boundary is an analytic convex curve. Uniqueness of the points on polygon sides implies strict convexity.

Clearly $N(A) \subset N(U_\lambda)$ proving one direction of the above equality. If $\xi \notin N(A)$, pick χ in the interior of $N(A)$. Let ℓ be the line segment between χ and ξ which must meet $\partial N(A)$ in a single point ζ. The tangent to $\partial N(A)$ at ζ must be a side of some $N(U_\lambda)$ so $\xi \notin N(U_\lambda)$. $\square$

Cramer's Rule

We want to prove that $\int \frac{1}{z-e^{i\theta}} d\nu(\theta) = \frac{\Phi_n(z)}{z\Phi_n(z)-\bar{\lambda}\Phi_n^*(z)}$. In a GGT basis, φ_n is the last basis element in the $\mathcal{P}_n$ basis, so we are asking for the lower right corner matrix element of the resolvent of the GGT unitary, U_λ. By Cramer's rule, this is a ratio of determinants. Recognizing that dropping the last row and column gives us A, we find that

$$\left\langle \varphi, \frac{1}{z-U_\lambda}\varphi \right\rangle = \frac{\det(z-A)}{\det(z-U_\lambda)}$$

Given the relation of determinants of A and U_λ to OPUC and POPUC, we get the required formula!

Blaschke Products

In my OPUC book, I prove and use that

$$B_n(z) = \frac{\Phi_n(z)}{\Phi_n^*(z)}$$

is a Schur function because it is analytic in $\mathbb{D}$ and of magnitude 1 on $\partial\mathbb{D}$. It enters in Khrushchev theory. A simple calculation of the Carathéodory function for λB_n and, from that, the M-function, shows the associated measure is $d\nu_\lambda$ as computed above.

Szegő recursion for Φ_{n-1} and Φ_{n-1}^* show that

$$B_n(z) = \frac{-\bar{\alpha}_{n-1} + zB_{n-1}(z)}{1 - \alpha_{n-1} z B_{n-1}(z)}$$

which is just the Schur algorithm. Thus, the Schur iterates are λB_{n-j} and one finds, by Geronimus' theorem, that

$$\alpha_j(d\nu) = -\lambda\bar{\alpha}_{n-1-j}; \quad j = 0, \dots, n-1 \qquad \alpha_n(d\nu) = \lambda$$

Wendroff for POPUC

Our proof depends on the observation that if $z\Phi_n - \bar{\lambda}\Phi_n^* = Q_{n+1}; z\Phi_n - \bar{\mu}\Phi_n^* = R_{n+1}$, then

$$\Phi_n(z) = \frac{\lambda Q_{n+1}(z) - \mu R_{n+1}(z)}{(\lambda - \mu)z}$$

This immediately implies uniqueness. It also shows that if Q_{n+1} and R_{n+1} have a common zero, then it is a zero of Φ_n. On the one hand this shows that if one move the zeros to go from z^n to Φ_n keeping all zeros inside $\mathbb{D}$, the deformed Q_{n+1} and R_{n+1} cannot have common zeros which means interlacing for the case where z^n implies interlacing in general.

Wendroff for POPUC

Conversely, one can note if a polynomial P_n has a zero on $\partial\mathbb{D}$, then P_n^* has a zero in the same place, so if Q_{n+1} defined as $zP_n - \bar{\lambda}P_n^*$ also has zero there. A deformation argument shows that if P_n is defined in terms of zeros Q_{n+1} and R_{n+1} and the first formula on the page, shows that P_n has its zeros inside $\mathbb{D}$ so we can use Wendroff for OPUC.

Wendroff for Second Kind POPUC

The key to proving this theorem is that $F(z) = \frac{\Psi^*_{n+1}}{\Phi^*_{n+1}}$ is the rational Carathéodory function of the measure associated to the POPUC. Such a function is pure imaginary between the poles with monotone imaginary part implying a single zero, hence the interfacing. Since it is pure imaginary, the reflection principle implies that $\overline{F\left(\frac{1}{\bar{z}}\right)} = -F(z)$. The product condition follows from this.

For the converse, one shows under the conditions on the zeros the function

$$f(z) = \frac{\prod_{j=1}^{n+1}(1 - \bar{y}_j z)}{\prod_{j=1}^{n+1}(1 - \bar{w}_j z)}$$

is a rational Carathéodory function whose measure has the given first and second kind POPUC's.

