

$$\mathcal{P}_N \subset [0, 1]^d \quad N\text{-pts}$$

$$\mathcal{D}_N(x) = \# \mathcal{P}_N \cap [0, x] - N x_1 \dots x_d$$

Roth $\| \mathcal{D}_N \|_2 \approx (\log N)^{\frac{d-1}{2}}$

$$\mathcal{D}_N \approx \sum_{|R| \approx \frac{1}{N}} \frac{\langle \mathcal{D}_N, h_R \rangle}{|R|} h_R$$

$$F = \sum_{\bar{r} \in H_n^d} f_{\bar{r}}$$

$$f_{\bar{r}} = \sum_{R \in \mathcal{D}_{\bar{r}}} \varepsilon_R h_R \quad \varepsilon_R = \pm 1$$

$$\mathcal{D}_{\bar{r}} = \{ R \in \mathcal{D}^d : |R_j| = 2^{-r_j} \}$$

$$H_n^d = \{ \bar{r} \in \mathbb{Z}_+^d : r_1 + \dots + r_d = n \}$$

① $\# H_n^d \approx n^{d-1}$

② $\exists f_{\bar{r}} : \langle \mathcal{D}_N, f_{\bar{r}} \rangle \geq c_d$

Ⓐ $\langle \mathcal{D}_N, F \rangle \geq n^{d-1}$

Ⓑ $\| F \|_2 \approx n^{\frac{d-1}{2}}$

$$\| \mathcal{D}_N \|_2 \approx \frac{\langle \mathcal{D}_N, F \rangle}{\| F \|_2} \approx n^{\frac{d-1}{2}}$$

Littlewood-Paley theory

Fourier series

$$f: \mathbb{T} \rightarrow \mathbb{C} \quad f \sim \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

$$\|f\|_2^2 = \sum |c_k|^2$$

$$f_{\pm} \sim \sum \pm c_k e^{2\pi i k x}$$

$$\|f_{\pm}\|_2 = \|f\|_2$$

BUT $p \neq 2 \quad \|f_{\pm}\|_p \neq \|f\|_p$

$$S^m f(x) = \left(\sum_{m=0}^{\infty} \left| \sum_{|2^m| \leq |k| < 2^{m+1}} c_k e^{2\pi i k x} \right|^2 \right)^{1/2}$$

square function

Theorem: (L^p) $\|S^m f\|_p \approx \|f\|_p \quad \forall 1 < p < \infty$

Walsh functions - almost same

$$Sf^2 = \sum_m |\Delta_m f|^2$$

Haar basis

$d=1$

+1



$$f \in L^2 \quad f \stackrel{L^2}{=} \sum_{I \in D_f} \frac{\langle f, h_I \rangle}{|I|} h_I$$

$$Sf(x) = \left(\left(\int f(x) dx \right)^2 + \sum_{I \in D} \frac{\langle f, h_I \rangle^2}{|I|^2} \mathbb{1}_I(x) \right)^{\frac{1}{2}}$$

$$= \left(\left(\int f(x) dx \right)^2 + \sum_{k=0}^{\infty} \left(\sum_{|I|=2^{-k}} \frac{\langle f, h_I \rangle}{|I|} h_I(x) \right)^2 \right)^{\frac{1}{2}}$$

$$f = \sum \alpha_I h_I$$

$$Sf = \left(\sum \alpha_I^2 \mathbb{1}_I \right)^{\frac{1}{2}}$$

Theorem: $\forall 1 < p < \infty \exists A_p, B_p > 0$

$$\left\| \mathbb{1}_I(x) \right\| = h_I^2(x)$$

$$A_p \|Sf\|_p \leq \|f\|_p \leq B_p \|Sf\|_p$$

$$B_p \approx \sqrt{p} \quad p\text{-large}$$

$$A_q = \frac{1}{B_p}$$

$$\frac{1}{q} + \frac{1}{p} = 1$$

Holds $f: [0,1] \rightarrow H$

Hilbert space



Alpenhotel Altmünster
Hauptstraße 28
4813 Altmünster
Tel: ++43/7612/87377
Fax: ++43/7612/88837
Email: alpenhotel@traunseehotels.at



