Expansion Complexity

$S = (s_n)_{n=0}^{\infty}$ sequence over $\mathbb{F}_q$, $(s_0, s_1, \dots, s_{N-1}) \neq \underline{0}$

$$E_N(S) = \min\left\{\deg h : h\left(X, \sum_{n=0}^{N-1} s_n X^n\right) \equiv 0 \bmod X^N\right\}$$
$$0 \neq h \in \mathbb{F}_q[X,Y]$$

a) Legendre sequence

$$\mathcal{L} = (l_n), \quad l_n = \begin{cases} 1, & n \text{ quad. res. mod } p \\ 0, & \text{otherw.} \end{cases} \quad \text{over } \mathbb{F}_2$$
$$p \equiv 3 \ (8),$$

<u>Known:</u>
$$E_N(\mathcal{L}) \begin{cases} = p+1, & N \geq p^2+p+1 \\ \geq \left\lceil \frac{N}{p+1} \right\rceil, & N \leq p^2+p \end{cases} \quad (*)$$

Problem 1. Improve $(*)$.

b) $\mathcal{J} = (j_n)$, $j_n \equiv n^{-1} \bmod p$ $(0^{-1} = 0)$ over $\mathbb{F}_p$

$$E_N(\mathcal{J}) \begin{cases} = p, & N \geq p^2-p+1 \\ \geq \left\lceil \frac{N}{p} \right\rceil & \text{otherw.} \end{cases} \quad (**)$$

<u>Problem 2:</u> Improve $(**)$.

polynomial Liouville function

$F \in \mathbb{F}_q[X]$, monic, $F = f_1^{e_1} \cdots f_\omega^{e_\omega}$,

f_i irred. over $\mathbb{F}_q$, pairw. dist.

$$\lambda(F) = (-1)^\omega$$

Correlation:

$$S(n,q) := \frac{\sum_{\deg F = n} \lambda(F)\,\lambda(F+1)}{q^n} \xrightarrow[q\to\infty]{n \text{ fixed}} 0$$

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Problem 3: fix q

$$S(n,q) \xrightarrow{n\to\infty} 0 \quad ?$$