The Tale of a Wrong Conjecture: Borg's Theorem for Periodic Jacobi Matrices on Trees

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Parameter Counting

It is always interesting to figure out how rare a rare thing is. If we look at points in $\mathbb{R}^n$, most will have unequal coordinates. We can ask the codimension of the set of points, $\{\mathbf{x} \mid x_i = x_j \text{ for some } i \neq j\}$, with not all distinct coordinates, clearly codimension 1 as a finite union of hyperplanes. We could quickly figure out what the answer is by looking at the simplest case, $n = 2$ where it is a single line and arguing that coincidence happens a pair at time so that should be the general case.

Harder, but not a lot more, is looking at self-adjoint matrices and asking for the codimension of those with a degenerate eigenvalue. Again we start with the simplest case, 2×2 matrices.

Parameter Counting

If we write the general 2×2 self-adjoint matrix as

$$\begin{pmatrix} a+b & c \\ \bar{c} & a-b \end{pmatrix}$$

with $a, b \in \mathbb{R}, c \in \mathbb{C}$, we see the set of such matrices has real dimension 4 and those with only one eigenvalue (so $b = c = 0$) dimension 1 so real codimension 3. If we only look at real matrices, c is real so the real codimension is 2. Again, eigenvalue coincidences occur in pairs, so we expect those are the right codimensions in general.

Wigner-von Neumann and Weyl

That this expectation is correct is a famous theorem of Wigner and von-Neumann published in 1929. They went to the same high school and were lifelong friends and, while postdocs, published two back-to-back papers in Physikalische Zeitschrift (the first on embedded eigenvalues is perhaps more famous among spectral theorists; the second with the theorem below is much more quoted in the physics community at large).

Theorem *In the* $\frac{n(n+1)}{2}$ *dimensional space of self-adjoint real* $n \times n$ *matrices, those with a degenerate eigenvalue are a variety of dimension* $\frac{n(n+1)}{2} - 2$. *In the* n^2 *dimensional space of self-adjoint complex* $n \times n$ *matrices, those with a degenerate eigenvalue are a variety of dimension* $n^2 - 3$.

Wigner-von Neumann and Weyl

The argument that WvN use is simple. They counted dimension by looking at the eigenvalues and at the fact that given the eigenvalues, you have to pick frames of eigenvectors (i.e. orthonormal eigenvectors up to phase). For example, one dimension of the lost two in the real case comes from the lower dimension of the set of distinct eigenvalues and the other one comes from the fact that if the last two eigenvalues are the equal ones, their eigenspace is determined as the space orthogonal to the first $n-2$ whereas if those last two are unequal, one has to choose a unit vector in a two dimensional space, an extra real parameter.

Wigner-von Neumann and Weyl

One can argue that while this result is attributed to Wigner-von Neumann in 1929, it is in essence in a 1926 work of Weyl (or even earlier work of Szegő). For it is easy to argue we must get the same answer for $U(n)$ as for $n \times n$ complex matrices and for $O(n)$ and real matrices. That the Weyl formula for the Haar measure projected onto the sets of eigenvectors has factors of $|\lambda_i - \lambda_j|^m$ with $m = 2$ for $U(n)$ and $m = 1$ for $O(n)$ (as made famous by random matrix theory) is precisely an expression of the extra lower codimensions.

Perturbation Theory

Rather than the WvN picture of direct dimension counting, one can use eigenvalue perturbation theory to understand where codimension 2 comes from. To see if a degenerate eigenvalue splits to first order, one looks at the projection, P, onto the unperturbed eigenspace and then at PVP where V is the perturbation. If this has distinct eigenvalues, there is splitting to first order. We saw above in this effective $2D$ case the codimensions were 2 and 3.

A final remark before leaving this subject. In quantum mechanics without magnetic fields, Hamiltonians commute with a complex conjugation (essentially by time reversal invariance) so the relevant codimension is 2. Once there is a magnetic field, things are effectively complex, so codimension 3.

Generic Continuum Schrödinger

Related to this theme is the following theorem that I proved in 1976:

Theorem Let Y be the Frèchet space of C^∞ period 1 functions on $\mathbb{R}$ with the seminorms $\|V\|_n \equiv \sup|V^{(n)}(x)|$. Then the set of V's so that $h = -\frac{d^2}{dx^2} + V(x)$ has all gaps open is a dense open set.

We recall that these periodic Hamiltonians have an integrated density of states, $k(E)$ (one definition is that $k(E)$ is the limit as $m \to \infty$ of m^{-1} times the number of eigenvalues less than E of h restricted to $[0, m]$ with periodic boundary conditions). In the periodic case, k is strictly monotone precisely on the spectrum of h with gaps in the spectrum where k is constant and that there is a potential gap at the energies where $k(E) = n$ for $n = 1, 2, \ldots$.

Generic Continuum Schrödinger

The proof is easy if one uses band theory. A closed gap corresponds to a degenerate periodic or antiperiodic eigenvalue and an explicit calculation shows such a degeneracy is removed in perturbation theory for some perturbations, so the set where a given gap is open is a dense open set. The magic of the Baire category theorem then completes the proof.

Generic Discrete Jacobi

Much more can be understood easily in the period p Jacobi case. In that case, the space of $2p$ Jacobi parameters supports the Toda dynamical system which is completely integrable. The parameter space foliates into isospectral tori of dimension 2ℓ, where ℓ is the number of gaps. This is a precise expression that one looses two dimensions each time a gap closes. In the Schrödinger case, there is also the KdV dynamical systems but since all dimensions are infinite, it is more complicated to discuss codimensions.

