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Trackability of high dim. integration

$D_d = [0,1]^d \subseteq \mathbb{R}^d$ $\qquad d \to \infty$

$I_d(f) = \int_{D_d} f(x)\, dx$

F_d class of integrable fcts. $f: D_d \to \mathbb{R}$

Algorithms $N(f) = (f(x_1), \dots, f(x_n))$

Information map

$A_n(f) = \varphi(f(x_1), \dots, f(x_n))$

$\varphi: \mathbb{R}^n \to \mathbb{R}$ arbitrary.

error $e(A_n) = \sup_{f \in F_d} |I_d(f) - A_n(f)|$

$e(n, F_d) = \inf_{A_n} e(A_n)$ optimal error.

$n(\varepsilon, F_d) = \min\{n : \exists A_n \text{ with } e(A_n) \le \varepsilon\}$

information complexity

= number of sample points needed to achieve error ε.

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Ex: $F_d^k = \{ f : [0,1]^d \to \mathbb{R} \mid \|D^\alpha f\|_\infty \le 1$ for all $|\alpha| \le k \}$

$$|\alpha| = \alpha_1 + \dots + \alpha_d$$

Classical Result (Bakhvalov)

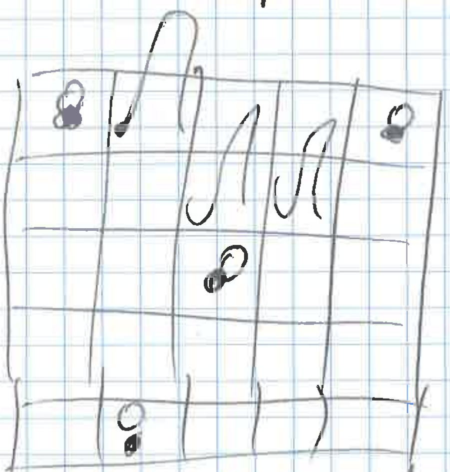
Fix $d \in \mathbb{N}$, there exist $a_d, b_d > 0$

$$a_d n^{-k/d} \le e(n, F_d^k) \le b_d n^{-k/d}$$

upper bound: provide algorithm using fct. values on a grid with coordinates $\frac{1}{2m}, \frac{3}{2m}, \dots, \frac{2m-1}{2m}$.

m^d grid points

lower bound: bump function technique



fooling function

PROBLEM: Behavior in d is hidden in constants.

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Capture complexity for large d

Tractability

- strong polynomial tractability $\quad n(\varepsilon, F_d) \le C\varepsilon^{-\alpha}$ for some $\alpha, C > 0$
- polynomial tract. $\quad n(\varepsilon, F_d) \le C\varepsilon^{-\alpha} d^{\beta}$
- Curse of dimension $\quad n(\varepsilon, F_d) \ge c(1+\gamma)^d$

$c, \gamma > 0$

for some $\varepsilon > 0$.

need exponentially many points.

Ex Nemirovski, Shapiro '71

$$F_d = \{ f : [0,1]^d \to \mathbb{R} \mid |f(x) - f(y)| \le \|x-y\|_\infty \}$$

$$\varepsilon = e(n, F_d) = \frac{d}{2d+2} n^{-1/d}$$

$$n = \left(\frac{2d+2}{d} \varepsilon \right)^{-d} \approx 2^d$$

$$\varepsilon = \frac{1}{4} \qquad \text{Curse}$$

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Ex star discrepancy

$$\varepsilon = \mathrm{disc}^*(n,d) \le 10 \sqrt{\frac{d}{n}}$$

$$n \le 100 \cdot \frac{d}{\varepsilon^2} = 100\, \varepsilon^{-2} \cdot d^{1}$$

polynomially tractable.

Examples of lower bounds for some F_d

- monotone functions
- convex functions
- smooth functions

some geometry involved.

(1) Monotone Functions H, N, W 2012

$$F_d^{\mathrm{mon}} = \{ f : [0,1]^d \to [0,1] \mid f \text{ is monotone increasing} \}$$

$$f(x) \le f(y) \Longleftarrow x \le y \text{ coordinatewise.}$$

initial error $n = 0$

$$e(0, F_d) = \inf_c \sup_{F_d} \left| \int f - c \right|$$

$$c = \frac{1}{2} \qquad e(0, F_d) = \frac{1}{2}$$

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well behaved, because the initial error is constant.

