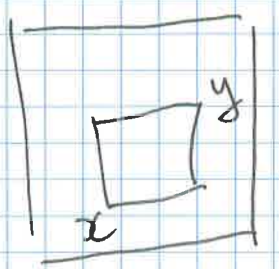
The largest empty box

d

The Problem

$P_{(n)} \subseteq [0,1]^d \qquad \# P_n = n$



$x \le y$

$\mathcal{B} = \{ [x,y) : x \le y \}$

$\mathrm{disp}(P_n) = \sup\{ |B| : B \cap P_n = \emptyset \}$ Dispersion

$\mathrm{disp}(n,d) = \inf\{ \mathrm{disp}(P_n) : \# P_n = n \}$ Rote/Tichy 1996

$n(\varepsilon,d) = \min\{ n : \mathrm{disp}(P_n) \le \varepsilon \} \qquad \varepsilon > 0$

$\boxed{Q}$ Study $\mathrm{disp}(n,d)$, $n(\varepsilon,d)$ $\quad d \to \infty$

- MER
- Approximation of high-dim rank 1 tensors

2014 Bachmayr

2015 Novak, Rudolf

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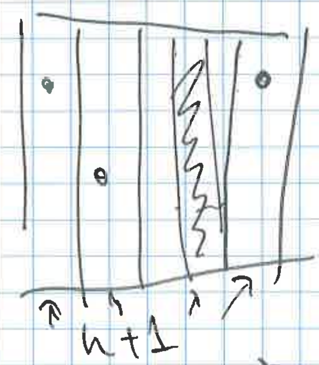
Plan

1. nontrivial lower bound (Aistleitner, Rudolf)
2. Gerhard Larcher: upper bound
3. periodic disp. (Mario Ullrich)

① $\operatorname{disp}(n,d) \geq \frac{1}{n+1}$ pigeon hole principle



$\frac{1}{n+1}$ $\qquad n(\varepsilon,d) \geq \frac{1}{\varepsilon} - 1$

- $\operatorname{disp}(n,d) \geq \frac{5}{4(n+5)}$ Dumitrescu/Jiang 2013.

$$n(\varepsilon,d) \geq \frac{5}{4\varepsilon} - 5$$

$\operatorname{disp}(4,d) \geq \frac{1}{4}$ + pigeon hole. $d \geq 2$

THM: $\operatorname{disp}(n,d) \geq \frac{\log_2 d}{4(n+\log_2 d)}$ $\qquad d \geq 32$

$$n(\varepsilon,d) \geq \frac{\log_2 d}{4\varepsilon} - \log_2 d$$

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Proof: 1. Ingredient Pigeon Hole:

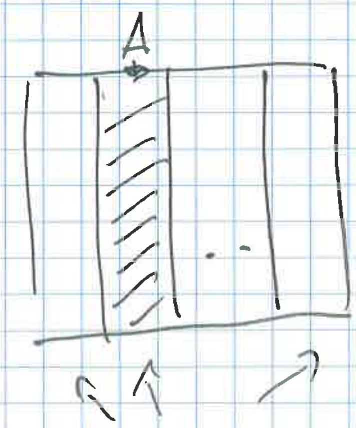
$$\mathrm{disp}(n,d) \geq \frac{l+1}{n+l+1} \boxed{|\mathrm{disp}(l,d)|} \qquad (*)$$

$$\mathrm{disp}(A,n,d) = \inf_{P_n \subseteq A} \sup_{B \subseteq A \setminus P_n} |B|$$

$A \subseteq [0,1]^d$ rect. box

$$\mathrm{disp}(A,l,d) \geq |A| \, \mathrm{disp}(l,d)$$

Pigeon hole: $\boxed{n = (l+1)k + r}$ $r \in \{0,1,..,k-1\}$



$k+1$ boxes

One of these boxes (A) contains $\leq l$ points

$$\mathrm{disp}(n,d) \geq \inf_{|A| = \frac{1}{k+1}} \mathrm{disp}(A,l,d) \geq \frac{1}{k+1} \mathrm{disp}(l,d)$$

$$k+1 = \frac{n-r}{l+1} + 1 \leq \frac{n+l+1}{l+1}.$$

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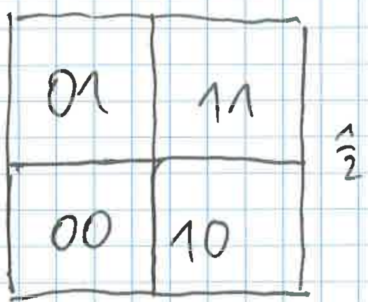
Ingredient 2: nontrivial bound for (not too) small ℓ – structure in the coords.

