What is the largest hole in a random point set?

Mathias Sonnleitner
JKU Linz

2021-01-22

JOHANNES KEPLER UNIVERSITÄT LINZ

Der Wissenschaftsfonds.

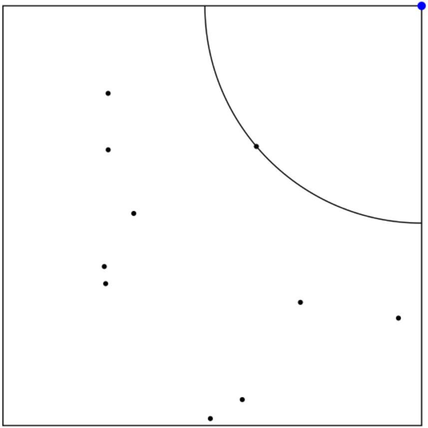
UNI GRAZ

Holes in random point sets

Let $D \subset \mathbb{R}^d$ bounded Lipschitz and $P_n = \{X_1, \dots, X_n\}$ i.i.d. uniform random points on D.

A hole is a ball $B_r(x) = \{y \in D : \|x - y\| < r\}$ with $x \in D$ and $B_r(x) \cap P_n = \emptyset$.

How big is the largest hole in P_n?



Holes in random point sets

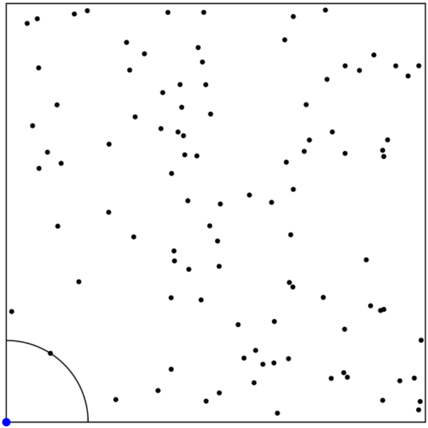
Let $D \subset \mathbb{R}^d$ bounded Lipschitz and $P_n = \{X_1, \ldots, X_n\}$ i.i.d. uniform random points on D.

A hole is a ball

$B_r(x) = \{y \in D : \|x - y\| < r\}$

with $x \in D$ and $B_r(x) \cap P_n = \emptyset$.

How big is the largest hole in P_n?

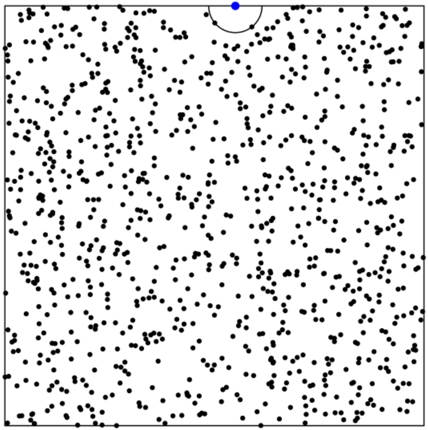


Holes in random point sets

Let $D \subset \mathbb{R}^d$ bounded Lipschitz and $P_n = \{X_1, \ldots, X_n\}$ i.i.d. uniform random points on D.

A hole is a ball $B_r(x) = \{y \in D : \|x - y\| < r\}$ with $x \in D$ and $B_r(x) \cap P_n = \emptyset$.

How big is the largest hole in P_n?

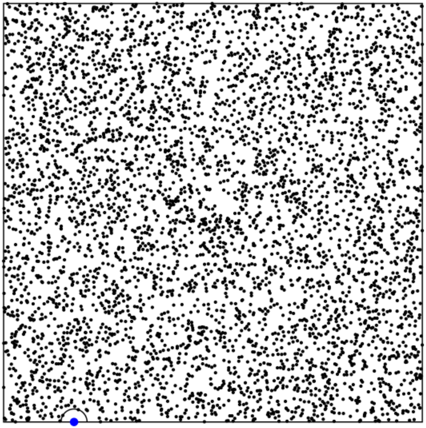


Holes in random point sets

Let $D \subset \mathbb{R}^d$ bounded Lipschitz and $P_n = \{X_1, \dots, X_n\}$ i.i.d. uniform random points on D.