Hotel Post
Ortsplatz 5
4801 Traunkirchen
Tel: ++43/7617/23070
Fax: ++43/7617/2809
Email: post@traunseehotels.at



Seehotel „Das Traunsee“
Klosterplatz 4
4801 Traunkirchen
Tel: ++43/7617/2216
Fax: ++43/7617/3496
Email: traunsee@traunseehotels.at

$d=2$

$$F(x) = \sum_{|R|=2^{-n}} \epsilon_R h_R(x) \quad \epsilon_R = \pm 1$$

$$F(x_1, \dots) = \sum \alpha_I h_I(x_1)$$

$$\alpha_I = \sum_{\substack{|R|=2^{-n} \\ R_1=I}} \epsilon_R \cdot \prod_{j=2}^n h_{R_j}(x_j)$$

$$\|F\|_q \leq B_q \left\| \left(\sum_{k=0}^n \left[\sum_{\substack{|R|=2^{-n} \\ |R_1|=2^{-k}}} \epsilon_R h_R(x) \right]^2 \right)^{\frac{1}{2}} \right\|_q$$

$d=2$

$$\|F\|_q \leq B_q \left\| \left(\sum_{k=0}^n 1 \right)^{\frac{1}{2}} \right\|_q = B_q \cdot (n+1)^{\frac{1}{2}}$$

$\frac{d-1}{2}$ ←

$d=3$

$$\vec{F}_2(x_2) = \sum_I \left\{ \sum_{\substack{|R|=2^{-n} \\ |R_1|=2^{-k}}} \varepsilon_R \prod_{j \neq 2} h_{R_j}(x_j) \right\}_{k=0}^n h_I(x_2)$$

→ $\|\vec{F}_2(x_2)\|_{l^2}$

$\|F\|_q \stackrel{\text{in } x_1}{\leq} B_q \|\vec{F}_2\|_q \leq$

$$\leq C B_q^2 \left\| \left(\sum_{k=0}^n \sum_{r=0}^n \left| \sum_{\substack{|R|=2^{-n} \\ |R_1|=2^{-k} \\ |R_2|=2^{-r}}} \varepsilon_R h_R \right|^2 \right)^{\frac{1}{2}} \right\|_q$$

$$(n^2)^{\frac{1}{2}} = n = n^{\frac{d-1}{2}}$$

$$f = \sum_{R} \alpha_R h_R$$

$$Sf = \left(\sum \alpha_R^2 \cdot 1_R \right)^{\frac{1}{2}}$$

Then $1 < p < \infty$

$$(A_p)^d \|Sf\|_p \leq \|f\|_p \leq (B_p)^d \cdot \|Sf\|_p$$

"Hyperbolic" $\underline{L-p}$

$$f = \sum_{|R|=2^{-n}} \alpha_R h_R$$

$$(A_p)^{d-1} \|Sf\|_p \leq \|f\|_p \leq (B_p)^{d-1} \|Sf\|_p$$

\uparrow
 $\approx p^{\frac{d-1}{2}}$

$$F = \sum_{|R|=2^{-n}} \epsilon_R h_R \quad \epsilon_R = \pm 1$$

$$\|F\|_p$$

$$1 < p < \infty$$

$$\approx \left\| \left(\sum_{|R|=2^{-n}} \epsilon_R^2 I_R(x) \right)^{\frac{1}{2}} \right\|_p$$

$$\text{const} = \# \mathcal{H}_n^d \approx n^{d-1}$$

$$\approx n^{\frac{d-1}{2}}$$

$$p=1$$

$$\|F\|_1 \leq \|F\|_2 \approx n^{\frac{d-1}{2}}$$

$$\|F\|_4 \approx n^{\frac{d-1}{2}}$$

$$n^{\frac{d-1}{2}} \approx \|F\|_2 \leq \|F\|_1^{\frac{1}{3}} \quad \|F\|_4^{\frac{2}{3}} \leq \|F\|_1^{\frac{1}{3}} \quad n^{\frac{d-1}{2} \cdot \frac{2}{3}} \leq n^{\frac{d-1}{3}}$$

