METHODS OF HARMONIC ANALYSIS IN DISCREPANCY THEORY.

Lecture 1.

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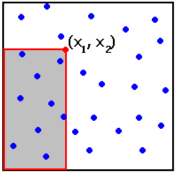
SFB Winter Schoool
on Complexity and Discrepancy
Traunkirchen, Austria
November 30, 2015

Discrepancy function

Consider a set $\mathcal{P}_N \subset [0,1]^d$ consisting of N points:



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$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - N x_1 x_2 \dots x_d$$

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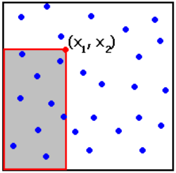

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Extremal discrepancy (star-discrepancy):

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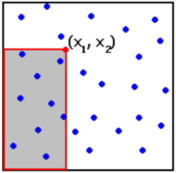

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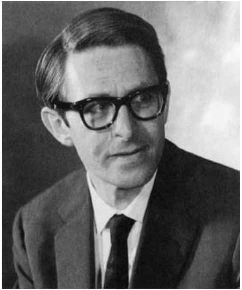
L^p discrepancy: $\|D_N\|_p = \left(\int_{[0,1]^d} |D_N(x)|^p \, dx \right)^{1/p}.$

Roth's Theorem



Klaus Roth, October 29, 1925 – November 10, 2015

Theorem (ROTH, K. F. On irregularities of distribution, Mathematatika 1 (1954), 73–79.)

There exists $C_d \geq 0$ such that for any N-point set $\mathcal{P}_N \subset [0,1]^d$

$$\|D_N\|_2 \geq C_d (\log N)^{\frac{d-1}{2}}.$$

Roth's Theorem

According to Roth himself, this was his favorite result.

- William Chen (private communication)
- Kenneth Stolarsky (private communication)
- Ben Green (comment on Terry Tao's blog)

12 comments

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12 November, 2015 at 9:55 am
Ben Green

I did meet Roth, in Inverness around 7 years ago. I asked him what his favourite proof (among his results was) and he said the lower bound for the L^2 discrepancy of point sets with respect to axis parallel boxes. It is a very elegant argument, nicely described in Bernard Chazelle's book "Discrepancy Theory". Later in his career he became quite interested in the "Heilbronn triangle problem", which came up in conversation the other day: given n points in the unit square, what's the smallest area of triangle they are guaranteed to span. I believe that $n^{-2+o(1)}$ is conjectured, and that Roth was the first to improve on the trivial bound $O(1/n)$. Subsequently bounds of the form $O(n^{-1-c})$ were obtained.

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Roth's Theorem: legacy

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- 4 papers by Roth (On irregularities of distribution. I–IV)
- 10 papers by W.M. Schmidt (On irregularities of distribution. I–X)
- 2 by J. Beck (Note on irregularities of distribution. I–II)
- 4 by W. W. L. Chen (On irregularities of distribution. I–IV)
- 2 by Beck and Chen (Note on irregularities of distribution. I–II)
- a book by Beck and Chen, "Irregularities of distribution".

References

Books

- Kuypers, Niederreiter
  "Uniform distribution of sequences"
- Beck, Chen
  "Irregularities of distribution"
- Drmota, Tichy
  " Sequences, discrepancies and applications"
- Matoušek
  "Geometric discrepancy"
- Dick, Pillichshammer
  "Digital nets and sequences"

References

References specific to these lectures

- *"Roth's orthogonal function method in discrepancy theory"*, Uniform Distribution Theory 6 (2011), no. 1, 143–184. (DB)
- Chapter *"Roth's Orthogonal Function Method in Discrepancy Theory and Some New Connections"* in the book "Panorama of Discrepancy Theory", Lecture Notes in Math 2017 Springer Verlag, 2014. pp. 71–158. (DB)
- *Small Ball and Discrepancy Inequalities*, Michael Lacey `http://arxiv.org/pdf/math/0609816.pdf`
- *The supremum norm of the discrepancy function: recent results and connections*, Monte Carlo and Quasi-Monte Carlo Methods 2012, Springer Proceedings in Math. and Stat. 65 Springer Verlag, 2013. (DB, M. Lacey)

Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0,1]^d$ *with* $\#\mathcal{P}_N = N$*:*

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

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Theorem (Davenport, 1956 ($d = 2$, $p = 2$); Roth, 1979 ($d \geq 3$, $p = 2$); Chen, 1982 ($p > 2$, $d \geq 3$); Chen, Skriganov, 2000's)

There exist sets $\mathcal{P}_N \subset [0,1]^d$ with

$$\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$$

Roth's orthogonal function method

- Dyadic intervals in $[0,1]$:

$$\mathcal{D} = \left\{ I = \left[\frac{m}{2^n}, \frac{m+1}{2^n} \right) : \, m, n \in \mathbb{Z}, \, n \geq 0, \, 0 \leq m < 2^n \right\}.$$

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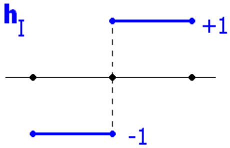
- L^∞ normalized Haar function on a dyadic Interval I:
$h_I = -\mathbf{1}_{I_{left}} + \mathbf{1}_{I_{right}}$



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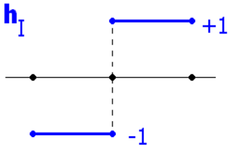

- Orthogonality:

$$\langle h_{I'}, h_{I''} \rangle = \int_0^1 h_{I'}(x) \cdot h_{I''}(x)\, dx = 0, \qquad I', I'' \in \mathcal{D},\, I' \neq I'',$$

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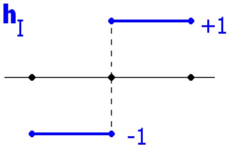

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- $f \in L^2([0,1])$ can be written as $f = \sum_{I \in \mathcal{D}_*} \frac{\langle f, h_I \rangle}{|I|} h_I$

Roth's orthogonal function method

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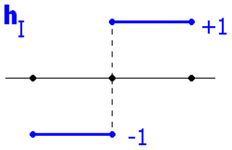

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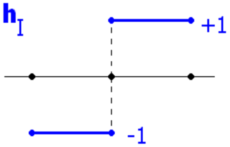
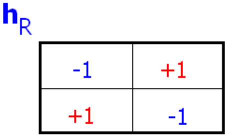


- For a dyadic rectangle $R = I_1 \times ... \times I_d \subset [0,1]^2$

$$h_R(x) := h_{I_1}(x_1) \cdot ... \cdot h_{I_d}(x_d)$$



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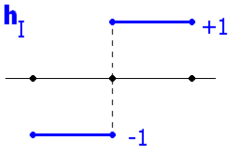

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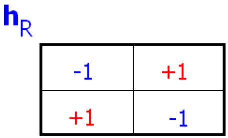

- $f \in L^2([0,1]^d)$: $\quad f = \sum_{R \in \mathcal{D}_*^d} \frac{\langle f, h_R \rangle}{|R|} h_R$

Roth's orthogonal function method

- Main idea:

$$D_N \approx \sum_{R:\,|R|\approx\frac{1}{N}} \frac{\langle D_N, h_R\rangle}{|R|} h_R$$

- Define the collection

$$\mathbb{H}_n^d = \{\vec{r} = (r_1, \dots, r_d) \in \mathbb{Z}_+^d : \|\vec{r}\|_1 = n\},$$

where the ℓ_1 norm is defined as $\|\vec{r}\|_1 = |r_1| + \cdots + |r_d|$.

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- These vectors will specify the shape of the dyadic rectangles in the following sense: for $R \in \mathcal{D}^d$, we say that $R \in \mathcal{D}_{\vec{r}}^d$ if $|R_j| = 2^{-r_j}$ for $j = 1, \dots, d$.