The Three Martini Problem

The same theme occurs in a more famous story, what is called the *Ten Martini Problem*. This concerns one of the most famous examples in mathematical physics which I named the almost Mathieu operator (acting on $\ell^2(\mathbb{Z})$)

$$H_{\lambda,\alpha,\theta}u(n) = u(n+1) + u(n-1) + 2\lambda\cos(\pi\alpha n + \theta)u(n)$$

This is periodic if α is rational but only almost periodic if α is irrational. If $\alpha = p/q$, then there is a possible gap when $k(E) = j/q;\ j = 1, \dots, q-1$ and so the spectrum has q (or fewer) bands. If α is irrational, one can prove (Johnson-Moser & Bellisard) that on any potential gap $k(E) = \{m\alpha\}$, the fractional part of $m\alpha$. If all gaps are open, the spectrum is a Cantor set (i.e. closed and nowhere dense).

The Three Martini Problem

Mark Kac and I discussed this situation at lunch one day in 1981 and agreed that it was an interesting conjecture to prove that $H_{\lambda,\alpha,\theta}$ had a Cantor spectrum for all irrational α and $\lambda \neq 0$ (if α is irrational, it is known (Avron-Simon) that the spectrum is θ independent). "That's a grand conjecture", said Mark, "I'll offer ten Martinis for its solution." He later repeated this offer at an AMS meeting and I popularized it as the ten Martini problem. Solved in full in 2004 by Avila-Jitomirskaya after an important partial result by Puig. This is weaker than the result that all gaps are open, something known as the dry form of the ten Martini problem.

The Three Martini Problem

A year after my lunch with Kac, Bellisard and I used the strategy of my periodic result. We first proved that if $\alpha = p/q$ is rational and $q\theta$ is not a multiple of π, then all gaps were open (i.e. the spectrum had $q-1$ gaps). This non-trivial analytic fact was proven using ideas motivated by the classical result of Ince that the continuum Mathieu operator $(-\frac{d^2}{dx^2} + \lambda\cos(x))$ has all gaps open. But once we knew that and had some continuity results on $k(E)$ of Avron-Simon, the magic of the Baire category theorem showed that for a Baire generic set of (α, λ), the spectrum is a Cantor set! It is remarkable that with one's Baire hands one can learn something about the irrational case (Cantor spectrum) by knowing something about the rational case even though, of course, in the rational case the spectrum is never Cantor.

The Three Martini Problem

When I told Mark about this on the phone admitting it wasn't the full result, he remarked "But it is still interesting! I'll give you three martinis for it." So I always think of this as the three Martini result. Alas, before we met again, Mark was dead of pancreatic cancer (the same disease that felled the other half of the Feynman-Kac formula).

Continuum Schrödinger

Related to these themes is the following 1946 theorem of Borg:

Theorem *Let V be a periodic function on $\mathbb{R}$ so that $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has spectrum $[\Sigma, \infty)$. Then V is constant.*

In other words, if V is not constant, at least one gap is open.

Jacobi Matrices

In 1975, Hochstadt proved the analog for Jacobi matrices

Theorem *Let* $\{a_n, b_n\}_{n\in\mathbb{Z}}$ *be periodic in* n *so that the corresponding two sided Jacobi matrix on* $\ell^2(\mathbb{Z})$ *has spectrum* $[a, b]$. *Then* a *and* b *are each constant.*

Hochstadt's Theorem

We recall, that if V is a function on $\mathbb{R}$ with period L, then gaps occur at energies where $k(E) = n/L$. So if we thought V had period L but really had a shorter period L/p, then the only gaps will be at n's divisible by p. In 1984, Hochstadt proved the following converse:

Theorem *Let V be a periodic function on $\mathbb{R}$ with period L so that, for some integer p, $-\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R}, dx)$ has gaps in its spectrum only at some subset of the points where $k(E) = pn/L, n = 1, 2, \ldots$. Then V has period L/p*

This is a strengthening of Borg in that it implies a Borg's theorem (since no gaps means the hypothesis holds for all p, so $V(x + m/p) = V(x)$ for all rational m/p). There is a Jacobi matrix version of this theorem.

The Bethe Sommerfeld Conjecture

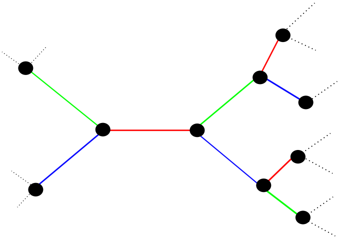
Before leaving the discussion of results for gaps in the spectrum of periodic Schrödinger operators, I should mention

Theorem *If V is a smooth, periodic function on $\mathbb{R}^\nu, \nu \geq 2$, then $-\Delta + V$ has only finitely many gaps in its spectrum*

By periodic, we mean invariant under a ν-dimension lattice of translations. This result is generally called the Bethe-Sommerfeld conjecture after its occurrence in a 1933 monograph of those authors (Bethe was Sommerfeld's student). It has a long involved literature. The definitive general result was proven by Parnovski in 2008. It says that 1D (which generically has infinitely many gaps) is very different from higher dimensions.

Regular Trees

The rest of this talk focuses on Jacobi matrices on infinite trees. We will mainly consider the fixed degree tree like the following degree 3 regular tree



Among spectral theorists, about the only literature on such operators is a lot on the random case (Klein, Aizenman-Warzel,...) and some results on rooted trees by Breuer and by Keller, Lenz and S. Warzel.

Graph Theory Formalism

A *graph* is a collection of points, aka *vertices*, and connectors, aka *edges*. Each edge has two ends which are vertices. There is a natural topological space associated to a graph and we always suppose it is connected. We want to allow edges that start and end at the same vertex (aka *self–loops*) and definitely want to allow multiple edges between a given pair of vertices.

A graph which is simply connected is called a *tree*. The *degree* of a vertex is the number of edges with that vertex as an end. A *leaf* is a vertex of degree one and we will normally only consider graphs with no leaves. Thus, our trees are always infinite. Of course, trees have no self loops and at most one edge between two vertices. A graph with constant degree is called *regular*.

We will most often consider regular graphs.