<u>THM</u>: $0<\varepsilon<1/2$. Then

$$n(\varepsilon, F_d^{mon}) \geq 2^d(1-2\varepsilon) \text{ for } d \in \mathbb{N}$$

<u>Curse</u>

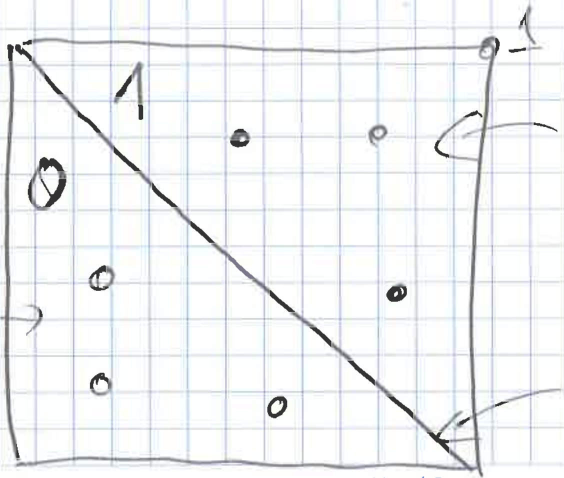
$$\iff 2\varepsilon \geq 1-n\cdot 2^{-d}$$

<u>Pf</u>: A_n fixed algorithm using fct. values at $x_1, \dots, x_n$.

<u>need</u> "fooling functions" $f^+, f^- \in F_d^{mon}$

with $N(f^+) = N(f^-)$

but $I_d(f^+) - I_d(f^-) \geq 1 - n\,2^{-d}$



k sample points

$\boxed{k+l=n}$

$H = \{z \in \mathbb{R}^d : \sum_1^d z_i = \frac{d}{2}\}$

l sample points

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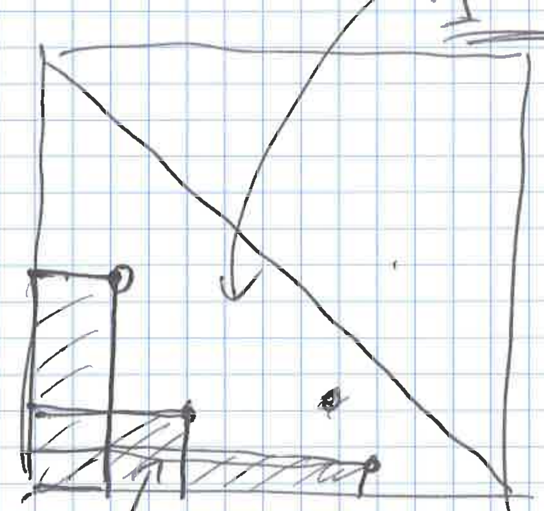
f^+ largest monotone fct.
with $f^+(x_i) = 1$ for x_i in
the upper half, $f^+(x_i)=0$ in lower half.

