L^1 discrepancy

$1 < p < \infty$: $\quad \| D_N \|_p \gtrsim (\log N)^{\frac{d-1}{2}}$

$\exists \mathcal{P}_N \quad \| D_N \|_p \lesssim (\log N)^{\frac{d-1}{2}}$

$$F = \sum_{|R| = 2^{-n}} \overset{\pm 1}{\varepsilon_R}\, h_R = \sum_{F \in \mathbb{H}_n^d} f_F \qquad \langle D_N, f_F \rangle \geq c$$

$$\# \mathbb{H}_n^d \approx n^{d-1}$$

$$\| F \|_2 \approx n^{\frac{d-1}{2}}$$

$$\| F \|_p \approx n^{\frac{d-1}{2}} \qquad \text{(Littlewood - Paley)}$$

$$\| F \|_{\exp(L^{\frac{2}{d-1}})} \approx n^{\frac{d-1}{2}}$$

$$\underbrace{\langle D_N, F \rangle \gtrsim n^{d-1}}$$

$p = 1$? $\qquad \| F \|_\infty$ may be $\approx n^{d-1}$

(2) $d \geq 3$ $\quad \exists \varepsilon > 0, \; c > 0$ s.t.

EITHER
$$\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}+\varepsilon}$$

OR
$$\|D_N\|_2 \geq \exp\left(c (\log N)^{\varepsilon}\right) \qquad (*)$$

(3) $d = 3$
$$\|D_N\|_1 \cdot \|D_N\|_{L \cdot \log L} \gtrsim (\log N)^2$$

(4) $d \geq 3$ $\quad \forall C_1 \; \exists C_2$

if $\|D_N\|_1 \lesssim C_1 \sqrt{\log N}$, then $\|D_N\|_2 \geq N^{C}$.

Sketch of Proof of (2)

Assume that (*) doesn't hold

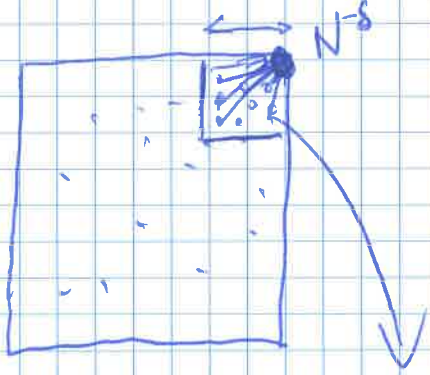
$$Y = F \cdot \mathbb{1}_{\{|F| < A\}} \qquad \|Y\|_\infty \leq A$$

$$\left|\{|F| > A\}\right| \lesssim e^{(-\ldots A^{\frac{2}{d-1}})}$$

$$\|F\|_{\exp(L^{\frac{2}{d-1}})} \lesssim \ldots$$

EXAMPLE: There exists $\mathcal{P}_N \subset [0,1]^d$

s.t. $\|D_N\|_1 \lesssim (\log N)^{\frac{d-1}{2}}$ $\quad \|D_N\|_2 \approx N^{\frac{1}{4}}$



Net $\quad N$ pts

$\|D_N\|_1 \lesssim \|D_N\|_2 \lesssim (\log N)^{\frac{d-1}{2}}$

volume $N^{-\delta d}$

$\approx N^{1-\delta d}$ pts.

$$\|D_N - \tilde{D}_N\|_1 \approx N^{1-\delta d} \cdot N^{-\delta d} = \underbrace{N^{1-2\delta d}}_{\approx \text{const}}$$

$$\|D_N - \tilde{D}_N\|_2^2 \approx N^{2-2\delta d} \cdot N^{-\delta d} = N^{2-3\delta d} \approx N^{\frac{1}{2}}$$

$$\delta = \frac{1}{2d}$$

$\|D_N\|_2$ or $\|\tilde{D}_N\|_2 \approx N^{\frac{1}{4}}$

Known $\quad \exists \mathcal{P}_N \quad \|D_N\|_1 \lesssim (\log N)^{\frac{d-1}{2}}$

$d=2$ $\quad \|D_N\|_1 \gtrsim (\log N)^{\frac{1}{2}}$ (Halász '82)
Riesz product

$d \geq 3$ $\quad \|D_N\|_1 \gtrsim (\log N)^{\frac{1}{2}}$

Conjecture $\quad \|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$

Amirkhanyan
DB, Lacey.