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$$\#\mathbb{H}_n^d = \binom{n+d-1}{d-1} \approx n^{d-1},$$

Roth's orthogonal function method

- A function f on $[0,1]^d$ is an r-function with parameter $\vec{r} \in \mathbb{Z}_+^d$ if f is of the form

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- Orthogonal for different r.

Haar coefficients of D_N: counting part

$$D_N(x) = \sum_{p \in \mathcal{P}_N} \mathbf{1}_{[p,\vec{1}]}(x) - N \cdot x_1 \cdot \dots \cdot x_d,$$

- In dimension $d = 1$: $\int \mathbf{1}_{[q,1]}(x) \cdot h_I(x)\, dx = \int_q^1 h_I(x)\, dx = 0$

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- For $p \in [0,1]^d$

$$\int_{[0,1]^d} \mathbf{1}_{[p,\vec{1}]}(x) \cdot h_R(x)\, dx = \prod_{j=1}^{d} \int_{p_j}^{1} h_{R_j}(x_j)\, dx_j = 0$$

when $p \notin R$.

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  when $p \notin R$.
- If $R \in \mathcal{D}^d$ is empty, i.e. $R \cap \mathcal{P}_N = \emptyset$:

$$\left\langle \sum_{p \in \mathcal{P}_N} \mathbf{1}_{[p,\vec{1}]},\, h_R \right\rangle = 0.$$

Haar coefficients of D_N: linear part

$$D_N(x) = \sum_{p \in \mathcal{P}_N} \mathbf{1}_{[p, \vec{1}]}(x) - N \cdot x_1 \cdot \dots \cdot x_d,$$

- Easy to compute

$$\langle N x_1 \dots x_d, h_R \rangle = N \prod_{j=1}^{d} \langle x_j, h_{R_j}(x_j) \rangle = N \cdot \frac{|R|^2}{4^d}.$$

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- If a rectangle $R \in \mathcal{D}^d$ does not contain points of $\mathcal{P}_N$

$$\langle D_N, h_R \rangle = -N|R|^2 4^{-d}.$$

- Let $R \subset [0,1]^2$ be an arbitrary dyadic rectangle of dimensions $2h_1 \times 2h_2$ which does not contain any points of $\mathcal{P}_N$

Haar coefficients of D_N: linear part (intuition)

- Let $R \subset [0,1]^2$ be an arbitrary dyadic rectangle of dimensions $2h_1 \times 2h_2$ which does not contain any points of $\mathcal{P}_N$
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- For any point $x = (x_1, x_2) \in R'$

$$D_N\big(x\big) - D_N\big(x + (h_1, 0)\big) + D_N\big(x + (h_1, h_2)\big) - D_N\big(x + (0, h_2)\big)$$
$$= -N \cdot h_1 h_2 = -N \cdot \frac{|R|}{4}$$

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- Integrate over R' to get $\langle D_N, h_R \rangle$.

Lemma

Let $\mathcal{P}_N \subset [0,1]^d$ and let $n \in \mathbb{N}$ be such that $2^{n-2} \leq N < 2^{n-1}$. Then, for any $\vec{r} \in \mathbb{H}_n^d$, there exists an r*-function $f_{\vec{r}}$*

$$\langle D_N, f_{\vec{r}} \rangle \geq c_d > 0.$$

Roth's orthogonal function method

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$$f_{\vec{r}} = \sum_{R \in \mathcal{D}^d_{\vec{r}}:\, R \cap \mathcal{P}_N = \emptyset} (-1) \cdot h_R + \sum_{R \in \mathcal{D}^d_{\vec{r}}:\, R \cap \mathcal{P}_N \neq \emptyset} \operatorname{sgn}\big(\langle D_N, h_R \rangle\big) \cdot h_R$$

