Subsequences of Automatic Sequences

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Automatic Sequences

Example (Thue-Morse sequence)

$$n = 22 = (10110)_2, \qquad u_{22} = 1$$

$$\mathrm{u} = (u_n)_{n \geq 0} = 011010011001\ldots$$

$$u_n = s_2(n) \bmod 2.$$

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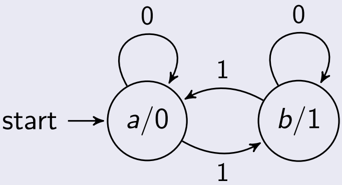

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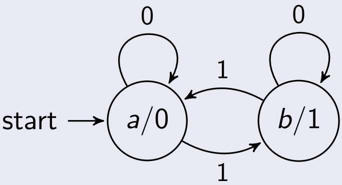

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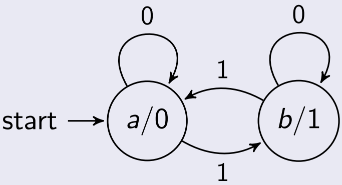

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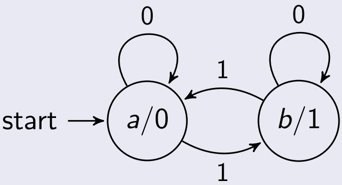

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Known Results

Theorem (Gelfond - 1967)

$(s_q(an+b))_{n\geq 0}$ is uniformly distributed modulo m (under natural conditions for q, m).

Theorem (Drmota, Mauduit, Rivat - 2019)

$(s_2(n^2))_{n\geq 0}$ is normal modulo 2.

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A notorious open problem

Show that $(s_2(n^3))_{n\geq 0}$ is uniformly distributed modulo 2.

A hopefully simpler problem

Show that $(s_2(n^3+2m^3))_{n,m\geq 0}$ is uniformly distributed modulo 2.

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- Rewriting the statement using exponential sums.
- Using "Cutting techniques" to reduce the number of relevant digits.
- Estimates for

$$\left|\sum_{n\leq 2^\nu} \mathrm{e}\left(\frac{1}{2}s_2(n) + n\alpha\right)\right| = 2^\nu \prod_{\lambda=0}^{\nu-1} \left|\sin(2^\lambda \pi\alpha)\right|. \tag{1}$$

New Approach?

Combination of the circle Method (e.g. Waring's problem) and estimates for (1).

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