Open problem: Robust lattice rules

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Consider the weighted Korobov space $\mathcal{H}_{d,\alpha,\gamma} \subseteq L_2([0,1]^d)$ with smoothness parameter $\alpha > 1$ and product weights $\gamma = (\gamma_j)_{j\geq 1}$.

Functions in the space are absolutely convergent Fourier series,

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^d} \widehat{f}(\boldsymbol{h}) e^{2\pi \mathrm{i} \boldsymbol{h}\cdot\boldsymbol{x}}.$$

Using the usual CBC construction, we can construct, for prime N, a generating vector $\boldsymbol{g} \in \{1, 2, \ldots, N-1\}^d$ of a rank-1 lattice rule such that

$$[e_{N,d,\alpha,\gamma}(\mathcal{H}_{d,\alpha,\gamma}, \boldsymbol{g})]^2 \leq \frac{2^{1/\lambda}}{N^{1/\lambda}} \prod_{j=1}^{d} \left(1 + \gamma_j^\lambda \zeta(\alpha\lambda)\right)^{1/\lambda}$$

for all $\lambda \in (1/\alpha, 1]$.

Now, make special choices for the weights γ_j:
Let $c > 0$ be a constant. For $\ell \in \mathbb{N}$, let $\mathcal{H}^{(\ell)}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}$ denote the Korobov space with product weights $\boldsymbol{\gamma}^{(\ell)} = (\gamma_j^{(\ell)})_{j\ge 1}$, where

$$\boldsymbol{\gamma}^{(\ell)} = \Big(\underbrace{\frac{c}{\ell}, \frac{c}{\ell}, \frac{c}{\ell}}_{\ell \text{ components}}, 0, 0, \ldots \Big),$$

i.e.,

$$\gamma_j^{(\ell)} = \begin{cases} \frac{c}{\ell} & \text{for } 1 \le j \le \ell, \\ 0 & \text{for } j > \ell. \end{cases}$$

Question: Asked by G. Larcher in 2020:

Can we find a single generating vector $\boldsymbol{g}^*$ that yields a low integration error for all spaces $\mathcal{H}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}^{(\ell)}$, $\ell \in \mathbb{N}$, simultaneously?

Answer: Yes and No.

Let $\mathcal{H}^{(0)}_{d,\alpha,\boldsymbol{\gamma}^{(0)}}$ denote the usual Korobov space with product weights $\boldsymbol{\gamma}^{(0)} = (\gamma_j^{(0)})_{j\geq 1}$, where

$$\boldsymbol{\gamma}^{(0)} = \left(\frac{c}{1}, \frac{c}{2}, \frac{c}{3}, \ldots\right),$$

i.e.,

$$\gamma_j^{(0)} = \frac{c}{j} \quad \text{for} \quad j \geq 1.$$

Note that we always have $\gamma_j^{(\ell)} \le \gamma_j^{(0)}$ for all $j \ge 1$ and all $\ell \ge 1$, and hence

$$\gamma_{\mathfrak{u}}^{(\ell)} := \prod_{j \in \mathfrak{u}} \gamma_j^{(\ell)} \le \prod_{j \in \mathfrak{u}} \gamma_j^{(\ell)} =: \gamma_{\mathfrak{u}}^{(0)}$$

for all $\mathfrak{u} \subseteq \{1, \ldots, d\}$.

This implies that, for any $\ell \in \mathbb{N}$,

$$e_{N,d,\alpha,\boldsymbol{\gamma}^{(\ell)}}(\mathcal{H}^{(\ell)}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}, \boldsymbol{g}) \le e_{N,d,\alpha,\boldsymbol{\gamma}^{(0)}}(\mathcal{H}^{(0)}_{d,\alpha,\boldsymbol{\gamma}^{(0)}}, \boldsymbol{g})$$

for any generating vector $\boldsymbol{g}$.

Use the CBC construction to obtain a $\boldsymbol{g}^*$ such that

$$[e_{N,d,\alpha,\boldsymbol{\gamma}^{(\ell)}}(\mathcal{H}^{(\ell)}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}, \boldsymbol{g}^*)]^2$$

$$\leq [e_{N,d,\alpha,\boldsymbol{\gamma}^{(0)}}(\mathcal{H}^{(0)}_{d,\alpha,\boldsymbol{\gamma}^{(0)}}, \boldsymbol{g}^*)]^2$$

$$\leq \frac{2^{1/\lambda}}{N^{1/\lambda}} \prod_{j=1}^{d} \left(1 + (\gamma_j^{(0)})^\lambda \zeta(\alpha\lambda)\right)^{1/\lambda}$$

for all $\lambda \in (1/\alpha, 1]$.

Note that

$$\sum_{j=1}^{d} \gamma_j^{(0)} = \sum_{j=1}^{d} \frac{c}{j} \simeq \log d,$$

so the upper bound yields polynomial tractability, simultaneously for all $\mathcal{H}^{(\ell)}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}$.

Question: Can we get a similar result for strong polynomial tractability?

Answer: Yes and No.

By using the fact that $\gamma_j^{(\ell)} = 0$ for any $j \geq \ell + 1$, we can sharpen the previous result. We can easily deduce the existence of a $\boldsymbol{g}^*$ such that

$$[e_{N,d,\alpha,\boldsymbol{\gamma}^{(\ell)}}(\mathcal{H}^{(\ell)}_{d,\alpha,\boldsymbol{\gamma}^{(\ell)}}, \boldsymbol{g}^*)]^2$$

$$\leq [e_{N,\ell,\alpha,\boldsymbol{\gamma}^{(0)}}(\mathcal{H}^{(0)}_{\ell,\alpha,\boldsymbol{\gamma}^{(0)}}, \boldsymbol{g}^*)]^2$$

$$\leq \frac{2^{1/\lambda}}{N^{1/\lambda}} \prod_{j=1}^{\ell} \left(1 + (\gamma_j^{(0)})^\lambda \zeta(\alpha\lambda)\right)^{1/\lambda}$$

for all $\lambda \in (1/\alpha, 1]$.

This bound holds for all $\ell \in \mathbb{N}$, and also for all $d \in \mathbb{N}$, so we have strong polynomial tractability for each $\ell \in \mathbb{N}$.

However...

...the constant for strong polynomial tractability depends on ℓ.

Open question:
Can we obtain a similar result, where the implied constant is independent of ℓ ?

(Much) more generally:
Given two Korobov spaces with two different weight sequences: when does the CBC construction for one space also work (in the sense of error convergence *and* tractability) for the other?

Some results by J. Dick and T. Goda, but far from complete answers.

Thanks for your attention.