$$\circ n^{\frac{d-1}{2} \cdot \frac{2}{3}}$$

$$\|F\|_1 \approx n^{\frac{d-1}{2}}$$

$$\|F\|_1 \approx n^{\frac{d-1}{2}}$$



Alpenhotel Altmünster
Hauptstraße 28
4813 Altmünster
Tel: ++43/7612/87377
Fax: ++43/7612/88837
Email: alpenhotel@traunseehotels.at



Hotel Post
Ortsplatz 5
4801 Traunkirchen
Tel: ++43/7617/23070
Fax: ++43/7617/2809
Email: post@traunseehotels.at



Seehotel „Das Traunsee“
Klosterplatz 4
4801 Traunkirchen
Tel: ++43/7617/2216
Fax: ++43/7617/3496
Email: traunsee@traunseehotels.at

"Signed" small ball conjecture

$$\uparrow \quad \|F\|_{\infty} \gtrsim n^{d/2}$$

"Dual" SSBC

$$\mathcal{H}_{>n} = \text{span} \{ h_R : |R| < 2^{-n} \}$$

$$\text{dist}_{L^1}(F, \mathcal{H}_{>n}) \lesssim n^{\frac{d-2}{2}}$$

$$\exists \tilde{F} \in \mathcal{H}_{>n} \quad \|F - \tilde{F}\|_1 \lesssim n^{\frac{d-2}{2}}$$

$$n^{d-1} = \|F\|_2^2 = \langle F, F \rangle = \langle F, F - \tilde{F} \rangle$$

$$\leq \|F\|_{\infty} \cdot \|F - \tilde{F}\|$$

$$\lesssim n^{\frac{d-2}{2}}$$

$$\|F\|_{\infty} \gtrsim n^{\frac{d-2}{2}} \quad \square$$

$$\underline{d=2}$$

$$\Psi = \prod_{k=0}^n \left(1 + \sum_{\substack{|R|=2^{-k} \\ |R|=2^{-k}}} \varepsilon_R h_R \right)$$

$$= \underbrace{1 + \sum_{|R|=2^{-n}} \varepsilon_R h_R}_F + \underbrace{\text{higher order}}_{\tilde{F}}$$

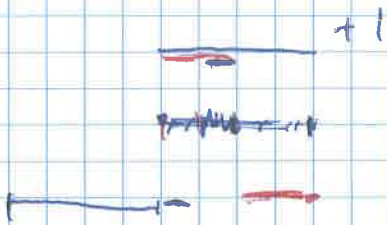
$$\|\Psi\|_1 = 1$$

$$\|F - \tilde{F}\| \leq \|\Psi\|_1 + 1 \leq 2$$

$$n^{\frac{d-2}{2}} = 1$$

$d=1$

$$\left\| \sum_{|R| \geq 2^{-n}} \epsilon_R h_R \right\|_{\infty} \geq n+1$$



$$\sum_{R \in \mathbb{R}} |\alpha_R|^2 |R| = \left\| \sum \alpha_R h_R \right\|_2^2 \approx n \rightarrow n^{2/3}$$

\Rightarrow SSBI in $d=2$

$d=1$

$$\left\| \sum_{|R| \geq 2^{-n}} \alpha_R h_R \right\|_{\infty} \gtrsim \sum |\alpha_R| \cdot |R|$$

$\alpha_R = +1$ or 0

NOT TRUE!

$n^{2/3}$

$$\sqrt{n} \approx \left\| \sum \alpha_R h_R \right\|_2$$

$$\left\| \sum \alpha_R h_R \right\|_{\infty} \approx n^{2/3}$$