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$$\langle D_N, f_{\vec{r}} \rangle \geq - \sum_{R \cap \mathcal{P}_N = \emptyset} \langle D_N, h_R \rangle = \sum_{R \cap \mathcal{P}_N = \emptyset} \langle N x_1 \ldots x_d, h_R \rangle$$

$$= \sum_{R \cap \mathcal{P}_N = \emptyset} N \cdot \frac{|R|^2}{4^d} \geq 2^{n-1} \cdot 2^{n-2} \cdot \frac{2^{-2n}}{4^d} = c_d.$$

Roth's theorem: Proof 1 (Duality)

- "Test function"

$$F = \sum_{\vec{r} \in \mathbb{H}_n^d} f_{\vec{r}}.$$

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- Orthogonality:

$$\|F\|_2 = \left(\sum_{\vec{r} \in \mathbb{H}_n^d} \|f_{\vec{r}}\|_2^2 \right)^{1/2} = (\#\mathbb{H}_n^d)^{1/2} \approx n^{\frac{d-1}{2}}.$$

- "Test function"
$$F = \sum_{\vec{r}\in\mathbb{H}_n^d} f_{\vec{r}}.$$
- Orthogonality:
$$\|F\|_2 = \Bigg(\sum_{\vec{r}\in\mathbb{H}_n^d} \|f_{\vec{r}}\|_2^2\Bigg)^{1/2} = (\#\mathbb{H}_n^d)^{1/2} \approx n^{\frac{d-1}{2}}.$$
- Previous lemma:
$$\langle D_N, F\rangle \geq (\#\mathbb{H}_n^d)\cdot c_d \approx n^{d-1}.$$

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$$F = \sum_{\vec{r} \in \mathbb{H}_n^d} f_{\vec{r}}.$$

- Orthogonality:

$$\|F\|_2 = \Bigg(\sum_{\vec{r} \in \mathbb{H}_n^d} \|f_{\vec{r}}\|_2^2 \Bigg)^{1/2} = (\#\mathbb{H}_n^d)^{1/2} \approx n^{\frac{d-1}{2}}.$$

- Previous lemma:

$$\langle D_N, F \rangle \geq (\#\mathbb{H}_n^d) \cdot c_d \approx n^{d-1}.$$

- Cauchy– Schwarz:

$$\|D_N\|_2 \geq \frac{\langle D_N, F \rangle}{\|F\|_2} \gtrsim n^{\frac{d-1}{2}} \approx \big(\log N \big)^{\frac{d-1}{2}}$$

$$\|D_N\|_2^2 \geq \sum_{|R|=2^{-n},\, R\cap\mathcal{P}_N=\emptyset} \frac{|\langle D_N, h_R\rangle|^2}{|R|}$$

Roth's theorem: Proof 1 (Orthogonality)

$$\|D_N\|_2^2 \geq \sum_{|R|=2^{-n},\, R\cap\mathcal{P}_N=\emptyset} \frac{|\langle D_N, h_R\rangle|^2}{|R|}$$

$$= \sum_{\vec{r}\in\mathbb{H}_n^d} \sum_{R\in\mathcal{D}_{\vec{r}}^d:\, R\cap\mathcal{P}_N=\emptyset} N^2 \cdot \frac{2^{-4n}}{2^{-n}\cdot 4^{2d}}$$

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$$\gtrsim (\#\mathbb{H}_n^d)\cdot 2^{n-1}\cdot 2^{2n-4}2^{-3n}$$

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\begin{aligned}
\|D_N\|_2^2 &\geq \sum_{|R|=2^{-n},\, R\cap\mathcal{P}_N=\emptyset} \frac{|\langle D_N, h_R\rangle|^2}{|R|} \\
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&\gtrsim (\#\mathbb{H}_n^d)\cdot 2^{n-1}\cdot 2^{2n-4}2^{-3n} \\
&\approx n^{d-1} \approx \big(\log N\big)^{d-1}.
\end{aligned}
$$

L^∞ estimates

Conjecture

$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$

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In dimension $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$