L^1 DICHOTOMY

(1) Fix $1 < p < \infty$ $\quad \forall C_1 \quad \exists C_2$

If $\mathcal{P}_N$ satisfies $\|D_N\|_p \leq C_1 (\log N)^{\frac{d-1}{2}}$

Then $\|D_N\|_1 \geq C_2 (\log N)^{\frac{d-1}{2}}$

Proof. $\quad 1 < q < p$

$$\|D_N\|_q \leq \|D_N\|_1^{\theta} \cdot \|D_N\|_p^{1-\theta} \qquad \frac{\theta}{1} + \frac{1-\theta}{p} = \frac{1}{q}$$

$$\|D_N\|_1 \geq \frac{\|D_N\|_q^{\frac{1}{\theta}}}{\|D_N\|_p^{\frac{1}{\theta}-1}} \gtrsim (\log N)^{\frac{d-1}{2}\cdot\cancel{\left(\frac{1}{\theta}-\left(\frac{1}{\theta}-1\right)\right)}} \quad \blacksquare$$

$\mathcal{P}_N \subset [0,1]^d \qquad \# \mathcal{P}_N = N$

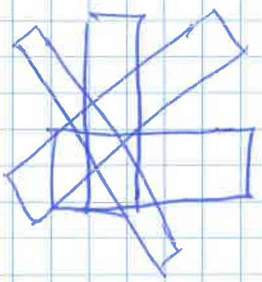
$A \subset [0,1]^d$

$$\mathcal{D}(\mathcal{P}_N, A) = \# \{ \mathcal{P}_N \in A \} - N \cdot vol(A)$$

$\mathcal{A}$ - family of sets

$$\mathcal{D}(\mathcal{P}_N, \mathcal{A}) = \sup_{A \in \mathcal{A}} | \mathcal{D}_{\mathcal{P}_N}(\mathcal{P}_N, A) |$$

→ $\mathcal{A}$ – rotated rectangles



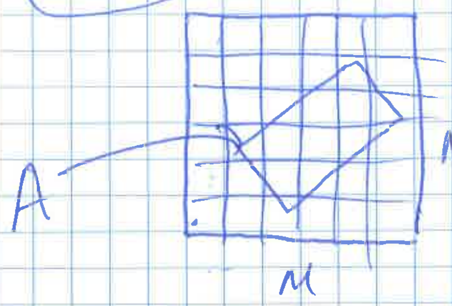
$$\mathcal{D}_N(\mathcal{A}) = \inf_{\mathcal{P}_N} \mathcal{D}(\mathcal{P}_N, \mathcal{A})$$

$$N^{\frac{1}{2} - \frac{1}{2d}} \lesssim \mathcal{D}_N(\mathcal{A}) \lesssim N^{\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}$$

Beck

JITTERED SAMPLING

$N = M^d$



N cubes Q_k

$vol(Q_k) = \frac{1}{N}$

iid random pts in each Q_k.

1) $Q_k \subset A$
$Q_k \subset A^c$ $\Longrightarrow$ Q_k contributes 0 to discrepancy

2) $$\#\{Q_k : Q_k \cap \partial A \neq \emptyset\} \lesssim \underbrace{M^{d-1} = N^{1-\frac{1}{d}}}_{\alpha_k}$$

$$D(P_N, A) = \sum_{Q_k \cap \partial A \neq \emptyset} \underbrace{\left(1\!\!1_{\{p_k \in A \cap Q_k\}} - N \cdot vol(A \cap Q_k)\right)}_{X_k}$$

$$X_k = \begin{cases} -\alpha_k & \text{with Prob } (1-\alpha_k) \\ 1-\alpha_k & \text{with Prob } \alpha_k \end{cases}$$

$E X_k = 0$

HOEFFDING

$$P\left(\left|\sum_k X_k\right| > \lambda\right) \leq 2 e^{-\frac{2\lambda^2}{m}}$$

m = # of terms.

$m \lesssim N^{1-\frac{1}{d}}$

$$\lambda = C N^{\frac{1}{2} - \frac{1}{2d}} \cdot \sqrt{\log N}$$

$$\mathbb{P}\left(D(\mathcal{P}_N, A) > C N^{\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N} \right)$$
$$\leq N^{-C_1}$$

ε- APPROXIMATING FAMILY
(ε-BRACKETING)

collection $\mathcal{C}$ s.t.

$\forall A \in \mathcal{A} \quad \exists C_1, C_2 \in \mathcal{C}$

$C_1 \subset A \subset C_2 \qquad |C_2 \setminus C_1| < \varepsilon$

$$|D(\mathcal{P}_N, A)| \leq \max \{ |D(\mathcal{P}_N, C_1)|, |D(\mathcal{P}_N, C_2)| \} + N \cdot \varepsilon$$

$$\varepsilon = \frac{1}{N}$$

Equal-area partition of the Sphere

Stolarsky (mentioned, no proof)

Beck & Chen (quoted Stolarsky)

Bourgain & Lindenstrauss (quoted Beck & Chen)