(**) $\operatorname{disp}(\ell, d) \geq \frac{1}{4}$ $\qquad 2^\ell - 1 \leq d$

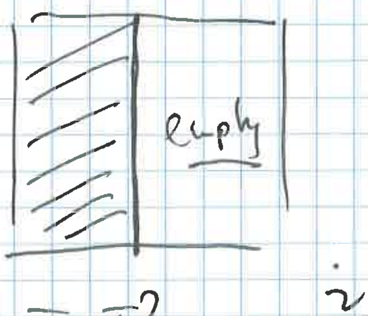
$P_\ell = \{t_1, \dots, t_\ell\} \subseteq [0,1]^d$

$t_m \longmapsto J_m \in \{0,1\}^d$



Case 1: $J_m(i) = \overset{1}{0}$ for some i
and all $m = \{1, \dots, \ell\}$

$\operatorname{disp} \geq \frac{1}{2}$



Case 2 $(J_1(i), \dots, J_m(i)) \in \{0,1\}^\ell \setminus \{\bar{0}, \bar{1}\}$

how many holes? $2^\ell - 2$

how many pigeons? $d \geq 2^\ell - 1$ $\qquad i = 1, \dots, d$

$\exists\ i \neq j \qquad J_m(i) = J_m(j) \qquad m = 1, \dots, \ell$

$\operatorname{disp} \geq \frac{1}{4}$

$$\mathrm{disp}(n,d) \overset{(*)}{\geq} \frac{\ell+1}{n+\ell+1}\,\mathrm{disp}(\ell,d)$$

$$\ell = \lfloor \log_2 d \rfloor \qquad \overset{(**)}{\geq} \frac{1}{4}\,\frac{\ell+1}{n+\ell+1}$$

$$= \frac{1}{4}\,\frac{\lfloor \log_2 d\rfloor + 1}{n+\lfloor \log_2 d\rfloor + 1}$$

$$\geq \frac{1}{4}\,\frac{\log_2 d}{n+\log_2 d} \qquad \frac{d}{n}$$

② Upper bound (Gerhard Larcher)

$$\mathrm{disp}(n,d) \leq \frac{2}{n}\,2^{cd}$$

$$\frac{1}{n}\left(2^d \prod_{i=1}^{d-1} p_i\right)$$

Rote, Tichy '96

(t,m,d) - digital nets.

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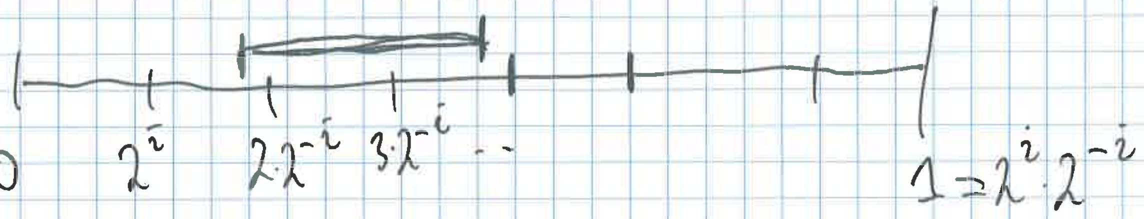
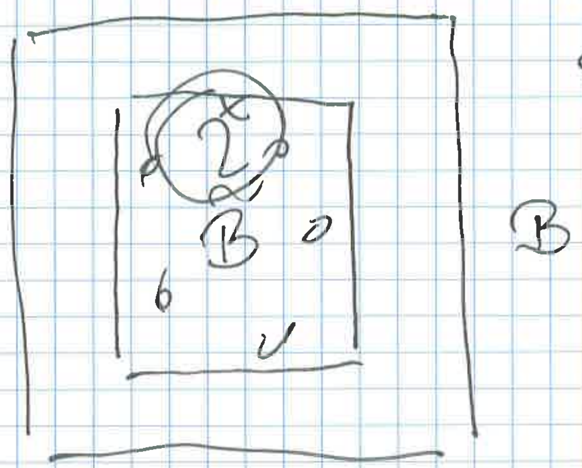
$b = 2$, $t \leq cd$ (Niederreiter-Xing '96)