1 here

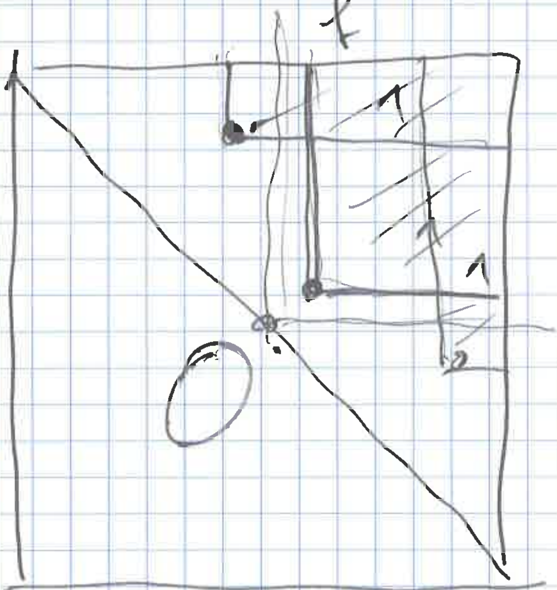


$f^+ = 0$

$$\int f^+ \geq 1 - \ell 2^{-d}$$

$$\int f^+ - \int f^- \geq 1 - \underbrace{(k+\ell)}_{n} 2^{-d}$$

f^-



$$\int f^- \leq k \cdot 2^{-d}$$

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2 Convex Functions

$$F_d^{con} = \{ f : [0,1]^d \to [0,1] \mid f \text{ convex} \}$$

initial error $1/2$

Thm: $0 < \varepsilon < \varepsilon_0$. Then ~~there~~

$$n(\varepsilon, F_d^{con}) \geq \frac{1}{d+1} \left(\frac{11}{10}\right)^d \left(1 - \frac{\varepsilon}{\varepsilon_0}\right)$$

Curse!

General Idea here

Fooling fct. $f^- = 0$

$$f^+(x_1) = \ldots = f^+(x_n) = 0$$

and then f^+ as large as possible.

$$\Longrightarrow \quad 2\varepsilon \geq \int f^+$$

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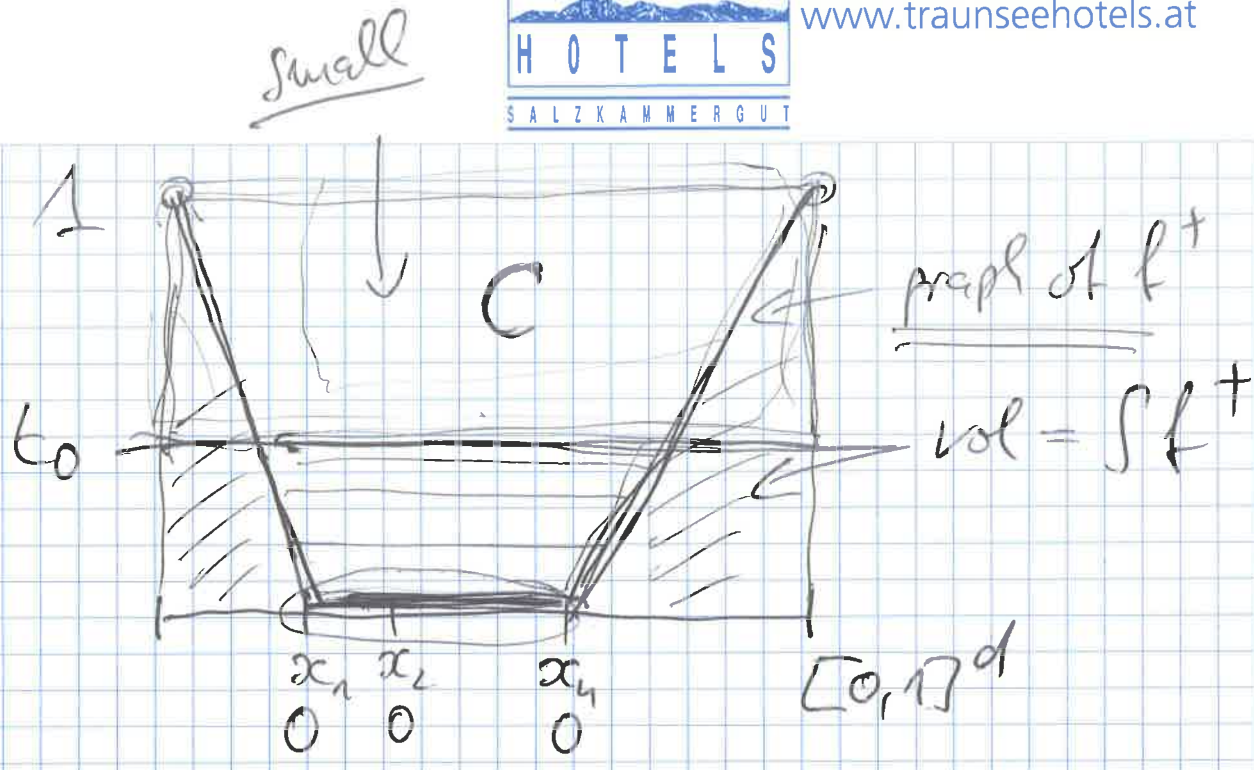

$$K = \operatorname{conv}\{x_1, \dots, x_n\}$$

$$C = \operatorname{conv}\{K \times \{0\} \cup [0,1]^d \times \{1\}\}$$

$$\int f^+ = 1 - \operatorname{vol} C$$

$$\operatorname{vol} C \leq (1 - t_0) + (d+1)\, n \cdot t_0 \left(\frac{10}{11}\right)^d$$

$$\varepsilon_0 = \frac{t_0}{2} \implies \text{THM.}$$

$$\boxed{\operatorname{vol}(K) \leq (d+1)\, n \cdot \left(\frac{6}{7}\right)^d}$$