Jacobi Matrices

A *Jacobi matrix on a graph*, $\mathcal{G}$, is associated to a set of real numbers $\{b_j\}_{j\in V}$ assigned to each vertex and strictly positive reals $\{a_\alpha\}_{\alpha\in E}$ assigned to each edge. Because we will only consider finite graphs or infinite trees with periodic parameters, the a's and b's are bounded sets. The Jacobi matrix acts on $\ell^2(V) \equiv \mathcal{H}(\mathcal{G})$, the vector space of square summable sequences indexed by the vertices of the graph. It has matrix elements

$$H_{jk} = \begin{cases} b_j, & \text{if } j = k; \\ \sum_\alpha a_\alpha, & \text{if } j \neq k \text{ are ends of one or more edges,} \\ & \qquad\qquad \alpha\text{, which we sum over;} \\ 0, & \text{if no edges have } i \text{ and } j \text{ as ends.} \end{cases}$$

If there are self–loops, one needs to modify this.

Covering Space Formalism

Let $\mathcal{G}$ be a finite graph (with no leaves). Its universal cover, $\mathcal{T}$ is a tree and if $\mathcal{G}$ has constant degree, so does $\mathcal{T}$, i.e. it is a *regular tree*.

Now let J be a Jacobi matrix on $\mathcal{G}$. There is a unique Jacobi matrix, H, on $\mathcal{T}$ so that if $\Xi : \mathcal{T} \to \mathcal{G}$ is the covering map and B_j, A_α the Jacobi parameters of J and b_j, a_α of H, then $b_j = B_{\Xi(j)}, a_\alpha = A_{\Xi(\alpha)}$. Any deck transformation, $G \in \Gamma$, the set of deck transformations on $\mathcal{T}$, induces a unitary on $\mathcal{H}(\mathcal{T})$ and these unitaries all commute with H. We call H a *periodic Jacobi matrix* and set p, the number of vertices of $\mathcal{G}$ to be its *period*, although, as I'll explain, there is some question if this is the right definition of period! We let q be the number of edges.

Free Groups

If $\mathcal{G}$ has m independent loops (equivalently, one can drop m edges and turn $\mathcal{G}$ into a connected finite tree), then the fundamental group of $\mathcal{G}$ is the free nonabelian group with m generators, $\mathcal{F}_m$. So that is the natural symmetry of our periodic trees.

The *free Jacobi matrix* on a tree is the one with all b's 0 and all a's 1. In this regard, there is a strange distinction between regular trees of constant degree d depending on whether d is even or odd! The graph with one vertex and k self loops has degree $d = 2k$. Its universal cover is the regular graph of degree $d = 2k$ and its free Laplacian is a period 1 Jacobi matrix. But there is no graph with a single vertex of odd degree, so, with our definition, the free Jacobi matrix on an odd degree homogenous tree is of period 2!!!

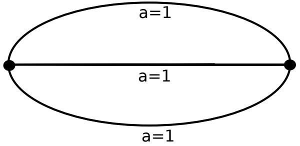
Free Groups

The point is the free group with k generators acts freely (i.e. no fixed point for non-identity elements) and transitively on the degree $2k$ regular tree. There is no such symmetry group on any odd degree regular tree although by looking at the cover of the two vertex, no self loop, d edge graph, one sees that $\mathcal{F}_{d-1}$ acts freely on the degree d regular tree but with two orbits rather than transitively. One can add an extra generator to get a transitive symmetry group but the action is no longer free.

As we'll see later, there is a transitive action on the vertices which will allow us to view the regular tree $\mathcal{T}_d$ as the points of a group.

Example 1: Free Jacobi Matrix on a Homogeneous Tree

It is illuminating to consider those few cases where we can compute the Green's function ($G_{jk}(z) = \langle \delta_j, (H-z)^{-1}\delta_k \rangle$) especially the diagonal case which is the Stieltjes transform of the spectral measure, $d\mu_j$. The IDS, $k(E)$, is defined by averaging the spectral measures over a unit cell (one point from each orbit) and integrating it from $-\infty$ to E. The simplest example is the d regular tree with all $a = 1$ and all $b = 0$. It has been extensively studied. Here is the underlying graph when $d = 3$ (set $a_1 = a_2 = a_3 = 1$)



There are related m-functions which we will discuss later and explain how to compute G from m.

Example 1: Free Jacobi Matrix on a Homogeneous Tree

The equation for m, which is independent of vertex and edge, is

$$m = \frac{1}{-z-(d-1)m} \Rightarrow m = \frac{-z+\sqrt{z^2-4(d-1)}}{2(d-1)}$$

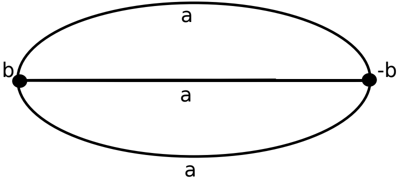
We take the plus sign on the square root to go to zero at ∞. Thus $\mathrm{spec}(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$. The formula for G, which is independent of vertex, is $(q \equiv d-1)$

$$G(z) = \frac{-(d-2)z + d\sqrt{z^2-4q}}{2(d^2-z^2)} \Rightarrow \frac{dk}{dE} = \frac{d\sqrt{4q-E^2}}{2\pi(d^2-E^2)}$$

the famed Kesten–McKay distribution, which arose first in random graph models, as the DOS for a large random degree d graph.