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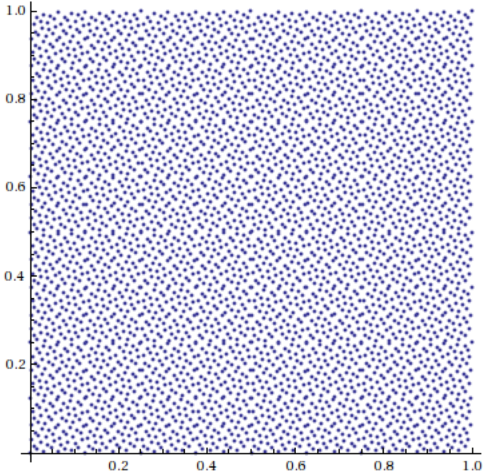
There exist $\mathcal{P}_N \subset [0,1]^2$ with $\|D_N\|_\infty \approx \log N$

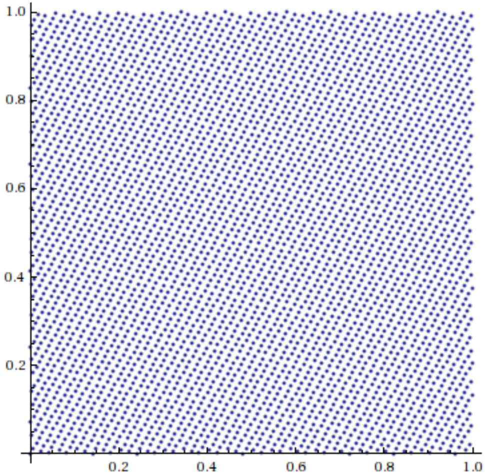
Low discrepancy sets



The van der Corput set with $N = 2^{12}$ points

$$\big(0.x_1x_2\ldots x_n,\ 0.x_nx_{n-1}\ldots x_2x_1\big), \quad x_k = 0 \text{ or } 1.$$

Discrepancy $\approx \log N$

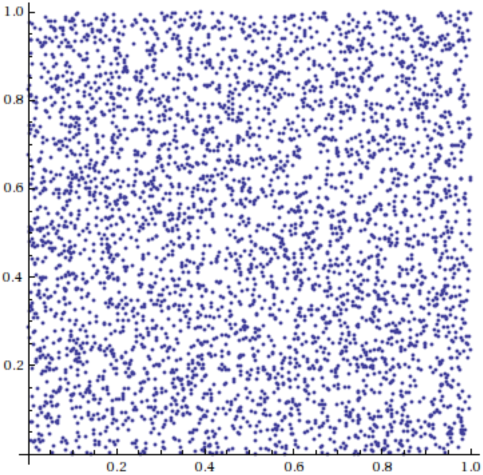


The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$$\big(n/N, \{n\alpha\}\big), \quad n = 0, 1, ..., N-1.$$

Discrepancy $\approx \log N$

Low discrepancy sets



Random set with $N = 2^{12}$ points

$$\text{Discrepancy} \approx \sqrt{N}$$

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$d \geq 3$, Halton, Hammersley (1960):

There exist $\mathcal{P}_N \subset [0,1]^d$ with $\|D_N\|_\infty \lesssim (\log N)^{d-1}$

Conjectures and results

Conjecture 1

$\|D_N\|_\infty \gtrsim (\log N)^{d-1}$

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Conjecture 2

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Theorem (DB, M.Lacey, A.Vagharshakyan, 2008)

For $d \geq 3$ there exists $\eta > 0$ such that the following estimate holds for all N-point distributions $\mathcal{P}_N \subset [0,1]^d$:

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}.$$

The small ball inequality

Instead of studying D_N we shall look at $\sum_{|R|=2^{-n}} \alpha_R h_R$

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- Connected to probability, approximation, discrepancy.
- Known: $\frac{d-1}{2} + \eta(d)$ for $d \geq 3$ (DB, Lacey, Vagharshakyan, 2008)