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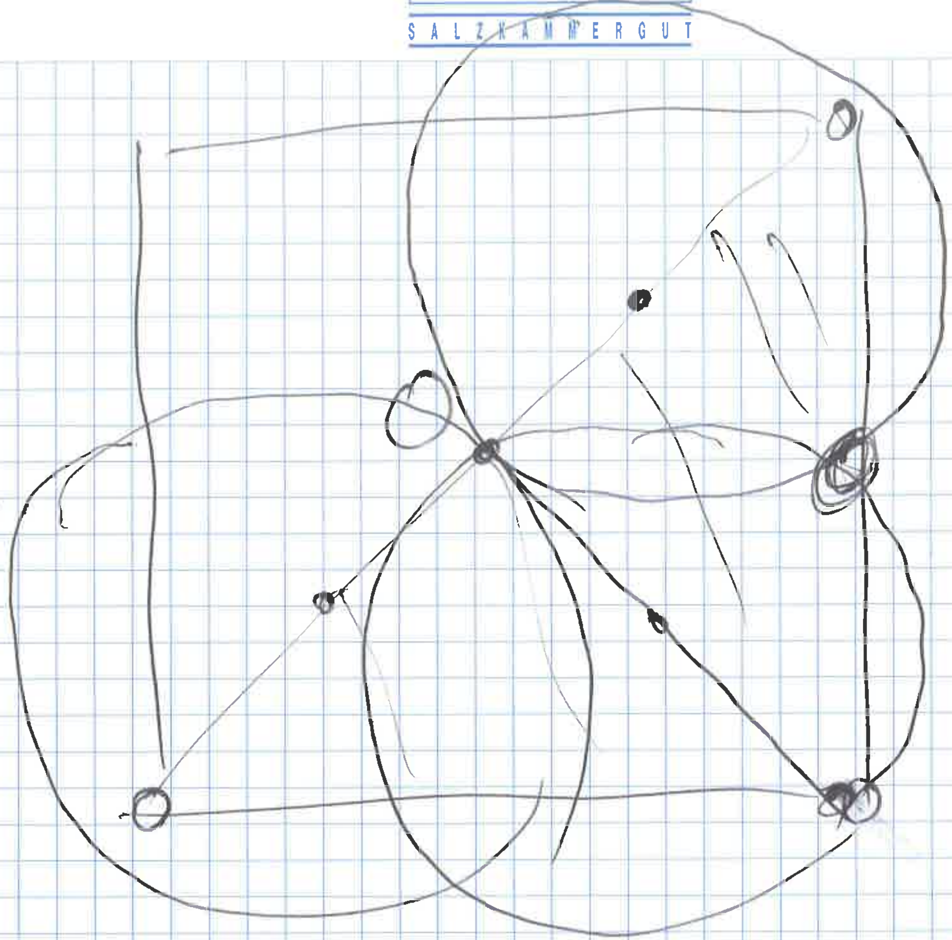
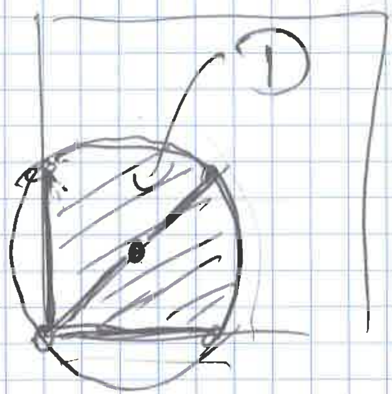


Elekes $\operatorname{conv}(K) \subseteq$ union of

$n(d+1)$ balls like $\cap [0,1]^d$

center $\left(\frac{1}{4}, \frac{1}{4}, \dots, \frac{1}{4}\right)$

radius $\frac{1}{4}\sqrt{d}$



AIM Prove $\operatorname{vol}(D) \le \left(\frac{6}{7}\right)^d$

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$$\mathrm{vol} = \mathbb{P}\left(\sum_1^d \left(X_i - \frac{1}{4}\right)^2 \le \frac{d}{16}\right) \overset{?}{\le} \left(\frac{6}{7}\right)^d$$

$X_1, \dots, X_d \sim [0,1]$ uniformly iid.