A hole is a ball $B_r(x) = \{y \in D : \|x - y\| < r\}$ with $x \in D$ and $B_r(x) \cap P_n = \emptyset$.

How big is the largest hole in P_n?



More precisely,...

What is the radius of the largest hole, $r_n := \sup_{x \in D} \min_{1 \le i \le n} \|x - X_i\|$?

Motivation: Quality of (random) points for num. approximation

Theorem (NWW '04, NT '06, KS '20)

Let $1 \le p \le q \le \infty$, $s > d/p$ and D be a bounded Lipschitz domain. Then there exist constants $c_1, c_2 > 0$ such that for any n and $P = \{x_1, \ldots, x_n\} \subset D$:

$$c_1 r_n^\alpha \le \inf_{\varphi} \sup_{\|f\|_{W_p^s(D)} \le 1} \|f - \varphi\big(f(x_1), \ldots, f(x_n)\big)\|_{L_q(D)} \le c_2 r_n^\alpha$$

with $\alpha = s - d(1/p - 1/q)$ and $\varphi : \mathbb{R}^n \to L_q(D)$ arbitrary.

Known results

Recall that $r_n = \sup_{x \in D} \min_{1 \le i \le n} \|x - X_i\|$ with $X_1, X_2, \ldots$ i.i.d.

Reznikov and Saff '16 gave bounds

$$0 < c_1(D) \le \mathbb{E}[r_n] \left(\frac{n}{\log n}\right)^{1/d} \le c_2(D) < \infty \text{ for all } n$$

and for nice D (e.g. smooth boundary) they showed

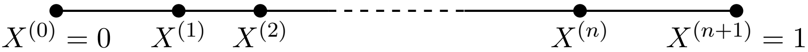
$$\lim_{n\to\infty} \mathbb{E}[r_n] \left(\frac{n}{\log n}\right)^{1/d} = \left(\frac{2(d-1)}{d} \frac{\mathrm{vol}(D)}{\mathrm{vol}(B_1(0))}\right)^{1/d}. \qquad (1)$$

With a different constant eq. (1) also holds for $[0,1]^d$ and 3-dim. polytopes.

Known results: The one-dimensional case

Let $P_n = \{X_1, X_2, \dots, X_n\}$ i.i.d. uniform random points on $[0, 1]$.

$$r_n = \max\left\{ \max_{i=2,\dots,n} \frac{X^{(i)} - X^{(i-1)}}{2}, X^{(1)}, 1 - X^{(n)} \right\}$$



From Reznikov and Saff: $\lim_{n\to\infty} \frac{n}{\log n} \mathbb{E}[r_n] = \frac{1}{2}$.

From Lévy (1939): for all $x \in \mathbb{R}$

$$\mathbb{P}[2(n-1)r_n - \log(n-1) < x] \to \exp(-e^{-x}) \text{ as } n \to \infty.$$

Open Problems

Let D be a bounded Lipschitz domain.

Problem 1: Show $\lim_{n\to\infty} \mathbb{E}[r_n] \left(\frac{n}{\log n}\right)^{1/d} = c(D)$.

Problem 2: Are there sequences $(a_n), (b_n)$ s.t. for all $x \in \mathbb{R}$

$$\mathbb{P}\left[\frac{r_n - b_n}{a_n} < x\right] \to \exp(-e^{-x}) \text{ as } n \to \infty?$$

Problem 3: Show a large deviations principle (LDP), that is find (s_n), $s_n \to \infty$ and $\mathbb{I} : \mathbb{R} \to [0, \infty]$ s.t. for all $x \in \mathbb{R}$

$$\mathbb{P}\left[r_n \left(\frac{n}{\log n}\right)^{1/d} > x\right] \approx e^{-s_n \mathbb{I}(x)}.$$