The L^2 estimate

$$\Big\| \sum_{R \in \mathcal{D}^d \colon |R| = 2^{-n}} \alpha_R h_R \Big\|_2 = \left(\sum_{|R|=2^{-n}} |\alpha_R|^2 2^{-n} \right)^{\frac{1}{2}}$$
$$\gtrsim \frac{\sum_{|R|=2^{-n}} |\alpha_R| 2^{-n/2}}{\left(n^{d-1} 2^n\right)^{\frac{1}{2}}}$$
$$= n^{-\frac{d-1}{2}} \cdot 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|.$$

The 'signed' version small ball inequality

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For dimensions $d \geq 2$, if all $\varepsilon_R = \pm 1$, we have

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- By orthogonality

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_2 = \Big(\sum_{\vec{r} \in \mathbb{H}_n^d} \|f_{\vec{r}}\|_2^2 \Big)^{1/2} = \sqrt{\#\mathbb{H}_n^d} \approx n^{\frac{d-1}{2}}$$

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For dimensions $d \geq 2$, we have for all choices of α_R

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Conjecture 2

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

The small ball conjecture and discrepancy

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- In both conjectures one gains a square root over the L^2 estimate.

Signed Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of $\alpha_R = \pm 1$

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Discrepancy estimates	**Small Ball inequality (signed)**
Dimension $d=2$	
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81)	$\Big\| \sum_{\lvert R\rvert=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n$ (Talagrand, '94; Temlyakov, '95)
Higher dimensions, L^2 bounds	
$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$	$\Big\| \sum_{\lvert R\rvert=2^{-n}} \varepsilon_R h_R \Big\|_2 \gtrsim n^{(d-1)/2}$
Higher dimensions, conjecture	
$\|D_N\|_\infty \gtrsim (\log N)^{d/2}$	$\Big\| \sum_{\lvert R\rvert=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}$
Higher dimensions, known results	
$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}$	$\Big\| \sum_{\lvert R\rvert=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d-1}{2}+\eta}$

$d = 2$: proof (V. Temlyakov, '95)

$$\mathcal{H}_n = \sum_{R:\, |R| = 2^{-n}} \alpha_R h_R, \qquad \alpha_R = \pm 1$$

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$$\Psi \stackrel{\text{def}}{=} \prod_{k=1}^{n} (1 + f_k)$$

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- **Thus**

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Structure of the Riesz product

$$\Psi \stackrel{\text{def}}{=} \prod_{k=0}^{n} (1 + f_k) = \begin{cases} 2^{n+1} & \text{if } f_k = +1 \text{ for all } k, \\ 0 & \text{otherwise.} \end{cases}$$

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- Assume that $\varepsilon_R = +1$ for all R.
- $f_k(x_1, x_2) = +1$ iff
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- $f_k(x_1, x_2) = +1$ iff
  $(k+1)^{st}$ binary digit of x_1 $=$ $(n-k+1)^{st}$ digit of x_2.
- if this holds for all $k = 0, 1, ..., n$:
  Van der Corput set with $N = 2^{n+1}$ points, i.e. the set of all points of the form

$$\left(0.x^{(1)}x^{(2)} \ldots x^{(n)}x^{(n+1)}, 0.x^{(n+1)}x^{(n)} \ldots x^{(2)}x^{(1)} \right).$$

Riesz product

Small ball inequality (d=2)

For $d = 2$, we have

$$\Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

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Sidon's theorem

If a bounded 2π-periodic function f has lacunary Fourier series $\sum_{k=1}^{\infty} a_k e^{i n_k x}$, $n_{k+1}/n_k > \lambda > 1$, then

