Sudler Product and Unbounded Continued Fraction Coefficients

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These kind of products have been studied in a variety of different fields/contexts:

- Partition theory
- Padé approximation
- Continued fractions
- Uniform distribution
- Discrepancy
- Mathematical physics

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For example:

- What can we say about $\lim_{n\to\infty} P_{q_n(\alpha)}(\alpha)$?
- What do we know about $\liminf_{N\to\infty} P_N(\alpha)$?
  (This was asked by Erdős and Szekeres 60 years ago.)

Basis for these results

Theorem (Mestel, Verschueren, 2016)

Let $\varphi = (\sqrt{5} - 1)/2$ and let $(F_n)_{n\in\mathbb{N}}$=(1,1,2,3,5,...) be the Fibonacci sequence. Then

$$\lim_{n\to\infty} P_{F_n}(\varphi) = \prod_{r=1}^{F_n} |2\sin(\pi r\varphi)| = c > 0.$$

Generalised version I (Grepstad, N.)

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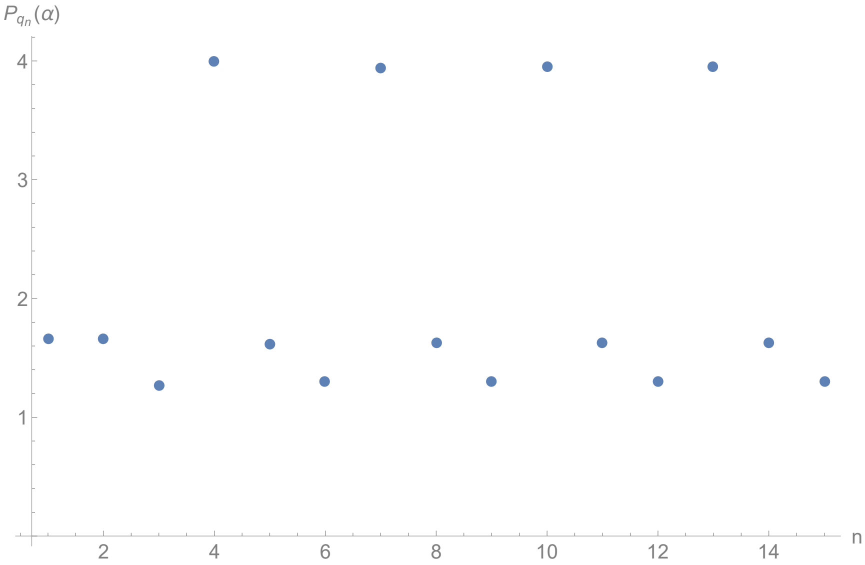
$\rightarrow$ This means $P_{q_n(\alpha)}(\alpha)$ has ℓ limit points.

$\alpha = [0; \overline{1, 2, 4}]$



Generalised version II (work in progress)

Consider:

- $\alpha \in (0,1)$ with **bounded c.f.c.**
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If one "adapts" $(n_i)_{i\in\mathbb{N}}$ to the structure of the c.f.e. of α then

$$\lim_{i\to\infty} P_{q_{n_i}}(\alpha) = \lim_{i\to\infty} \prod_{r=1}^{q_{n_i}(\alpha)} |2\sin(\pi r\alpha)| = c > 0.$$

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Consider: $\mathrm{e} = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1 \ldots]$

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We know: $\lim_{i\to\infty} P_{q_{3i+2}}(\mathrm{e}) = 0 \quad (a_{3i+2} < a_{3(i+1)+2})$

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But:

$$\lim_{i\to\infty} P_{q_{3i}}(\mathrm{e}) =? \quad \text{and} \quad \lim_{i\to\infty} P_{q_{3i+1}}(\mathrm{e}) =?$$

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Conjecture:

$$\lim_{i\to\infty} P_{q_{3i}}(\mathrm{e}) = \infty \text{ and } \lim_{i\to\infty} P_{q_{3i+1}}(\mathrm{e}) = 0$$

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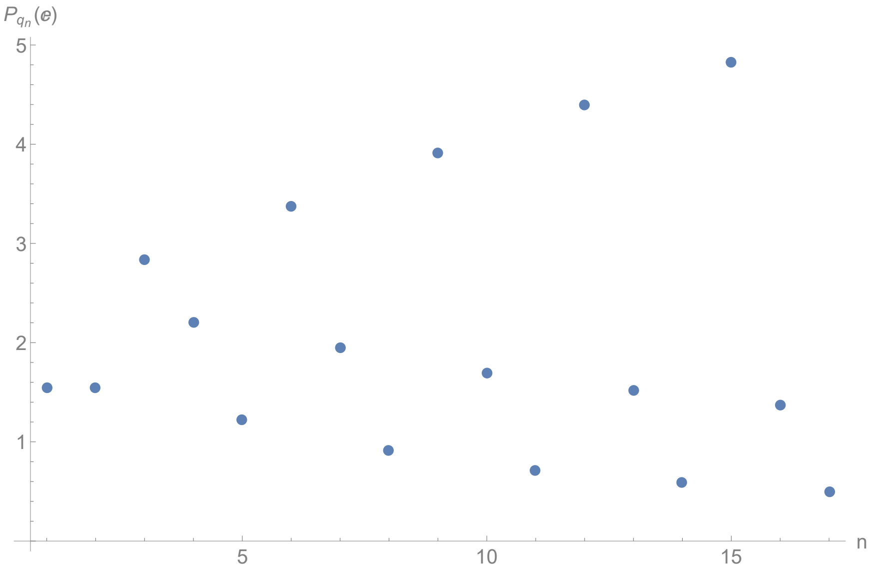

Thank you for your attention!