Example 2: Bipartite Degree 3

Consider a graph with two vertices and three edges between them. All the $a = 1$ and the two b's are b and $-b$ as shown here



There are two m-functions, $m_\pm$. A direct calculation gets equations they each obey which are quadratic in the m and quartic in z and one finds that

$$m_\pm(z) = -\frac{(z^2 - b^2) - \sqrt{(z^2 - b^2)^2 - 8(z^2 - b^2)}}{4(z \mp b)}$$

Example 2: Bipartite Degree 3

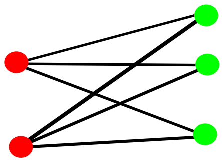
If $P(z)$ is the polynomial in the square root, one find that P vanishes at $z = \pm b, z = \pm\sqrt{b^2 + 8}$ so

$$\operatorname{spec}(H) = \left[-\sqrt{b^2+8}, -b\right] \cup \left[b, \sqrt{b^2+8}\right]$$

If $b \neq 0$, there is a single gap which is *always* open. This is a strong hint that something like Borg's Theorem might hold.

Example 3: The rg Model

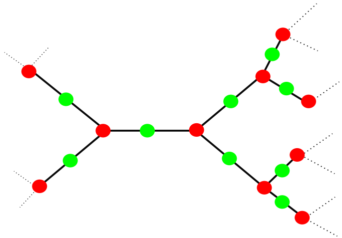
Aomoto found a very interesting example which we have dubbed the rg model. r and g are two positive integers. The underlying finite graph has $r+g$ vertices which we think of as r red vertices and g green. There are rg edges one between each pair of vertices of different colors, so the red vertices have degree g and the green degree r. $b \equiv 0$ and $a = 1$ on all edges. Here is the graph if $r = 2, g = 3$.



Aomoto showed that if $r \neq g$, this model always has an eigenvalue at $E = 0$. He analyzed some Green's function equations he had, to prove there must be a pole at $z = 0$.

Example 3: The rg Model

Avni, Breuer and I wrote down an explicit eigenfunction that is illuminating. One needs to display an explicit ℓ^2 function with the property that for each vertex, the sum of the values at all the neighbors is 0. In the above case, here is the tree



The function is zero at all red vertices so the eigenfunction equation holds trivially at every green vertex. The value at the green vertices depends only on the distance from the central vertex.

Example 3: The rg Model

It has the value 1 at the center and must have the value $-1/2$ are the vertices a distance 2 away and inductively $(-1/2)^k$ are vertices a distance $2k$ from the center. The number of such vertices is $2(2)^k$ so the ℓ^2 norm is $1+\sum_{k=1}^{\infty} 2^{k+1}(1/2)^{2k} < \infty$. Avni-Breuer-Simon found explicit formulae for the Green's function and showed that 0 is an isolated point of the spectrum which if $r < g$ has DOS weight $g - r/g + r$. There are two symmetric bands each of whose DOS weight is $r/r + g$.

Christiansen-Simon-Zinchenko (in prep) have analyzed this further and showed that the function I described and its translates span the eigenspace.

Sunda's Theorem

So far there are three big theorems known for these families of interesting operators

Theorem 1 (Sunada, 1992) *For a period p periodic Jacobi matrix on a tree, $k(E)$ in any gap has a value which is a multiple of $1/p$. This implies the spectrum has at most p bands.*

Sunada doesn't discuss discrete models explicitly but instead discusss continuum models on hyperbolic manifolds and remarks *A discrete (graph-theoretical) analogue of periodic Schrödinger operators can be treated in much the same way.* Recently (2020 preprint), Garza-Vargas and Kulkarni found an alternate proof of Sunada's theorem using free probability.

Aomoto's Theorem

Theorem 2 (Aomoto, 1991) *A period p periodic Jacobi matrix on a homogeneous tree has no eigenvalues*

This is, of course, of especial interest because in the same paper, Aomoto described what we call the rg model and proved that it did have eigenvalues. We find Aomoto's proof extremely mysterious. He has several strange looking calculations which in the end lead to an equality that implies the result.

No Singular Continuous Spectrum

Theorem 3 (Avni-Breuer-Simon, 2020) *All period p periodic Jacobi matrices on trees have no singular continuous spectrum*

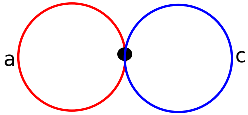
We prove this by showing that the Green's functions are algebraic functions (i.e. near infinity they solve $P(G(z), z) = 0$ for a polynomial in two variables). It follows that as they approach the real axis, the only possible singular points are finitely many poles and/or branch points. While he makes no mention of singular continuous spectrum, in several papers, Aomoto claims that the Green's functions are algebraic because he finds a series of p linked algebraic equations in z and the p diagonal Green's functions (with square roots) that they obey. But there is a gap in the proof in that he never proves that they are independent.

First Guess

There is an obvious first guess of how one might guess Borg's Theorem extends to trees.

Conjecture *If a periodic Jacobi matrix has no gaps in its spectrum, then a and b are each constant*

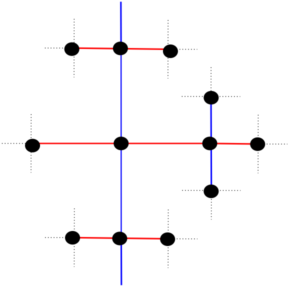
We've been working on these problems for about 5 years and for a while we thought this was a reasonable conjecture, but then we realized that



has period 1!

First Guess

So the tree



which definitely has non-constant a also has period 1 and so no gap. Clearly, a similar phenomenon works on any homogeneous tree with even degree. If $b = 0$ and the $2k$ values of a are equal in pairs, we have period 1 and no gap!

The Conjectures

After thinking about this, we decided it was a decent guess that this was the only counterexample so our published paper has the following

Conjecture 1. *Let $\mathcal{T}$ be a regular tree of odd degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then b is constant and a is constant.*

Conjecture 2. *Let $\mathcal{T}$ be a regular tree of even degree. If $H(\mathcal{T})$ is a periodic Jacobi matrix with no gaps in its spectrum, then the period is 1.*

That means, $\mathcal{G}$ has a single b and $\deg(\mathcal{T})/2$ self loops.

Conjecture 3. *Let $\mathcal{T}$ be a tree which is not regular. If $H(\mathcal{T})$ is a periodic Jacobi matrix, then it must have gaps in its spectrum.*

Actually, these are a single conjecture that no gaps implies period 1!

