Dispersion of Minkowski Lattices

Jaspar Wiart
Joint work with Thomas Lachmann

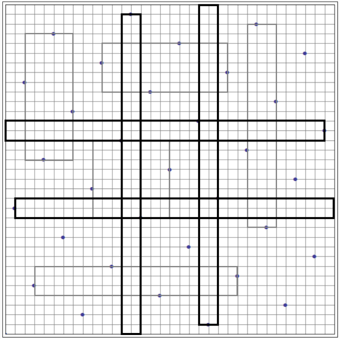
Johannes Kepler University Linz

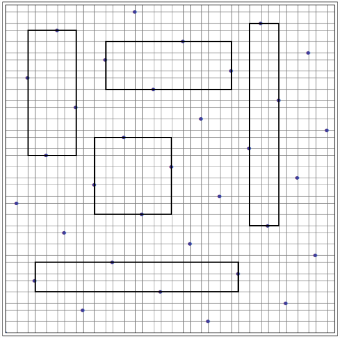
Dispersion

Definition

Let $\mathcal{B} = \{\prod_{j=1}^{d}(a_i, b_i) : 0 \leq a_j < b_j \leq 1\}$ and let $P \subset [0,1]^d$ be a point set. The dispersion of P is

$$\mathrm{disp}(P) = \sup\{\mathrm{Vol}(B) : B \in \mathcal{B},\ B \cap P = \emptyset\}.$$





Dispersion

- $\mathrm{disp}(n, d) = \inf\{\mathrm{disp}(P) : P \subset [0,1]^d,\ |P| = n\}$
- $\Omega(d/n) \leq \mathrm{disp}(n, d) \leq O(d^2 \log(d)/n)$
- Goal of the project is to improve the upper bound.
- We are confident that $\lim_{n\to\infty} n \cdot \mathrm{disp}(n, 2) = 1.89443$.
- Thomas observed that this is achieved by the lattice Λ_2 with generating matrix

$$\begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

- Known as the Minkowski lattice of the algebraic number field $\mathbb{Q}[\sqrt{5}]$.

Algebraic number fields

Definition

An (algebraic) number field F is a finite degree field extension of $\mathbb{Q}$, (i.e. finite dimension $[F : \mathbb{Q}] = d$ as V.S. over $\mathbb{Q}$).

Definition

The algebraic integers in F form a ring, denoted $\mathcal{O}_F$, called the ring of integers of F.

Example: Quadratic fields

The ring of integers of $\mathbb{Q}[\sqrt{a}]$ is $\mathbb{Z}[\frac{1+\sqrt{a}}{2}]$ when $a \equiv 1 \mod 4$ and $\mathbb{Z}[\sqrt{a}]$ when $a \equiv 2, 3 \mod 4$.

Example: Cyclotomic fields

The ring of integers of the $\mathbb{Q}[\zeta_n]$ is $\mathbb{Z}[\zeta_n]$.

The Minkowski lattice of a number field

Definition

Let $F \subset \mathbb{R}$ be a number field of degree $[F : \mathbb{Q}] = d$. Let $\sigma_1, \ldots, \sigma_d$ be the d embeddings of F into $\mathbb{R}$. The Minkowski lattice of F is $\Lambda_F = \{(\sigma_1(\alpha), \sigma_2(\alpha), \ldots, \sigma_d(\alpha)) \in \mathbb{R}^d : \alpha \in \mathcal{O}_F\}$.

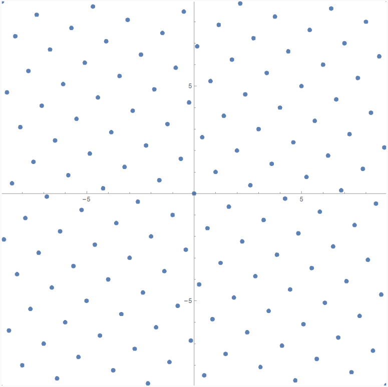
- If $\{\beta_1, \beta_2, \ldots, \beta_d\}$ is a basis for $\mathcal{O}_F$ over $\mathbb{Z}$, then a generating matrix for Λ_F is

$$\begin{pmatrix} \sigma_1(\beta_1) & \sigma_1(\beta_2) & \cdots & \sigma_1(\beta_d) \\ \sigma_2(\beta_1) & \sigma_2(\beta_2) & \cdots & \sigma_2(\beta_d) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_d(\beta_1) & \sigma_d(\beta_2) & \cdots & \sigma_d(\beta_d) \end{pmatrix}.$$

