Dispersion of Sequences

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Dispersion and Discrepancy

Let $(x_n)_n \subseteq [0,1]^s$ be a sequence in the s-dimensional unit cube and $X_N = \{x_1, \ldots, x_N\}$.
$J = \{\prod_{i=1}^{d}(a_i, b_i) \subseteq [0,1]^s \colon 0 \leq a_i < b_i \leq 1\}$ the set of all axis-aligned boxes. For $B \in J$ we write $A(B; X_N)$ for the number of points of X_N contained in B and $\mathrm{vol}(B)$ the Lebesgue measure of this box, i.e. $\prod_{i=1}^{s}(b_i - a_i)$.

Dispersion and Discrepancy

Definition (Discrepancy)

The discrepancy of X_N is defined as

$$D(X_N) = \sup_{B \in J} \left| vol(B) - \frac{A(B; X_N)}{N} \right| .$$

Definition (Dispersion)

The dispersion of X_N is defined as

$$\mathrm{disp}(X_N) = \sup_{B \in J, A(B; X_N) = 0} \left| \mathrm{vol}(B) \right| .$$

Disclaimer

Not to be confused with another definition of dispersion of a sequence, which is in some way a special case of the previous definition.

Definition

With the metric $d_{\max}(y, z) = \max_{1 \leq i \leq s} |y_i - z_i|$ another 'dispersion' is given by

$$d(X_N) = \sup_{x \in [0,1]^s} \min_{1 \leq n \leq N} d(x, x_n).$$

Low-Discrepancy

There are some constructions known, e.g. Halton sequence, which achieve

$$D(X_N) \leq C_s \frac{(\log N)^s}{N}$$

with some constant C_s
This is conjectured to be optimal.

Question

What can be achieved for dispersion?

Expectation

If we do not work with sequences but a fixed number of points and we can freely distribute them every time, then we can achieve

$$D(X_N) \leq C'_s \frac{(\log N)^{s-1}}{N}$$

which can be achieved with the Hammersley set.
For the dispersion it was recently shown that in the same setting one can achieve (shown by B. Bukh and T.-W. Chao)

$$\mathrm{disp}(X_N) \leq 8000 \frac{s^2 \log s}{N}$$

For sequences one could expect a similar difference.

Dispersion Monotone Sequence

Corresponding to the question before, but this time we have the contraint that

$$\mathrm{disp}(X_{N+1}) < \mathrm{disp}(X_N),$$

i.e. the dispersion is supposed to decrease with every new point. *Alternatively.* Every new point has to be set into a currently biggest box.

Question

Is it necessary to have a worse dispersion in between to achieve optimal dispersion in the long run?

Additional Questions and Summary

Some additional questions:

Question

Are good discrepancy sequences also good dispersion sequences, resp. which of these are good? And vice versa.

Question

Which sequences can minimize the dispersion and the discrepancy simultaneously?

And our previous questions:

Question

Which upper bounds for dispersion can be achieved for sequences?

Question

Is it best to proceed in a 'greedy' behaviour when constructing sequences to minimize dispersion?