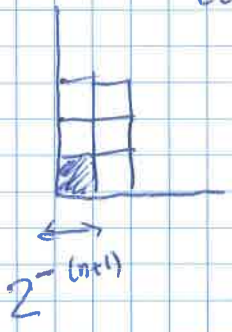
WEDNESDAY 2.12

Sharpness of signed SBC

$$\exists\, \varepsilon_R = \pm 1 \; : \quad \Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \lesssim n^{d/2}$$

$$\varepsilon_R = \text{iid } \pm 1$$

cubes $Q_1, Q_2, \dots, Q_M \qquad M = 2^{(n+1)d}$



$$X_k = \sum \varepsilon_R h_R \Big|_{Q_k}$$

X_k – a sum of $\approx n^{d-1}$ iid ± 1

$$\mathbb{E}\, |X_k| \approx n^{\frac{d-1}{2}}$$

$$\boxed{\mathbb{E} \max_{1 \le k \le M} |X_k| \lesssim \mathbb{E}\, |X_k| \cdot \sqrt{\log M}}$$

HOEFFDING INEQUALITY

$$\mathbb{P}\left(|X_k| > t\right) \le 2 \exp\left(-\frac{t^2}{c n^{d-1}}\right)$$

$\Downarrow$

$$\mathbb{E}\left(\exp(tX)\right) \le \exp\left(c n^{d-1} t^2\right)$$

Jensen

$$\exp\Big(t\,\mathbb{E}\max_{1\le k\le M}|X_k|\Big) \le \mathbb{E}\max_{1\le k\le M}\exp(t|X_k|)$$

$$\le \mathbb{E}\sum_{k=1}^{M}\exp(t|X_k|)$$

$$\le 2M\exp(cn^{d-1}t^2)$$

$$\mathbb{E}\max_{1\le k\le M}|X_k| \le \frac{1}{t}\cdot\log 2M + c\cdot n^{d-1}\cdot t$$

$$t^2 = \frac{\log 2M}{cn^{d-1}} \qquad t = \frac{\sqrt{\log 2M}}{c'\,n^{\frac{d-1}{2}}}$$

$$\mathbb{E}\max_{1\le k\le M}|X_k| \lesssim \sqrt{\log(2M)}\cdot n^{\frac{d-1}{2}}$$

$$M = 2^{(n+1)d}$$

$$\log M \approx n$$

in our case

$$\mathbb{E}\,\|\Sigma\,\varepsilon_R h_R\|_\infty \lesssim$$

$$\lesssim \sqrt{n}\cdot n^{\frac{d-1}{2}} = n^{d/2}$$

(Exponential) Orlicz spaces.

$\psi : \mathbb{R}_+ \to \mathbb{R}_+$ $\quad \psi \geq 0 \quad \psi(0) = 0 \quad$ convex

$$\|f\|_{L^\psi} = \inf \left\{ K > 0 : \int_\Omega \psi\left(\frac{|f(x)|}{K}\right) dx \leq 1 \right\}$$

$\psi(x) = x^p \quad p \geq 1 \quad L^\psi = L^p$

$\psi(x) = e^{x^\alpha} - 1 :$ $\exp(L^\alpha)$ – exponential Orlicz space

$\alpha = 2$ subgaussian functions

$$\|f\|_{\exp(L^\alpha)} \overset{\int f = 0}{\approx} \sup_{t > 0} \frac{\log \mathbb{P}(|f| > t)}{-t^{+\alpha}}$$

$$\approx \sup_{p > 1} p^{-\frac{1}{\alpha}} \cdot \|f\|_p$$

$L^\infty \subset \exp(L^\alpha) \subset L^p$

$L^1 \supset L(\log L)^\beta \supset L^p \qquad p > 1$

$$F = \sum_{|R|=2^{-n}} \varepsilon_R \, h_R$$

$$SF = const \approx n^{\frac{d-1}{2}}$$

Littlewood – Paley (hyperbolic)

$$\|F\|_p \lesssim p^{\frac{d-1}{2}} \|SF\|_p$$

$$\le p^{\frac{d-1}{2}} \, n^{\frac{d-1}{2}}$$

$$\|F\|_{\exp(L^{\frac{2}{d-1}})} \lesssim n^{\frac{d-1}{2}}$$

$$\Downarrow$$

$$\|\mathcal{D}_N\|_{L\cdot(\log L)^{\frac{d-1}{2}}} \gtrsim (\log N)^{\frac{d-1}{2}}$$

$$\psi(x) = x\cdot(1+\log \underbrace{\cancel{x}}_{|x+1|})^{\frac{d-1}{2}}$$

1dim d-1 dim

$$X = (x', \tilde{x})$$

$$R = R' \times \tilde{R}$$

$$\boxed{\varepsilon_R = \varepsilon_{R'} \cdot \varepsilon_{\tilde{R}}}$$

Theorem (Karslidis, 15)