The Conjectures

But we wish to emphasize the different forms and the proofs may be different. As for all gaps open, we made two conjectures. Let $\mathcal{G}$ be a finite graph. Let $\mathcal{P}(\mathcal{G})$ be the set of allowed Jacobi parameters. It is an open orthant of $\mathbb{R}^{p+q}$ since $p+q$ is the number of vertices plus the number of edges. We say a period p Jacobi matrix has all gaps open if the spectrum has p bands. It is easy to see the set of Jacobi parameters for which all gaps are open is an open set.

Conjecture 4. *The set of parameters with all gaps open is a dense open set in the set of allowed parameters.*

We at least know the set is non-empty, for if all b are different and $\sum a < \min_{i\neq j} |b_i - b_j|$, then all gaps are open. Thinking Wigner-von Neumann, we conjectured

Conjecture 5 *The set of parameters where all gaps are not open is a variety of codimension 2.*

An interesting email

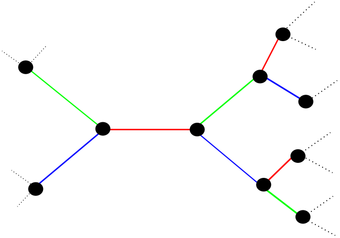
I'm sure you've heard lots of complaints about it taking too long from acceptance of a paper to publication. But I doubt you've heard a complaint about it taking too little time. It was literally 11 days between submission of the final version of our accepted paper for *Advances in Math* and the appearance of proofs in our mailbox. Not surprisingly, there were no changes in our paper so we returned proofs rapidly and 2 days after we received proofs, the paper appeared "published" online. And 16 days after that, we received an email from two graduate students at Berkeley, Jorge Garza Vargas and Achit Kulkarni, with counter examples to several of our recently published conjectures!!!!

An interesting email

Their manuscript was a second version of a preprint that they posted in the arXiv, a few weeks after our paper (about 7 months before). That original version didn't know of our work nor of Sunada's much earlier work. They had rediscovered Sunada's gap labelling theorem with a new proof using free probability theory. They found our preprint in the meantime, realized their main result wasn't new and found our Borg conjectures which they determined were false!

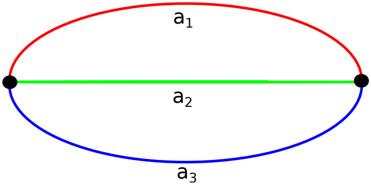
FTS Model

For counterexamples to Conjectures 1 and 2, the heavy lifting had been done earlier. Jacobi matrices on the degree d homogeneous tree with $b = 0$ and the each vertex with the same three a values are connected to random walks on certain groups if the a's from a single vertex sum to 1. In 1985, Tim Steeger proved the main result we'll need in his PhD thesis and the thesis only appeared in print in a 1994 AMS Memoir jointly with his thesis advisor Alessandro Figà-Talamanca. First their model when $d = 3$



FTS Model

This model is the lift of the 2 point



so the period is 2 and there is at most one gap open. Their important result (which we will prove later)

Theorem (Figà-Talamanca-Steger, 1985/1994) *Let H be the Jacobi matrix on the degree d homogeneous tree with $b = 0$ and $a_1 \geq a_2 \ldots \geq a_d$ at each vertex. Then $0 \in \operatorname{spec}(H)$ if and only if*

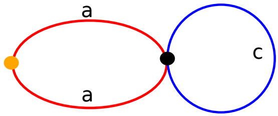
$$a_1^2 \leq \sum_{j=2}^{d} a_j^2$$

FTS Model

GVK realized that because the unitary, U, that flips signs at odd vertices obeys $UHU^{-1} = -H$, if there is a gap, then 0 must be in the gap, so not in the spectrum and conversely, if there is no gap, then the spectrum is $[-c, c]$, so $0 \in \text{spec}(H)$ so the last equation gives a necessary and sufficient condition for there to be no gap. Thus the gap is closed, for example, if $d-1$ a's have the value 1 and the other a has the value β with $0 < \beta \leq \sqrt{d-1}$ providing lots of examples where there is no gap for a period 2 model with a not constant. Moreover, one has $p = 2$ and $q = d$, so noting that the gap remains closed if the equation holds and $b_1 = b_2$, we get a set with no gap of codimension 1 contradicting Conjecture 5!

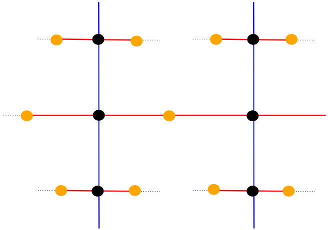
GVK Model

GVK also showed that the model with $p = 2$, $q = 3$ with graph



has no gap providing a counter example to Conjecture 3 (since one vertex has degree 2 and one has degree 4). Here is the tree

GVK Model



To see that $0 \in \mathrm{spec}(H)$ for this model for any values of a and c, consider φ be the function on the vertices that vanishes at all vertices off the central axis and at the black vertices on the central axis and alternates signs ± 1 at the yellow vertices on that line. Formally $H\varphi = 0$ and if one truncates to n consecutive yellow vertices and normalizes, unit vectors, φ_n, with $\|H\varphi_n\| = \sqrt{2/n} \to 0$. So, H is not invertible and the gap is closed.

Latest Borg Attempt

All the counterexamples we know have constant b. The most conservative conjecture to make is

Conjecture 6 *Let H be a periodic Jacobi matrix on the degree d homogeneous tree with all $a = 1$. If H has no gaps in its spectrum, then b is constant*

Braver is to conjecture a stronger result

Conjecture 7 *Let H be a periodic Jacobi matrix on some tree. If H has no gaps in its spectrum, then b is constant*

Note that Borg only considered the potential whose analog is b so one can claim that all along this is the correct analog of Borg.