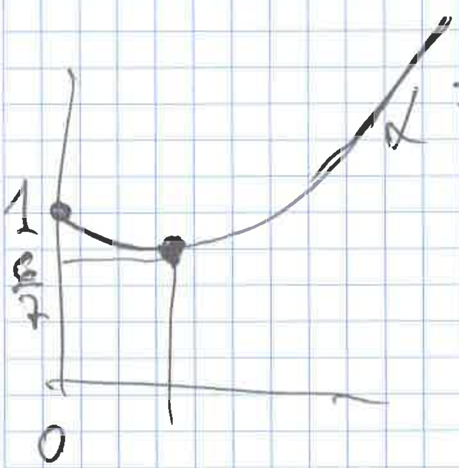
$$\sum_1^d \left(\frac{X_i}{2} - X_i^2\right) \ge 0$$

$$e^{\alpha \sum (\quad)} \ge 1 \qquad \alpha > 0$$

$$\| \quad \prod_{i=1}^d e^{\alpha\left(\frac{X_i}{2} - X_i^2\right)}$$

$$\mathbb{P}(\,\cdot \ge 1) \le \mathbb{E} \prod_1^d e^{\alpha\left(\frac{X_i}{2} - X_i^2\right)}$$

$$= \prod_1^d \mathbb{E}\, e^{\alpha\left(\frac{X_i}{2} - X_i^2\right)}$$



$$= \left(\int_0^1 e^{\alpha\left(\frac{x}{2} - x^2\right)} dx\right)^d$$

☺

$$\varphi(\alpha) \quad \cancel{\varphi(\alpha)} = \le \frac{6}{7}.$$

done

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3 Smooth fct.

here smoothness 2

$$F_d = \left\{ f \in C^1([0,1]^d) \;\middle|\; \begin{array}{l} \|f\|_\infty \le 1 \\ Lip(f) \le L_0^d \\ Lip(\nabla f) \le L_1^d \end{array} \right\}$$

THM: Integration suffers from the Curse $\iff \limsup \sqrt{d}\, L_0^d > 0$

& $\limsup d \cdot L_1^d > 0$

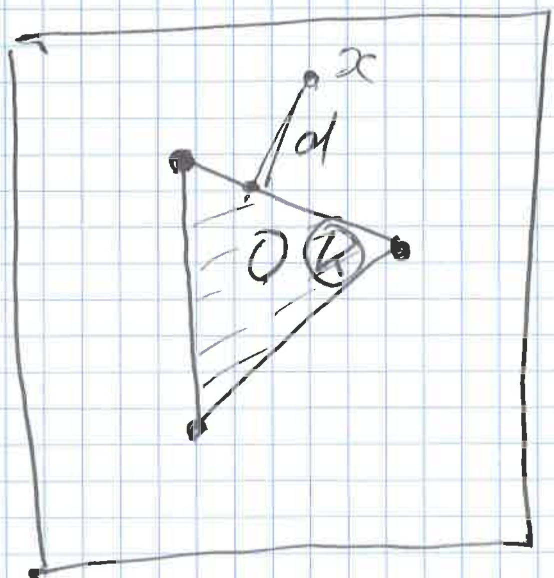
Lower Bound

Fooling Fct

$f = 0$ on

$K = \operatorname{conv}\{x_1, \ldots, x_n\}$.