P_{2^m}: any elementary box of volume 2^{t-m} contains exactly 2^t points.





$\underline{d=1}$ $|\tilde{B}| \geq \frac{1}{2^d} |B|$

$\| $

2^{t-m}

$$|B| \geq 2^{t-m} \, 2^d \Rightarrow |\tilde{B}| \geq 2^{t-m}$$

$$\mathrm{disp}(P_{2^m}) \leq 2^{t-m+d} = \frac{1}{2^m} 2^{t+d}$$

$$\leq \frac{1}{2^m} 2^{(1+c)d}$$

$$= \frac{1}{2^m} 2^{6d}$$

$2^m \leq n < 2^{m+1}$

$$\mathrm{disp}(P_n) \leq \mathrm{disp}(P_{2^m}) \leq \frac{2}{n} 2^{6d}$$

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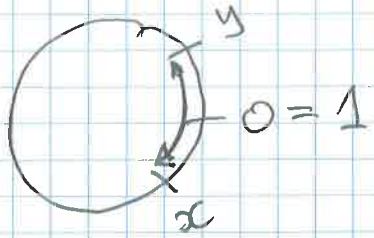
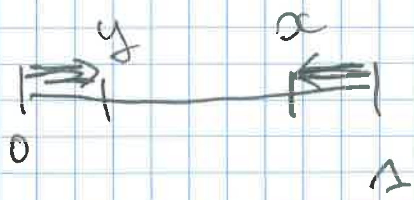
2a Periodic Dispersion

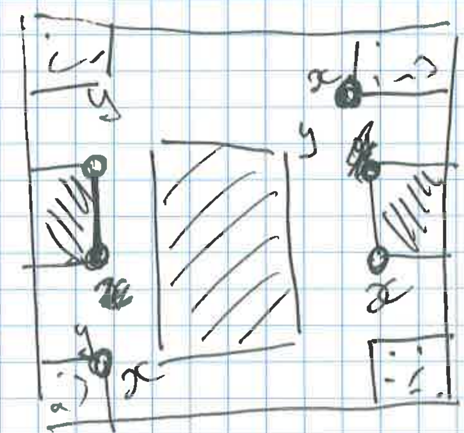
$[0,1] = \mathbb{T}$ torus





tensor product.

$$\mathcal{B}^{\mathbb{T}} = \{[x,y) : x,y \in \mathbb{T}^d\}$$



$$\mathrm{disp}^{\mathbb{T}}(P_n) = \sup_{P_n \cap B \neq \emptyset} \{|B| : B \in \mathcal{B}^{\mathbb{T}}\}$$

$\mathrm{disp}^{\mathbb{T}}(n,d)$, $n^{\mathbb{T}}(\varepsilon,d)$

Upper bound holds: $\leq \frac{2}{n} 2^{6d}$.