$$\|f\|_\infty \gtrsim \sum_{k=1}^{\infty} |a_k|$$

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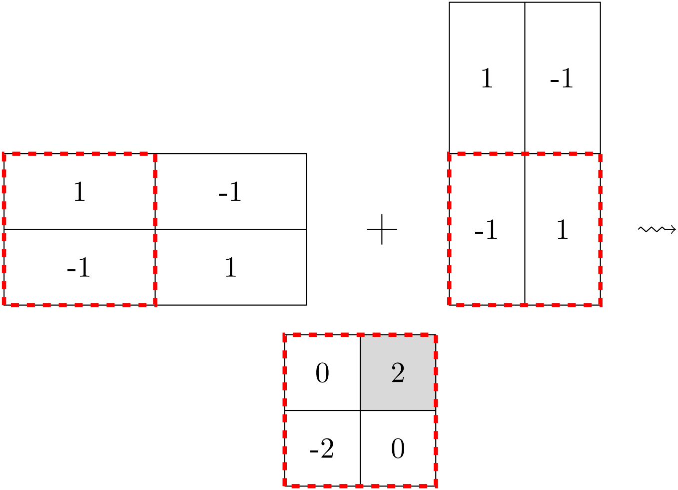
- Riesz product: $P_K(x) = \prod_{k=1}^{K} (1 + \varepsilon_k \cos n_k x)$

Discrepancy function $D_N(x) = \#\{\mathcal{P}_N \cap [0,x)\} - N x_1 x_2$	**Lacunary Fourier series** $f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_k x$, $\frac{n_{k+1}}{n_k} > \lambda > 1$
$\|D_N\|_2 \gtrsim \sqrt{\log N}$ (Roth, '54)	$\|f\|_2 \equiv \sqrt{\sum \lvert c_k\rvert^2}$
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81) Riesz product: $\prod (1 + c f_k)$	$\|f\|_\infty \gtrsim \sum \lvert c_k \rvert$ (Sidon, '27) Riesz product: $\prod (1 + \cos(n_k x + \phi_k))$
$\|D_N\|_1 \gtrsim \sqrt{\log N}$ (Halász, '81) Riesz product: $\prod \left(1 + i \cdot \frac{c}{\sqrt{\log N}} f_k\right)$	$\|f\|_1 \gtrsim \|f\|_2$ (Sidon, '30) Riesz product: $\prod \left(1 + i \cdot \frac{\lvert c_k \rvert}{\|f\|_2} \cos(n_k x + \theta_k)\right)$

Table: Discrepancy function and lacunary Fourier series

$d = 2$ proof (DB, Feldheim, '15)



A new proof in $d = 2$: signed case

- Let $\mathcal{D}_k = \{R = R_1 \times R_2 : |R_1| = 2^{-k}, |R_2| = 2^{-(n-k)}\}$

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$$F_k(x) = \sum_{R \in \mathcal{D}_k^2} \varepsilon_R h_R(x) + \sum_{R \in \mathcal{D}_{n-k}^2} \varepsilon_R h_R(x)$$

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- Start with $k = \frac{n+1}{2}$ (if n is odd)

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- In the end we have 2^{n+1} cubes Q_j of size $2^{-(n+1)} \times 2^{-(n+1)}$, on which all $F_k = +2$. Then on each Q_j

$$\sum_{|R|=2^{-n}} \varepsilon_R h_R(x) = \sum_{k=\frac{n+1}{2}}^{n} F_k(x) = \frac{n+1}{2} \cdot 2 = n+1.$$

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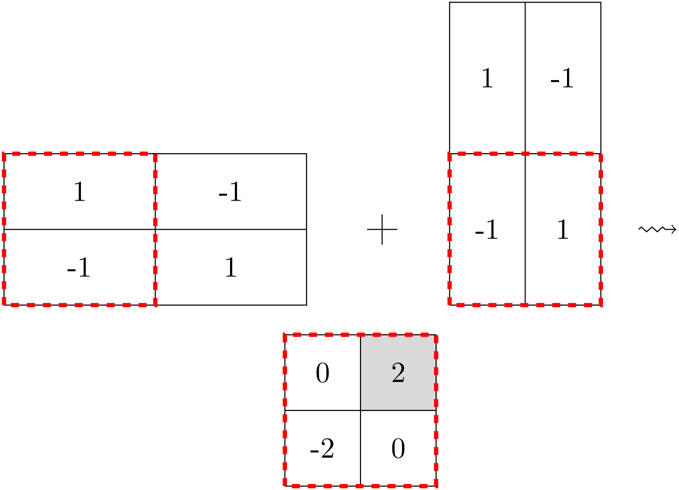