From $N(A)$ to the Eigenvalues

We haven't discussed the eigenvalues as foci, so it is interesting to note that what have discussed provides two ways to go from $N(A)$ to the eigenvalues for $A \in S_n$. Firstly, we can take an point on $\partial N(A)$ and by drawing the tangents get a circumscribed Poncelet $(n+1)$–gon. The intersection points gives us w_j and the sides give us m_{j+1}/m_j and so via $\sum_{j=1}^{n+1} m_j = 1$, the m_j. So we can construct the measure $d\nu = \sum_{j=1}^{n+1} m_j \delta_{w_j}$. Because of $\int \frac{1}{z-e^{i\theta}} d\nu(\theta) = \frac{\Phi_n(z)}{z\Phi_n(z) - \bar{\lambda}\Phi_n^*(z)}$, we see the eigenvalues are the zeros of the function defined by this integral.

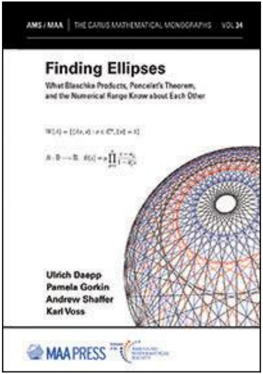
Alternatively, we can construct two circumscribed Poncelet $(n+1)$–gons and use Wendroff's theorem for POPUC to get the Verblunsky coefficients, so Φ_n and the eigenvalues as its zeros.

Two Matrix Representations

The refined standard proof that the eigenvalues determine $A \in S_n$ rely on a canonical matrix representation realizing A as a compressed shift using model theory. The representation is upper triangular in what is known as a Takenaka-Malmquist-Walsh basis from work going back to 1925. Our proof depends on a different matrix realization as a Hessenberg matrix. It would be interesting to see if the TMW basis is useful in OPUC and if the GGT basis is useful in model theory.

Announcement from the AMS

As I was preparing this talk, I got an announcement of a new book from the AMS/MAA



They forgot OPUC!

Poncelet's Theorem

OPUC on One Toe

Main Results

The Class S_n

Numerical Range

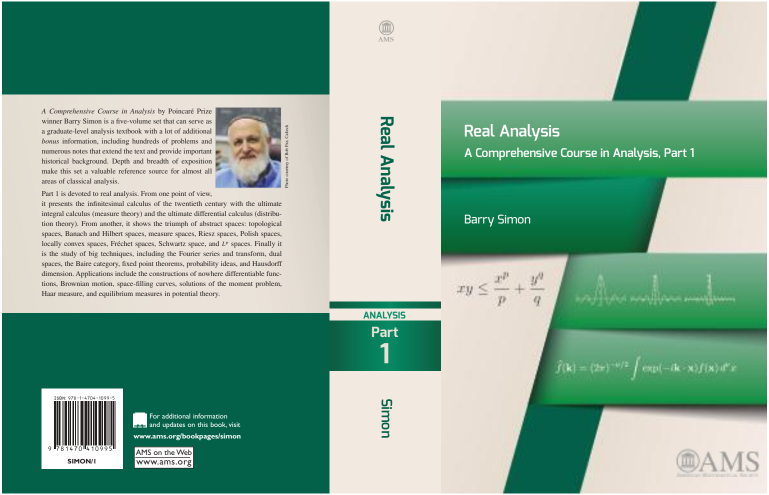
The M-Function for POPUC

The Schur functions Associated to OPUC

Wendroff's Theorem for POPUC

Final Remarks

And Now a Word from Our Sponsor



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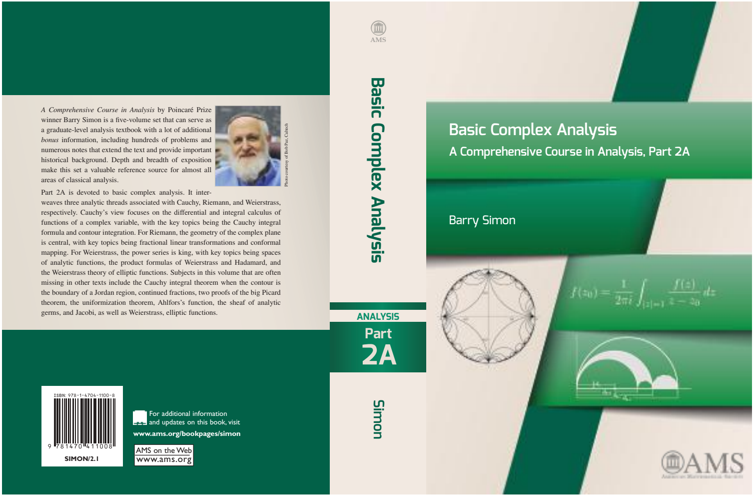

