$\mathcal{P}_N \subset [0,1]^d$ N-pts

$$\mathcal{D}_N(x) = \# \mathcal{P}_N \cap [0,x] - N x_1 \dots x_d$$

Roth $\quad \| \mathcal{D}_N \|_2 \gtrsim (\log N)^{\frac{d-1}{2}}$

$$\mathcal{D}_N \approx \sum_{|R| \approx \frac{1}{N}} \frac{\langle D_N, h_R \rangle}{|R|} h_R$$

$$\left[F = \sum_{\bar{r} \in \mathbb{H}_n^d} f_{\bar{r}} \right.$$

$$f_{\bar{r}} = \sum_{R \in \mathcal{D}_{\bar{r}}} \varepsilon_R h_R \qquad \varepsilon_R \to \pm 1$$

$$\mathcal{D}_{\bar{r}} = \{ R \in \mathcal{D}^d : |R_j| = 2^{-r_j} \}$$

$$\mathbb{H}_n^d = \{ \bar{r} \in \mathbb{Z}_+^d : r_1 + \dots + r_d = n \}$$

(1) $\# \mathbb{H}_n^d \approx n^{d-1}$

(2) $\exists f_{\bar{r}} : \langle D_N, f_{\bar{r}} \rangle \geq c_d$

(A) $\langle D_N, F \rangle \gtrsim n^{d-1}$

(B) $\| F \|_2 \approx n^{\frac{d-1}{2}}$

$$\Bigg\} \quad \| \mathcal{D}_N \|_2 \gtrsim \frac{\langle D_N, F \rangle}{\| F \|_2} \approx n^{\frac{d-1}{2}}$$

Littlewood–Paley Theory

Fourier series

$f : \mathbb{T} \to \mathbb{C} \qquad f \sim \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$

$$\|f\|_2^2 = \sum |c_k|^2$$

$$f_\pm \sim \sum \pm c_k e^{2\pi i k x}$$

$$\|f_\pm\|_2 = \|f\|_2$$

BUT $p \neq 2 \quad \|f_\pm\|_p \not\approx \|f\|_p$

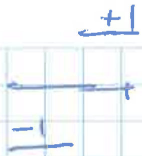
$$Sf(x) = \left(\sum_{m=0}^{\infty} \Big| \sum_{\lfloor 2^{m-1} \rfloor \le |k| < 2^m} c_k e^{2\pi i k x} \Big|^2 \right)^{1/2}$$

square function

Theorem (L-P): $\|Sf\|_p \approx \|f\|_p \quad \forall\, 1 < p < \infty$

Walsh functions – almost same

$$Sf^2 = \sum_m |\Delta_m f|^2$$



Haar basis, $d=1$

$f \in L^2 \qquad f \overset{L^2}{=} \sum_{I \in \mathcal{D}_*} \frac{\langle f, h_I \rangle}{|I|} h_I$

$$Sf(x) = \left(\left(\int f(x)\,dx \right)^2 + \sum_{I \in \mathcal{D}} \frac{\langle f, h_I \rangle^2}{|I|^2} \mathbb{1}_I(x) \right)^{\frac{1}{2}}$$

$$= \left(\left| \int f(x)\,dx \right|^2 + \sum_{k=0}^{\infty} \left| \sum_{|I| = 2^{-k}} \frac{\langle f, h_I \rangle}{|I|} h_I(x) \right|^2 \right)^{\frac{1}{2}}$$

$f = \sum \alpha_I h_I \qquad Sf = \left(\sum \alpha_I^2 \mathbb{1}_I \right)^{\frac{1}{2}}$

$\mathbb{1}_I(x) = h_I^2(x)$

Theorem: $\forall\, 1 < p < \infty \;\; \exists\, A_p, B_p > 0$

$$A_p \|Sf\|_p \leq \|f\|_p \leq B_p \|Sf\|_p$$

$B_p \approx \sqrt{p} \qquad p$-large

$A_q = \frac{1}{B_p} \qquad \frac{1}{q} + \frac{1}{p} = 1$

Holds $f : [0,1] \to H$ Hilbert Space

$d \geq 2$

$$F(x) = \sum_{|R| = 2^{-n}} \varepsilon_R \, h_R(x) \qquad \varepsilon_R = \pm 1$$

$$F(x_1, \ldots) = \sum \alpha_I \, h_I(x_1)$$

$$\alpha_I = \sum_{\substack{|R| = 2^{-n} \\ R_1 = I}} \varepsilon_R \cdot \prod_{j=2}^{n} h_{R_j}(x_j)$$

$$\| F \|_q \leq B_q \left\| \left(\sum_{k=0}^{n} \Big[\sum_{\substack{|R| = 2^{-n} \\ |R_1| = 2^{-k}}} \varepsilon_R \, h_R(x) \Big]^2 \right)^{\frac{1}{2}} \right\|_q$$

d = 2

$$\| F \|_q \leq B_q \left\| \left(\sum_{k=0}^{n} 1 \right)^{\frac{1}{2}} \right\|_q$$

$$= B_q \cdot (n+1)^{\frac{1}{2}} \qquad \leftarrow \frac{d-1}{2}$$

$d = 3$

$$\vec{F}_2(x_2) = \sum_I \left\{ \sum_{\substack{|R| = 2^{-n} \\ |R_1| = 2^{-k}}} \varepsilon_R \prod_{j \neq 2} h_{R_j}(x_j) \right\}_{k=0}^{n} \cdot h_I(x_j)$$