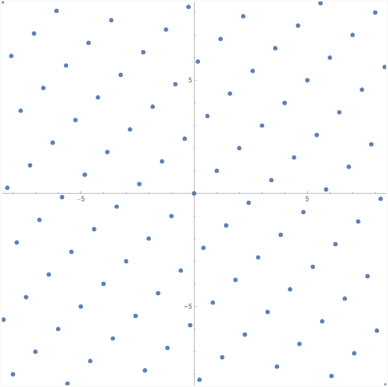
- We always have $\mathrm{disp}(\Lambda_F) < \infty$ and $\mathrm{disp}(n, d) \leq \frac{\mathrm{disp}(\Lambda_F)}{n \det(\Lambda_F)}$.

Examples

$\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$



$\mathbb{Z}[\sqrt{2}]$



The field $\mathbb{Q}[2\cos(\frac{\pi}{2d+1})]$

- Let $2d+1$ be prime. The Maximal real subfield of the cyclotomic field $\mathbb{Q}[\zeta_{2d+1}]$ is $\mathbb{Q}[2\cos(\frac{\pi}{2d+1})]$.
- Denote its Minkowski lattice by Λ_d.
- The generating matrix can be made to be anti-circulant.
- Example: Generating matrices for Λ_2 and Λ_3 are

$$\begin{pmatrix} 2\cos(\frac{\pi}{5}) & 2\cos(\frac{3\pi}{5}) \\ 2\cos(\frac{3\pi}{5}) & 2\cos(\frac{\pi}{5}) \end{pmatrix}, \begin{pmatrix} 2\cos(\frac{\pi}{7}) & 2\cos(3\frac{\pi}{7}) & 2\cos(5\frac{\pi}{7}) \\ 2\cos(3\frac{\pi}{7}) & 2\cos(5\frac{\pi}{7}) & 2\cos(\frac{\pi}{7}) \\ 2\cos(5\frac{\pi}{7}) & 2\cos(\frac{\pi}{7}) & 2\cos(3\frac{\pi}{7}) \end{pmatrix}$$

- If B is a maximal empty box amidst Λ_2, then $\mathrm{Vol}(B)/\det(\Lambda_2)$ is 1.89443.
- If B is a maximal empty box amidst Λ_3, then $\mathrm{Vol}(B)/\det(\Lambda_3)$ is either 2.74224, 2.92038, or 3.38423, so $\mathrm{disp}(n,3) \leq \frac{3.38423}{n}$.

Equivalent Boxes

Theorem

Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be one of the following transformations:

- $\mathbf{x} \mapsto \mathbf{x} + \alpha$, where $\alpha \in \Lambda_d$,
- $\mathbf{x} \mapsto (\sigma_1(u)x_1, \sigma_2(u)x_2, \ldots, \sigma_d(u)x_d)$, where $u \in \mathcal{O}_d$ is a unit,
- a cyclic shift of the coordinates of $\mathbb{R}^d$.

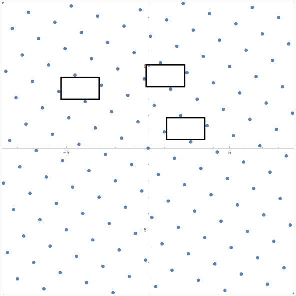
Then $T : \Lambda_d \to \Lambda_d$ and for an axis-parallel B,

- $T(B)$ is an axis-parallel box,
- $\mathrm{Vol}(B) = \mathrm{Vol}(T(B))$, and
- $|B \cap \Lambda_d| = |T(B) \cap \Lambda_d|$.

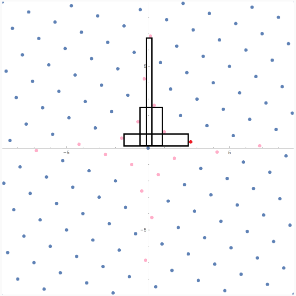
Finitely many possible volumes for maximal empty boxes!

Example: Equivalent boxes

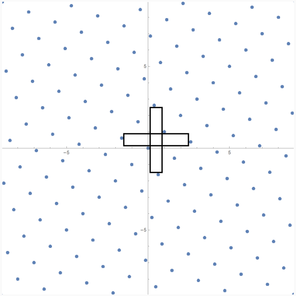
Translations



Multiplication by units



Cyclic shift



The End

We would welcome more collaborators!