$$\mathrm{disp}(n,d) \leq \mathrm{disp}^{\mathbb{T}}(n,d)$$

$$\mathrm{disp}^{\mathbb{T}}(n,d) \geq \min\{1, \tfrac{d}{n}\}$$

$$n^{\mathbb{T}}(\varepsilon,d) \geq d/\varepsilon \qquad \varepsilon < 1.$$

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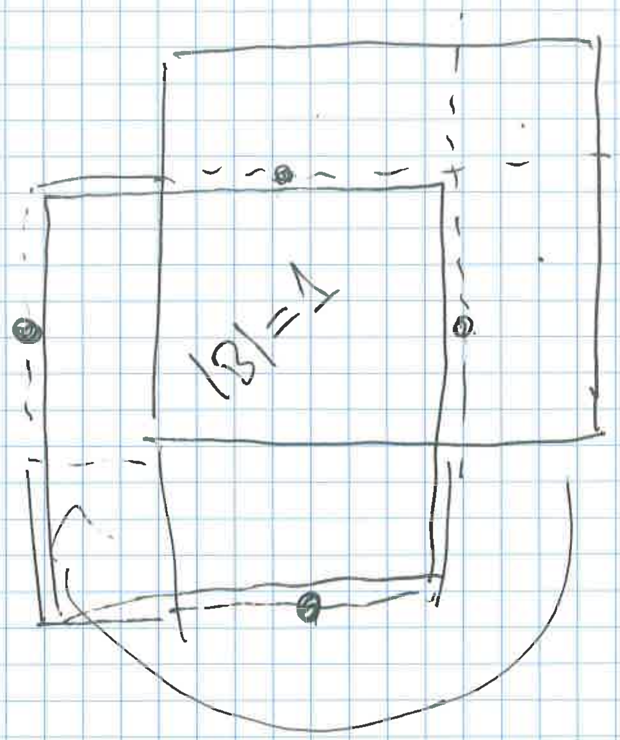
Argument 1) ~~d ≤ n~~ $n \leq d$, $n = d$

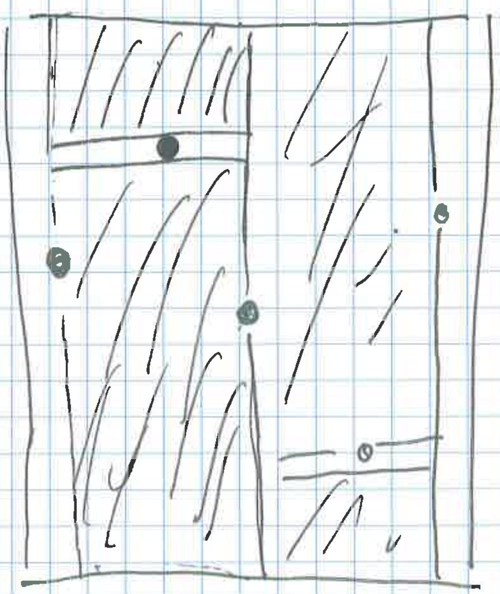


$d = 2 = n$

$|B| = 1$

disp $= 1$ if $n \leq d$.

$n > d$



$t_1 \quad \dots \quad t_n$

$t_1(1) < t_2(1) < \dots \; t_n(1)$

$2 = \sum_1^n$ empty boxes

One empty box $\geq \frac{2^{-d}}{n}$

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2 Summary:

$$\frac{\log_2 d}{4(n+\log_2 d)} \le \mathrm{disp}(n,d) \le \frac{2}{n} 2^{6d}$$

$$\min\{1, \frac{d}{n}\} \le \mathrm{disp}^{\pi}(n,d) \le \text{ — '' — }$$

$$\frac{\log_2 d}{4\varepsilon} - \log_2 d \le n(\varepsilon,d) \le \frac{1}{\varepsilon} 2^{6d+1}$$

$$\frac{d}{\varepsilon} \le n^{\pi}(\varepsilon,d) \le \text{ — '' — } \qquad \varepsilon < 1$$

Compare this to the situation for L_∞-discrepancy.

$$\mathrm{disc}^*(n,d) = \inf_{\# P_n = n} \|\Delta_{P_n}\|_\infty$$

$$n^*(\varepsilon,d)$$

$$c_0 \min\{1, \frac{d}{n}\} \le \mathrm{disc}^*_\infty(n,d) \le 10 \sqrt{\frac{d}{n}}$$

$$c_1 \frac{d}{\varepsilon} \le n^*_\infty(\varepsilon,d) \le 100 \frac{d}{\varepsilon^2} \qquad \varepsilon \le \varepsilon_0$$

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