Working title:
The structure of low-discrepancy sequences in $[0, 1)$

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The van der Corput sequence

Let

$$n = \varepsilon_0(n)2^0 + \varepsilon_1(n)2^1 + \varepsilon_2(n)2^2 + \cdots$$

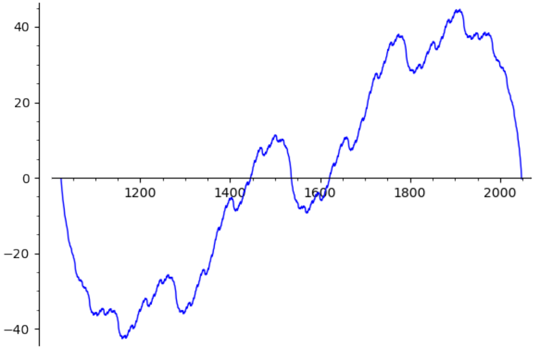
be the base-2 expansion of n and

$$\omega(n) = \sum_{j\ge 0} \varepsilon_j(n)2^{-j-1}$$

the van der Corput sequence in base 2.
The discrepancy $d(N) = ND_N(\omega)$ of $\omega(0), \ldots, \omega(N-1)$ satisfies

$$d(1) = 1, \quad d(2N) = d(N), \quad d(2N+1) = \frac{d(N) + d(N+1) + 1}{2}.$$

The function d has a fractal appearance, which we can see from (a linear transformation of) the partial sums.



($\rightsquigarrow$ S., UDT)

Research problems

Drmota, Larcher, and Pillichshammer showed that $d(N)$ satisfies a central limit theorem:

$$\frac{1}{M}\#\left\{N < M : d(N) \le \frac{1}{4}\log_2 N + y\frac{1}{\sqrt{48}}\sqrt{\log_2 N}\right\} = \Phi(y) + o(1).$$

Problem 1. Study the limiting distribution of the discrepancy of certain low-discrepancy sequences in one dimension.

What about the $(2,3)$-Halton sequence? (Lower bound due to Levin...)

Problem 2. Prove more detailed results (e.g. tail estimates) for the distribution of $d(N)$.

Can we find hints concerning the structure of general low-discrepancy sequences from such theorems?

Strong irregularities of distribution

Tijdeman and Wagner showed that a sequence has "almost nowhere small discrepancy":

Let ω be a sequence in $[0,1)$, and $M \geq 0$, $N > 1$ be integers. Then $D_N(\omega) < \log N/100$ for at most $2N^{5/6}$ integers $n \in (M, M+N]$.
(Sós considered this question for $\{n\alpha\}$-sequences.) — *strong irregularities of distribution*.

We could for example ask how the exponent ("5/6") changes if we change the constant ("1/100"). T–W proved in fact that it converges to zero with the constant.

Exact moments for the van der Corput-discrepancy

For the van der Corput sequence, we can find exact expressions for certain moments of the discrepancy:

$$m_{k,\ell,\lambda} = \frac{1}{2^\lambda} \sum_{n=2^\lambda}^{2^{\lambda+1}-1} d(n)^k d(n+1)^\ell.$$

Clearly, $m_{0,0,\lambda} = 1$ and it is easy to show that $m_{1,0,\lambda} = 1 + \lambda/4$.
The matrix powers that appear for the higher moments appear to have simple explicit descriptions . . . . . .

Summarizing: For sure, there is more to say about the discrepancy of the van der Corput sequence, and maybe we even learn something about the usual behaviour of low-discrepancy sequences.

Thank you!

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