Feige - Schechtman

Leopardi

$S_1, \dots, S_N$

$\sigma(S_i \cap S_j) = 0 \qquad \bigcup_{1}^{N} S_i = S^d$

$\sigma(S_i) = \frac{1}{N}$

normalized surface measure

$\mathrm{diam}(S_i) \lesssim N^{-\frac{1}{d}}$

Lower bound

$d=2$ Rotated squares $\qquad \mathcal{D}_N(A) \geq N^{\frac{1}{2}-\frac{1}{2d}} = N^{\frac{1}{4}}$

Discrepancy measure

$$D(A) = \sum_{p\in\mathcal{P}_N} \delta_p(A) - N\cdot \overbrace{\mathrm{Vol}(A)}^{\mathrm{vol}(A\cap[0,1]^2)}$$

Fix A symmetric $\quad A=-A$

$$\mathcal{D}(A+x) = \int_{\mathbb{R}^2} \mathbb{1}_{-A+x}(y)\, dD(y)$$

$\Delta_A(x)$

$$\int \mathbb{1}_{+A}(y-x)\, dD(y)$$

$$= \mathbb{1}_A * D \quad (x)$$

$$\widehat{\Delta}_A(\xi) = \widehat{\mathbb{1}}_A(\xi)\cdot\widehat{D}(\xi)$$

$$\|\Delta_A\|_2^2 = \|\widehat{\Delta}_A\|_2 = \int_{\mathbb{R}^2} |\widehat{\mathbb{1}}_A(\xi)|^2\cdot|\widehat{D}(\xi)|^2\, d\xi$$

$A_r = \left[-\frac{r}{2}, \frac{r}{2}\right]^2$

Trivial estimate

$r = \frac{1}{2\sqrt{N}}$ $\qquad \|\Delta_A\|_2 \geq \text{const}$

$N \cdot \text{vol}(A_r) = \frac{1}{4}$

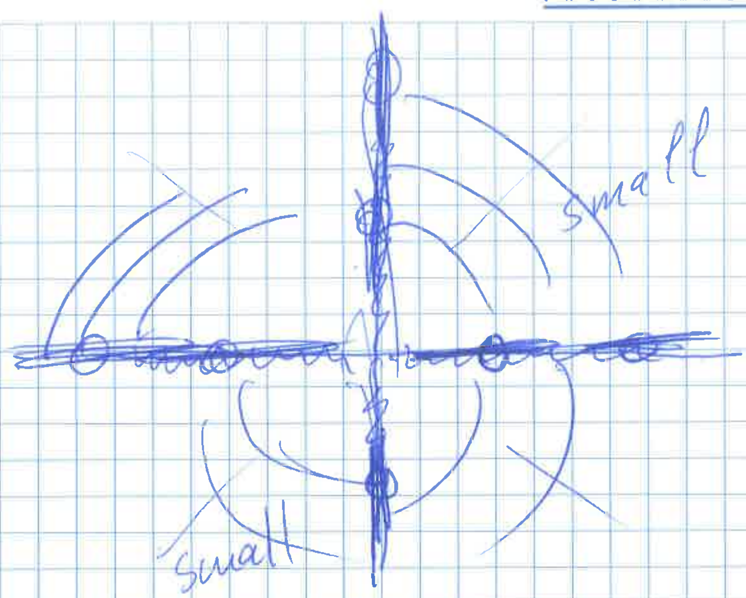
<u>IF</u> $\left|\widehat{1\!\!1}_{A_r}(\xi)\right|^2 \gtrsim \frac{r}{r_0} \left|1\!\!1_{A_{r_0}}(\xi)\right|^2$

<u>THEN</u> $r_0 = \frac{1}{2\sqrt{N}} \longrightarrow r_0 \approx \frac{1}{2}$

$$\|\Delta_{A_r}\|_2^2 \gtrsim \sqrt{N} \cdot \underbrace{\|\Delta_{A_{r_0}}\|_2^2}_{\text{const}} = N^{\frac{1}{2}}$$

BUT

$$\left|\widehat{1\!\!1}_{A_r}(\xi)\right|^2 = \left(\frac{\sin \pi \xi_1 r}{\pi \xi_1}\right)^2 \cdot \left(\frac{\sin \pi \xi_2 r}{\pi \xi_2}\right)^2$$



$$\int_0^{2\pi} \int_{\frac{1}{4}}^{\frac{1}{2}} \left| \widehat{\mathbb{1}}_{A_{r,\theta}}(\xi) \right|^2 dr \, d\theta$$

$\forall \mathcal{P}_N$

$\exists$ a ball of radius $\frac{1}{4}$ or $\frac{1}{2}$

s.t. $|\mathcal{D}(\mathcal{P}_N, \text{ball})| \gtrsim N^{\frac{1}{2} - \frac{1}{2d}}$

just $\frac{1}{4}$?