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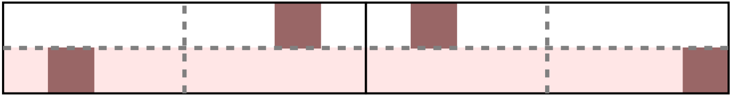

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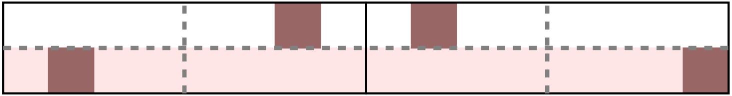

- We further choose a sub square in each of those and they have to lie in the opposite quarters of R.

Connection to binary (t, m, d)-nets

Definition

A set $\mathcal{P}$ of $N = b^m$ points in $[0,1)^d$ is called a (t, m, d)-net in base b if every b-adic box of volume b^{-m+t} contains exactly b^t points of $\mathcal{P}$.

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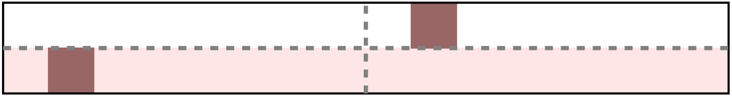

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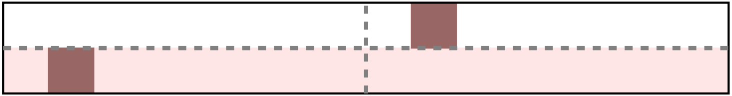

- Since in every such R these points lie in opposite quarters, it is actually a $(0, n+1, 2)$-net in base $b = 2$.

Examples of two-dimensional nets

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- Each dyadic $(0, m, 2)$-net $\mathcal{P}$ may be obtained this way
- The total number of different binary $(0, m, 2)$-nets is

$$2^{m2^{m-1}}$$

(Xiao, 1996)

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$$F_k(x) = |\alpha_{R'}| + |\alpha_{R''}|.$$

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&= 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|
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Extension to b-adic nets

- A box $R \in \mathcal{D}_b^2$ of dimensions $b^{-m_1} \times b^{-m_2}$ is a union of a $b \times b$ array of b-adic boxes of dimensions $b^{-(m_1+1)} \times b^{-(m_2+1)}$.

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- Define the family of functions $\mathcal{H}_R$. The function $\phi_R \in \mathcal{H}_R$ iff
  - ϕ_R takes values ± 1 on R and vanishes outside R.
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- $\# \mathcal{H}_R = b!$.
- If $b = 2$, then $\mathcal{H}_R = \{\pm h_R\}$ and $\# \mathcal{H}_R = 2$.

Small ball inequality and b-adic nets

Theorem

Fix the scale $m \in \mathbb{N}$ and an integer base $b \geq 2$. For each b-adic box $R \in \mathcal{D}_b^2$ with $|R| = b^{-(m-1)}$, choose a function $\phi_R \in \mathcal{H}_R$.

(i) *A b-adic analogue of the signed small ball inequality holds:*

$$\max_{x \in [0,1)^2} \sum_{|R| = b^{-(m-1)}} \phi_R(x) = m.$$

(ii) *The set on which the maximum above is achieved has the form*

$$\mathcal{P} + \left[0, b^{-m}\right)^2,$$

where $\mathcal{P}$ is a standard $(0, m, 2)$-net in base b.

(iii) *Each $(0, m, 2)$-net $\mathcal{P}$ in base b may be obtained this way.*

(iv) *The number of different $(0, m, 2)$-nets in base b is $(b!)^{mb^{m-1}}$.*