$$\longrightarrow \left| \vec{F}_2(x_2) \right|_{\ell^2}$$

in x_1

$$\| F \|_q \leq B_q \| \vec{F}_2 \|_q \leq$$

$$\leq C B_q^2 \left\| \left(\sum_{k=0}^{n} \sum_{r=0}^{n} \Big| \sum_{\substack{|R| = 2^{-n} \\ |R_1| = 2^{-k} \\ |R_2| = 2^{-r}}} \varepsilon_R h_R \Big|^2 \right)^{\frac{1}{2}} \right\|_q$$

$$(n^2)^{\frac{1}{2}} = n = n^{\frac{d-1}{2}}$$

$$f = \sum_{R} \alpha_R h_R$$

$$Sf = \left(\sum \alpha_R^2 \cdot 1_R \right)^{\frac{1}{2}}$$

Then $1 < p < \infty$

$$(A_p)^d \| Sf \|_p \le \| f \|_p \le (B_p)^d \cdot \| Sf \|_p$$

"Hyperbolic" $L-P$

$$f = \sum_{|R| = 2^{-n}} \alpha_R h_R$$

$$(A_p)^{d-1} \| Sf \|_p \le \| f \|_p \le \underbrace{(B_p)^{d-1}}_{\approx p^{\frac{d-1}{2}}} \| Sf \|_p$$

$$F = \sum_{|R|=2^{-n}} \varepsilon_R h_R \qquad \varepsilon_R = \pm 1$$

$$\|F\|_p \qquad 1 < p < \infty$$

$$\approx \Big\| \Big(\sum_{|R|=2^{-n}} \underbrace{\varepsilon_R^2}_{=1} \mathbb{1}_R(x) \Big)^{\frac{1}{2}} \Big\|_p$$

$$\text{const} = \# \mathbb{H}_n^d \approx n^{d-1}$$

$$\approx n^{\frac{d-1}{2}}$$

$p = 1$

$$\|F\|_1 \le \|F\|_2 \approx n^{\frac{d-1}{2}}$$

$$\|F\|_4 \approx n^{\frac{d-1}{2}}$$

$$n^{\frac{d-1}{2}} \approx \|F\|_2 \le \|F\|_1^{\frac{1}{3}} \|F\|_4^{\frac{2}{3}} = \|F\|_1^{\frac{1}{3}} \cdot n^{\frac{d-1}{2} \cdot \frac{2}{3}}$$

$$\|F\|_1 \gtrsim n^{\frac{d-1}{2}}$$

$$\boxed{\|F\|_1 \approx n^{\frac{d-1}{2}}}$$

"Signed" small ball conjecture

$$\|F\|_\infty \gtrsim n^{d/2}$$

$\Uparrow$

"Dual" SSBC

$$\mathcal{H}_{>n} = \operatorname{span}\{h_R : |R| < 2^{-n}\}$$

$$\operatorname{dist}_{L^1}(F, \mathcal{H}_{>n}) \lesssim n^{\frac{d-2}{2}}$$

$\exists \tilde{F} \in \mathcal{H}_{>n}$ $\|F - \tilde{F}\|_1 \lesssim n^{\frac{d-2}{2}}$

$$n^{d-1} = \|F\|_2^2 = \langle F, F \rangle = \langle F, F - \tilde{F} \rangle$$

$$\leq \|F\|_\infty \cdot \underbrace{\|F - \tilde{F}\|}_{\lesssim n^{\frac{d-2}{2}}}$$

$$\|F\|_\infty \gtrsim n^{\frac{d}{2}} \quad \blacksquare$$

d = 2

$$\Psi = \prod_{k=0}^{n} \Big(1 + \sum_{\substack{|R|=2^{-n} \\ |R_1|=2^{-k}}} \varepsilon_R h_R\Big)$$

$$= 1 + \overbrace{\sum_{|R|=2^{-n}} \varepsilon_R h_R}^{F} + \overbrace{\text{higher order}}^{-\tilde{F}}$$

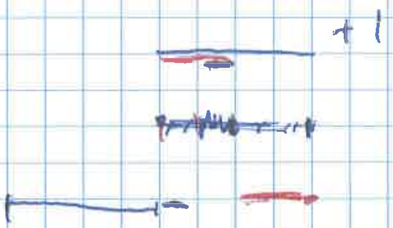
$$\|\Psi\|_1 = 1$$

$$\|F - \tilde{F}\| \leq \|\Psi\|_1 + 1 \leq 2$$

$$n^{\frac{d-2}{2}} = 1$$

$d=1$

$$\Big\| \sum_{|R| \geq 2^{-n}} \varepsilon_R h_R \Big\|_\infty \geq n+1$$



$$\sum |\alpha_R|^2 \cdot |R| =$$

$$= \Big\| \sum \alpha_R h_R \Big\|_2^2 \simeq n \gg n^{\frac{2}{3}}$$

$\Longrightarrow$ SSBI in $d=2$

$d=1$

$$\Big\| \sum_{|R| \geq 2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim \sum |\alpha_R| \cdot |R|$$

$\alpha_R = +1$ OR 0

NOT TRUE!

$n^{\frac{2}{3}}$

$$\sqrt{n} \approx \Big\| \sum \alpha_R h_R \Big\|_2$$

$$\Big\| \sum \alpha_R h_R \Big\|_\infty \lesssim n^{\frac{2}{3}}$$