What about Wigner-von Neumann

Once we accept the fact that gap opening depends on b and a's don't help, one can understand why Conjecture 5 failed for $p = 2$. Given that adding a constant to all $b's$ doesn't change which gaps are open, when $p = 2$, there are only two b's and one free parameter so, of course, only codimension 1. So we can be really brave and

Conjecture 8 *If the period $p \geq 3$, the set of parameters where all gaps are not open is a variety of codimension 2.*

The viewer should decide if this like grasping at straws or random walking towards successful conjectures!

G and M

Let H be a bounded Jacobi matrix on a tree, $\mathcal{T}$. If α is an edge with ends j, k, then removing the edge α disconnects $\mathcal{T}$ into two components, $\mathcal{T}_j^\alpha$ and $\mathcal{T}_k^\alpha$, containing j and k respectively. They are also trees although if either vertex has degree 2, they may have a leaf. We let $H(\mathcal{T}_j^\alpha)$ be the obvious Jacobi matrix acting on $\ell^2(\mathcal{T}_j^\alpha)$ and similar for $H(\mathcal{T}_k^\alpha)$. Define

$$G_j(z) = \langle \delta_j, (H-z)^{-1}\delta_j \rangle \qquad m_j^\alpha = \langle \delta_j, (H(\mathcal{T}_j^\alpha)-z)^{-1}\delta_j \rangle$$

and similarly for m_k^α. These are defined as analytic functions on $\mathbb{C} \setminus (A, B)$ if A and B are the bottom and top of $\operatorname{spec}(H)$. They are also analytic at infinity and in the gaps of the suitable spectra. One can show that the three operators have the same essential spectra, so all are meromorphic on $\mathbb{C} \setminus \operatorname{ess\,spec}(H)$.

Schur Complements

We want to derive the equations for G and m. These have often appeared in the literature on trees, especially on random discrete Schrödinger operators on trees, albeit many times with incorrect signs! A particularly clean method involves Banachiewicz' formula from the theory of Schur complements. One has a Hilbert space that is a direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ so that any $N \in \mathcal{L}(\mathcal{H})$ can be written

$$N = \begin{pmatrix} X & Z \\ Z^* & Y \end{pmatrix}$$

where, for example, $X \in \mathcal{L}(\mathcal{H}_1)$. Given such an N with Y invertible, we define the *Schur complement* of Y as $S = X - ZY^{-1}Z^*$. Let

$$L = \begin{pmatrix} \mathbf{1} & 0 \\ -Y^{-1}Z^* & \mathbf{1} \end{pmatrix} \text{ so } L^{-1} = \begin{pmatrix} \mathbf{1} & 0 \\ Y^{-1}Z^* & \mathbf{1} \end{pmatrix}$$

Schur Complements

A simple calculation shows that

$$L^*NL = \begin{pmatrix} S & 0 \\ 0 & Y \end{pmatrix}$$

so

$$\begin{aligned} N^{-1} &= L \begin{pmatrix} S^{-1} & 0 \\ 0 & Y^{-1} \end{pmatrix} L^* \\ &= \begin{pmatrix} S^{-1} & -S^{-1}ZY^{-1} \\ -Y^{-1}Z^*S^{-1} & Y^{-1} + Y^{-1}Z^*S^{-1}ZY^{-1} \end{pmatrix} \end{aligned}$$

which proves Banachiewicz' formula $(N^{-1})_{11} = S^{-1}$. We'll also need the off-diagonal $(N^{-1})_{12} = -S^{-1}ZY^{-1}$ below.

Formulae for G and M

For a tree, we fix $j \in \mathcal{T}$ and can write $\ell^2(\mathcal{T}) = \mathbb{C} \oplus \ell^2(\cup_{\alpha=(jk)} \mathcal{T}_k^\alpha)$ corresponding to singling out the site j. Then $(N^{-1})_{11}$ is a number, X is b_j, $Y = \oplus_{\alpha=(jk)} H(\mathcal{T}_k^\alpha)$ and Z is the various a_α. The result is

$$G_j(z) = \frac{1}{-z + b_j - \sum_{\alpha=(jk)} a_\alpha^2 m_k^\alpha(z)}$$

Similarly, if $\beta = (rj)$ is an edge in $\mathcal{T}$, we have that

$$m_j^\beta(z) = \frac{1}{-z + b_j - \sum_{\alpha=(jk);\, k \neq r} a_\alpha^2 m_k^\alpha(z)}$$

Note that if q is the number of edges in the underlying graph, $\mathcal{G}$, then there are $2q$ m–functions.

Formulae for G

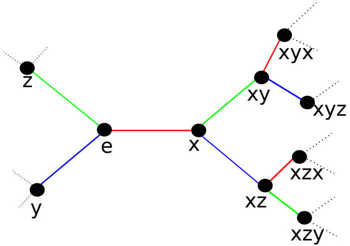
For vertices, $x, y \in \mathcal{T}$, let $G_{x,y}(z)$ be the matrix element of $(H - z)^{-1}$. Suppose that the last edge in the simple path from x to y is α from y to w. From the off-diagonal formula above, we get that

$$G_{xw}(z) = -a_\alpha m_w^\alpha(z) G_{xy}(z)$$

which let's one inductively compute all off-diagonal elements of G from the m's and diagonal elements of G.