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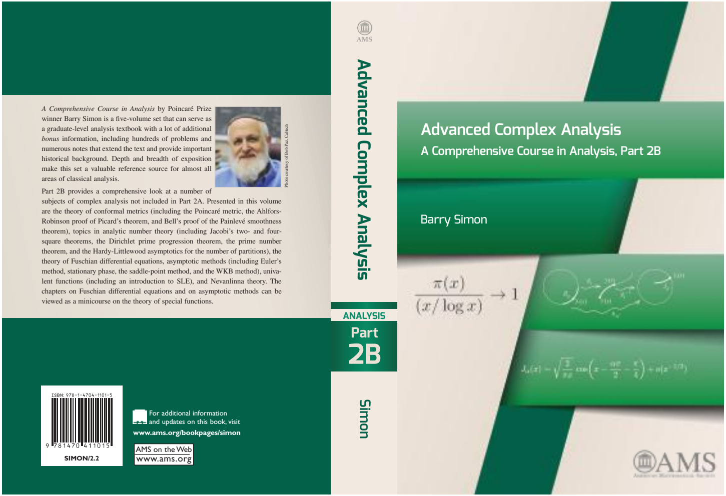

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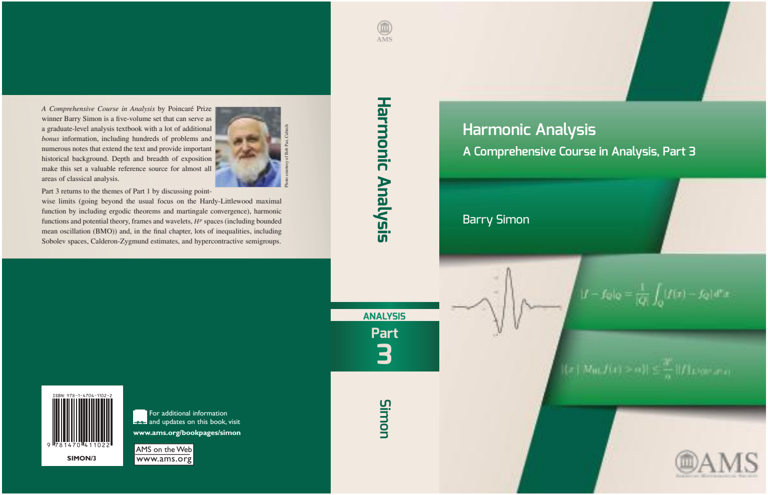

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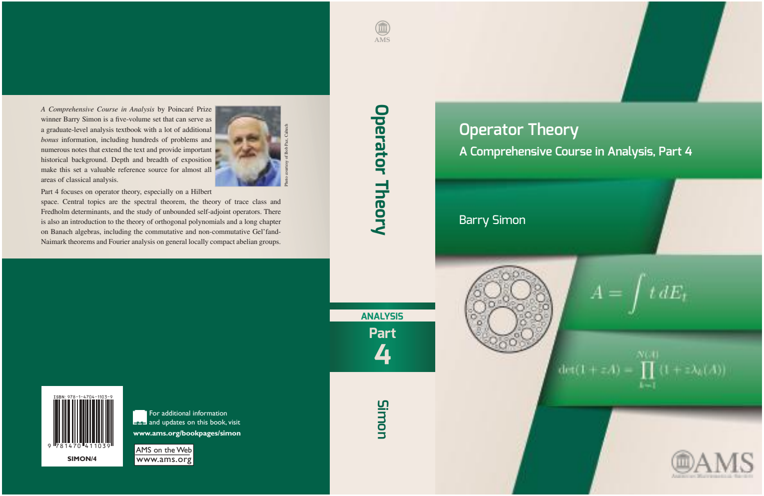

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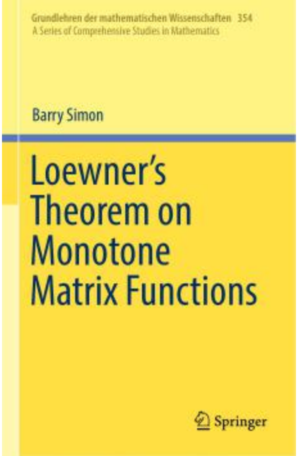

And tada, the latest book