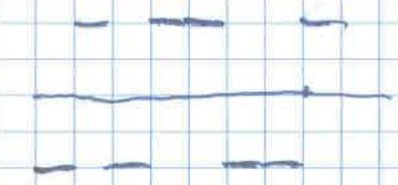
$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}$$

Fix $\tilde{x}$ 1-dim

$$\Big\| \sum \varepsilon_R h_R \Big\|_{L^\infty(x')} = \Big\| \sum_{|R'| \geq 2^{-n}} \Big[\sum \varepsilon_{\tilde{R}} h_{\tilde{R}}(\tilde{x}) \Big] \overset{\varepsilon_{R'}}{h_{R'}(x')} \Big\|_\infty$$

$$\uparrow \quad |\tilde{R}| = \frac{2^{-n}}{|R'|}$$

$$= \Big\| \sum_{k=0}^{n} \sum_{|R'|=2^{-k}} \Big[\sum_{|\tilde{R}|=2^{-(n-k)}} \varepsilon_{\tilde{R}} h_{\tilde{R}}(\tilde{x}) \Big] \varepsilon_{R'} h_{R'}(x') \Big\|_\infty$$



iid ±1
for $k = 0, 1, \dots, n$

$$= \sum_{k=0}^{n} \Big| \sum_{|\tilde{R}|=2^{-k}} \varepsilon_{\tilde{R}} h_{\tilde{R}}(\tilde{x}) \Big|$$

iid ±1

$$\Big\| \sum \alpha_k \overset{\downarrow}{r_k} \Big\|_\infty = \sum_i |\alpha_k|$$

$$\Big\| \sum \varepsilon_R h_R \Big\|_\infty = \Big\| \sum_{k=0}^{n} \Big| \sum_{|\tilde R| = 2^{-k}} \varepsilon_{\tilde R} h_{\tilde R}(\tilde x) \Big| \Big\|_{L^\infty(\tilde x)}$$

$$\geq \Big\| \sum \big| \quad -''- \quad \big| \Big\|_{L^1(\tilde x)}$$

$$= \sum_{k=0}^{n} \Big\| \sum_{|\tilde R| = 2^{-k}} \varepsilon_{\tilde R} h_{\tilde R}(\tilde x) \Big\|_{L^1}$$

$$\gtrsim \sum_{k=0}^{n} \Big\| \qquad \Big\|_{L^2}$$

$$\approx \sum_{k=0}^{n} k^{\frac{d-2}{2}} \approx n^{\frac{d-2}{2}} \cdot n$$

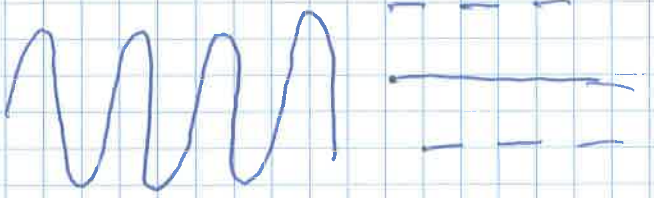
$$= n^{d/2} \quad ▨$$

Sidon

1) $\| \underbrace{\sum_i a_k \sin(2\pi 2^k x)}_{f} \|_\infty \gtrsim$

$\gtrsim \sum |a_k|$

2) $\|f\|_1 \gtrsim \|f\|_2$



Skriganov 2014

For any $\mathcal{P}_N \subset [0,1]^d$

there exist a digit shift T

$\mathcal{P}_N \oplus T$ satisfies

$$\mathbb{E}_T \, \| D_{\mathcal{P}_N \oplus T} \|_\infty \gtrsim \underbrace{(\log N)^{d/2}}$$

Gilbert – Varshamov bound

$\mathcal{C}$ – binary code of length m: $\mathcal{C} \subset \{0,1\}^m$

Hamming distance:

$$d(x,y) = \#\{1 \le i \le m : x_i \neq y_i\}$$

Lemma Let C be the maximal binary code of length m with min. Hamming distance d.

Then
$$\#C \geq \frac{2^m}{\sum_{k=0}^{d} \binom{m}{k}}$$

$$B_H(x,d) = \{y \in \{0,1\}^m : d_H(x,y) \le d\}$$

$$\#B_H = \sum_{k=0}^{d} \binom{m}{k}$$

$$\#C \cdot \#B_H(x,d) \geq 2^m$$