$\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ - d fold

Recall that we are studying the homogeneous tree of degree d with $b \equiv 0$ and each vertex having d edges with values $a_1 \geq a_2 \geq \ldots \geq a_d$. It is notationally useful to think of this as a d-fold free product of $\mathbb{Z}_2$, that is a group $\mathcal{G}_d$ with d generators, $x_1, \ldots, x_d$ whose only relations are $x_1^2 = x_2^2 = \ldots = e$. Thus $\mathcal{G}_d$ consists of all words $y_{j_1} \ldots y_{j_m}$ where each y_j is one of the generators with $y_{j_\ell} \neq y_{j_{\ell+1}}$ for all ℓ and the obvious product. We can thus view our Hilbert space as $\ell^2(\mathcal{G}_d)$



$\mathbb{Z}_2 * \ldots * \mathbb{Z}_2$ - d fold

The group acts on the Hilbert space via the left regular representation $(U(y)f)_w = f_{y^{-1}w}$ and since $H_{y,yx_j} = a_j$, we have that U commutes with H. U defines a free transitive action on the vertices of $\mathcal{G}_d$, even for odd d (but not freely on the edges). By the transitive action, we see that there is a single diagonal Green's function $G(z) = G_{yy}(z)$ and d m-functions where are vertex independent, $\{m_\alpha\}_{\alpha=1}^d$. The basic equations are

$$G(z) = \frac{1}{-z - \sum_{\alpha=1}^d a_\alpha^2 m_\alpha(z)}$$

$$m_\alpha(z) = \frac{1}{-z - \sum_{\beta \neq \alpha} a_\beta^2 m_\beta(z)}$$

$$G_{e,yx_j}(z) = -a_j m_j(z) G_{e,y}(z)$$

When $0 \notin \operatorname{spec}(H)$

We will need special subsets, $S_j, j = 1, \ldots, d$, of the group $\mathcal{G}$ of all words of odd length the form $x_j x_{k_1} x_j \ldots x_j x_{k_m} x_j$, $k_q \neq j$ (a word of length $2m+1$). We claim that when $0 \notin \operatorname{spec}(H)$

1. $G(0) = 0$, indeed, $G_{ey}(0) = 0$ whenever $\rho(y)$ is even
2. All the m_j functions have meromorphic continuations across a real neighborhood of $z = 0$
3. There is a single j so that m_j has a pole at z=0. All the other m_k's obey $m_k(0) = 0$
4. $m_j(z)G(z)$ has a removable singularity at $z = 0$ with value $-1/a_j^2$
5. For $k \neq j$, $m_j(z)m_k(z)$ has a removable singularity at $z = 0$ with value $-1/a_j^2$

When $0 \notin \mathrm{spec}(H)$

Thus

$$G_{ey}(0) = \begin{cases} \frac{1}{a_j} \prod_i^m - \left(\frac{a_{k_i}}{a_j}\right), & \text{if } y = x_j x_{k_1} x_j \dots x_{k_m} x_j \in S_j \\ 0, & \text{if } y \notin S_j \end{cases}$$

Therefore

$$\sum_{y \,|\, \rho(y) = 2M+1} |G_{ey}(0)|^2 = a_j^{-2} \left[a_j^{-2} \sum_{k \neq j} a_k^2 \right]^M$$

Since $\sum_y |G_{ey}|^2 < \infty$, we see that $a_j^{-2} \sum_{k \neq j} a_k^2 < 1 \Rightarrow j = 1$ and $a_1^2 > \sum_{j=2}^d a_j^2$. So, $0 \notin \mathrm{spec}(H) \Rightarrow a_1^2 > \sum_{j=2}^d a_j^2$.

When $a_1^2 > \sum_{j=2}^d a_j^2$

For the converse, we need the following result of Haagerup about convolution operators on $\ell^2(\mathcal{G}_d)$:
Theorem For $f, g \in \ell^2$, define $f * g(x) = \sum_{y \in G} f(xy^{-1})g(y)$. Then

$$\|f * g\|_2 \leq \|g\|_2 \sum_{r=0}^{\infty} |n+1| \left(\sum_{\rho(y)=r} |f(y)|^2 \right)^{1/2}$$

Conversely suppose that $a_1^2 > \sum_{j=2}^d a_j^2$. Let f_j be the function we wrote down that included $f_j(y) = G_{ey}(0) = \frac{1}{a_j} \prod_i^m -\left(\frac{a_{k_i}}{a_j}\right)$, if $y = x_j x_{k_1} x_j \dots x_j x_{k_m} x_j \in S_j$. Then an easy calculation shows that

$$\sum_{k=1}^{d} a_k f_j(y x_k) = \delta_{ye}$$

When $a_1^2 > \sum_{j=2}^d a_j^2$

that is f_j is a formal Green's function of H at energy 0. To see this, note that if $yx_j \in S_j$ with $y \neq e$, the sum has two terms which cancel and if $yx_j \notin S_j$, all terms in the sum are zero. Finally, if $y = e$, there is only one term in the sum which is $a_j(1/a_j) = 1$. If $a_1^2 > \sum_{j=2}^d a_j^2$, then by Haagerup's Theorem, convolution with f_1 is bounded on ℓ^2. By the last relation, $H(f_1 * h) = h$ for any h of finite support. It follows that $0 \notin \mathrm{spec}(H)$.

Thus we have shown that $0 \notin \mathrm{spec}(H) \iff a_1^2 > \sum_{j=2}^d a_j^2$ which is the contrapositive of the result I called the theorem of Figà-Talamanca-Steger.