$\varphi(x) = \operatorname{dist}(x, K)^2$



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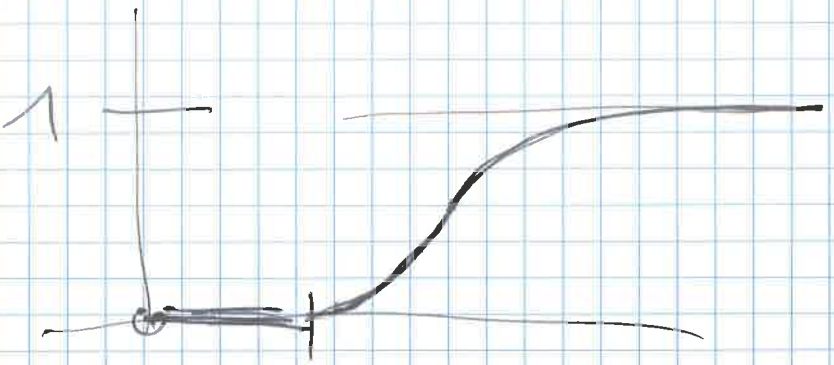
$H \neq N, U, W$



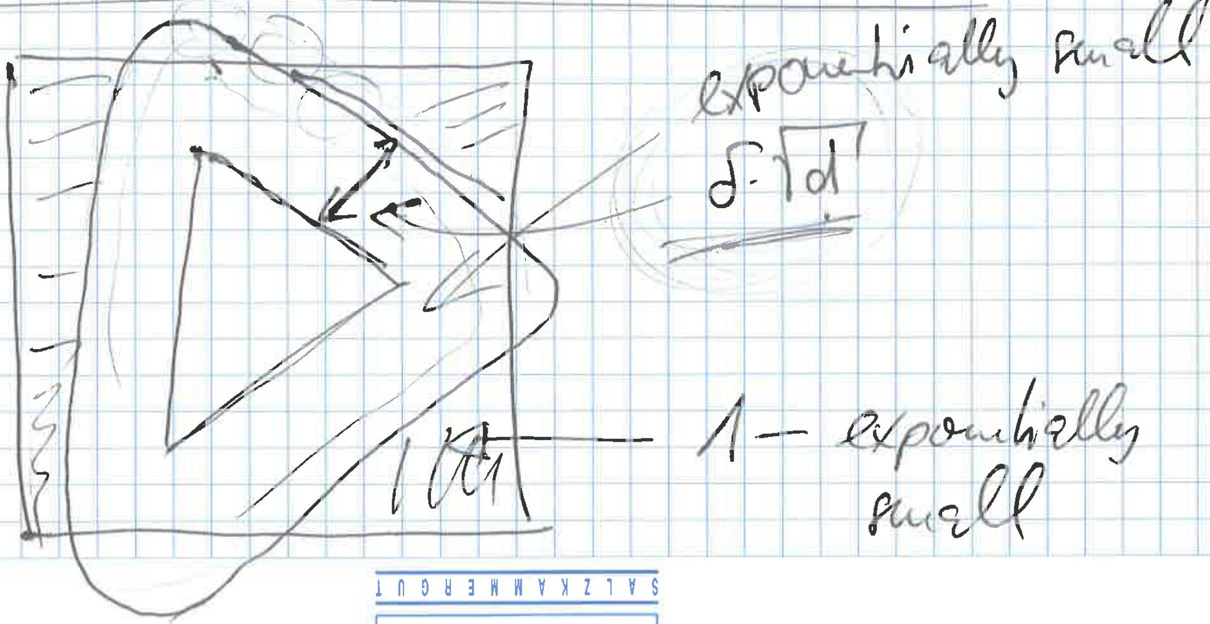
$$\varphi(x) = \operatorname{dist}(x, K)^2 = \|x - P_K(x)\|_2^2$$

$$\nabla\varphi = 2(x - P_K(x))$$

$$\|\nabla\varphi(x) - \nabla\varphi(y)\|_2 \le 2\|(x - P_K(x)) - (y - P_K(y))\|$$

$$\le 2\|x - y\|_2 + 2\underbrace{\|P_K(x) - P_K(y)\|_2}_{\|x-y\|_2}$$

$$= 4\|x - y\|_2$$



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smoothness $k \geq 2$ possible,
but condition is not if and only if
convolution.
C^∞ fcts.

$$\underline{F_d} = \{ f : [0,1]^d \to \mathbb{R} \mid f \in C^\infty, \|f^{\alpha}\|_\infty \leq 1 \text{ for all } \alpha \in \mathbb{N}_0^d \}$$

don't know : Curse of Dimensionality holds.

$\updownarrow$

<u>know</u> not strongly tractable.
(Jakub Wojtaszczyk)

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