Intractability of the anchored L_p discrepancy

SFB Workshop

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For $n, d \in \mathbb{N}$, $x \in [0,1]^d, t_1, \ldots, t_n \in [0,1]^d$ and $a \in \mathbb{R}^n$ define the discrepancy function

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as well as

$$\operatorname{disc}_p(n, d) = \inf_{t_1, \ldots, t_n, a_1, \ldots, a_n} \operatorname{disc}_p(\{t_i\}, \{a_i\}).$$

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Consider

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The conjecture is that $n(\epsilon, d)$ is bounded below by an expression that grows exponentially in d, i.e. that $\mathrm{disc}_p(n, d)$ is intractable.

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The conjecture is that $n(\epsilon, d)$ is bounded below by an expression that grows exponentially in d, i.e. that $\mathrm{disc}_p(n, d)$ is intractable. A closely related question is that of the intractability of the integration problem in a certain function space.

Intractability of the anchored L_p discrepancy

For $p \in [1, \infty], \alpha \in [0, 1]$, consider the function space

$F_{1,p}^{\alpha} = \{f \colon [0,1] \to \mathbb{R} \mid f \text{ absolutely continuous}, \partial f \in L_p[0,1], f(\alpha) = 0\}$

with norm $\|f\|_{F_{1,p}^{\alpha}} = \|\partial f\|_{L_p}$.

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The d-fold tensor product $F^\alpha_{d,p} = \bigotimes_{i=1}^d F^\alpha_{1,p}$ is the Sobolev space of functions on $[0,1]^d$ with dominating mixed smoothness, $f(x) = 0$ if $x_i = \alpha$ for any i, and $\|f\|_{F^\alpha_{d,p}} = \|\partial^{(1,\dots,1)} f\|_{L_p}$.

Intractability of the anchored L_p discrepancy

Consider the integration problem in $F^{\alpha}_{d,p}$, i.e. appoximating the integration funtional

$$INT_d(f) = \int_{[0,1]^d} f(x)\,\mathrm{d}x$$

by an algorithm

$$Q_{n,d}(f) = \sum_{i=1}^{n} a_i f(t_i).$$

Intractability of the anchored L_p discrepancy

For the worst-case error we have

$$e_{F_{d,p}^{\alpha}}(Q_{n,d}) = \sup_{f \in F_{d,p}^{\alpha}, \|f\| \le 1} |\operatorname{INT}_d(f) - Q_{n,d}(f)| = \|\operatorname{INT}_d - Q_{n,d}\|_{(F_{d,p}^{\alpha})^*},$$

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It turns out, using Hlawska-Zaremba identity, that we have $(1/p + 1/q = 1)$

$$\operatorname{disc}_p(\{t_i\}, \{a_i\}) = \mathrm{e}_{F^{1}_{d,q}}(Q_{n,d}),$$

$$\operatorname{disc}_p(n,d) = \mathrm{e}_{F^{1}_{d,q}}(n,d).$$

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2. $p = 2, \alpha \in (0, 1)$: intractable, because
$$\mathrm{e}_{F_{d,p}^{\alpha}}(n, d) \geq (1 - n\beta^d)_+^{1/2} \mathrm{e}_{F_{d,p}^{\alpha}}(0, d), \qquad \text{for a } \beta \in [1/2, 1),$$

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   proved by decomposing $F^{\alpha}_{1,p} = F^{\alpha}_{(1),p} \oplus F^{\alpha}_{(2),p}$, and further $F^{\alpha}_{d,p}$ into 2^d subspaces of functions with disjoint support $F_{(b)}$ and

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This works for $p \neq 2$ analogously.

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   The proof relies on an orthogonal decomposition
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   The proof relies on an orthogonal decomposition
   $F^{\alpha}_{1,p} = G^{\alpha}_{(1),p} \oplus G^{\alpha}_{(2),p}$, s.t. one of the subspaces can be further decomposed as in 2.
   Difficult to generalize for $p \neq 2$. Alternative proof for $p = 2$ using Parseval's identity and a basis of Haar functions:

$$\mathrm{e}(Q_{n,d})^2 = \|\,\mathrm{INT}_d - Q_{n,d}\|^2 = \sum_{b \in \mathcal{H}^d} \frac{1}{\|b\|_2^2} \left|\langle \mathrm{INT}_d - Q_{n,d}, b\rangle\right|^2 .$$