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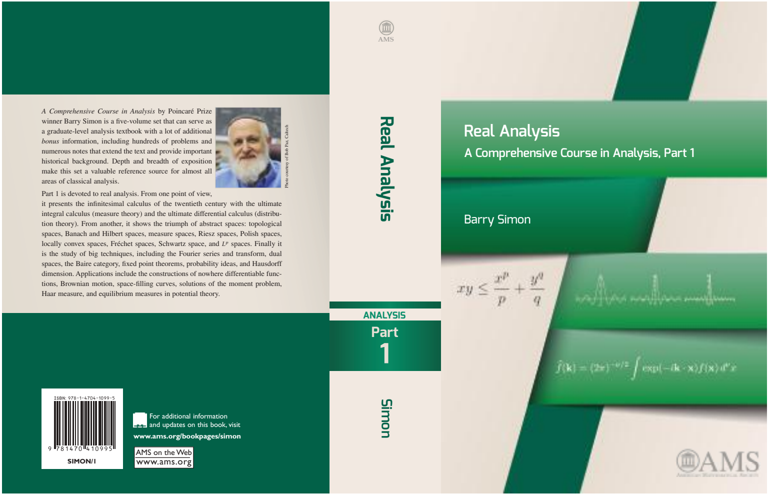

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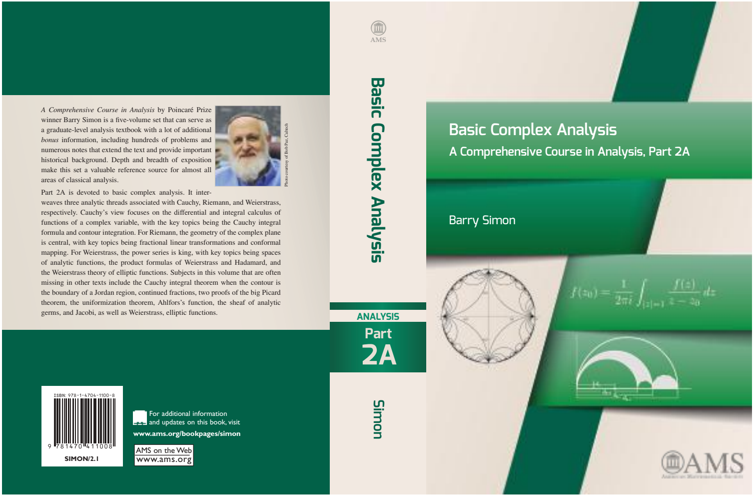

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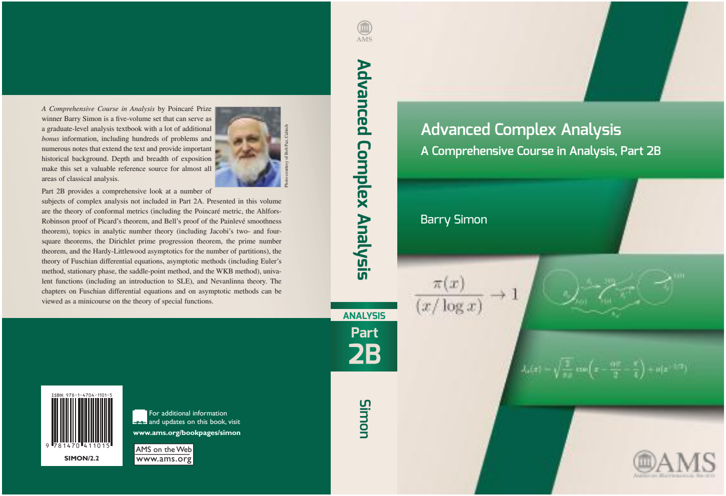

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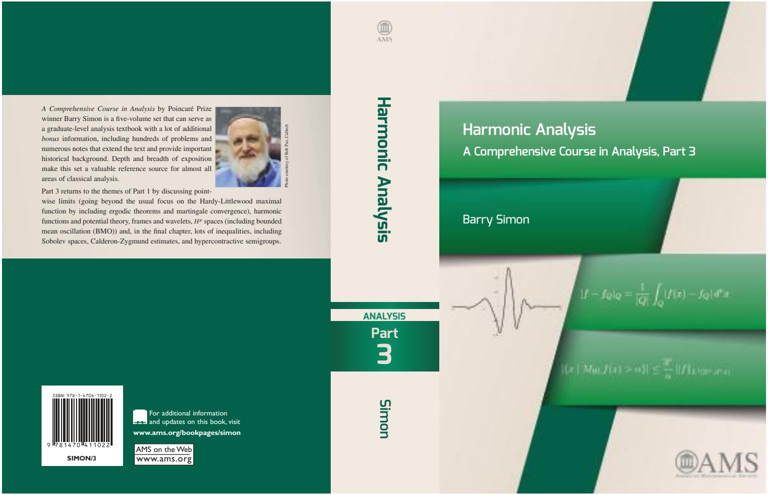

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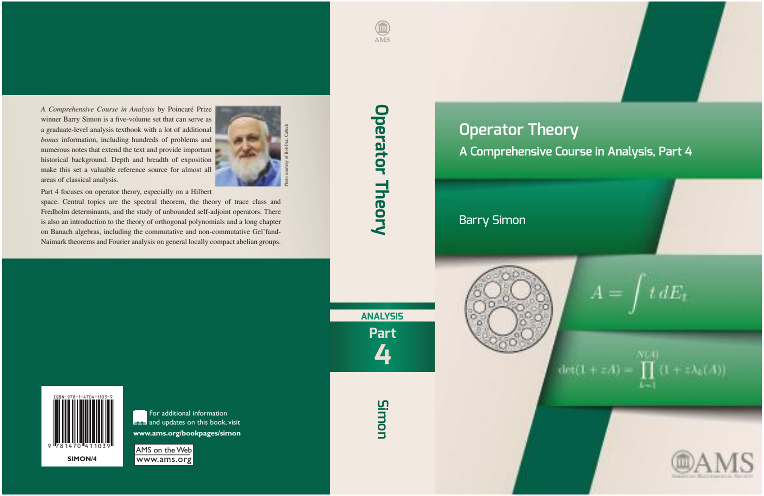

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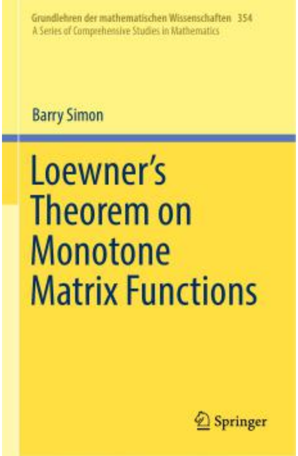

And tada, the latest book