METHODS OF HARMONIC ANALYSIS IN DISCREPANCY THEORY.

Lecture 3b.

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The small ball inequality

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$n^{\frac{d-2}{2}} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

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Signed Small Ball Conjecture

For dimensions $d \geq 2$, if all $\varepsilon_R = \pm 1$, we have

$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d}{2}}$$

Let $B : [0,1]^d \longrightarrow \mathbb{R}$ be the Brownian Sheet, i.e. a centered Gaussian process with covariance $\mathbb{E}B(s)B(t) = \prod_{k=1}^{d} \min\{s_k, t_k\}$.

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- Lower bound is known in $d = 2$ (Talagrand)

- Let $\{u_k\}$ be an orthonormal basis of $L^2([0,1]^d)$.

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- Then

$$\mathbb{P}(\|B\|_\infty < \epsilon) \leq \mathbb{P}\left(\Big\| \sum_{|R|=2^{-n}} g_R \eta_R \Big\|_\infty < \epsilon \right)$$
$$\leq \mathbb{P}\left(2^{-3n/2} n^{-\frac{1}{2}(d-2)} \sum_{|R|=2^{-n}} |g_R| < \epsilon \right)$$

Metric entropy of mixed smoothness classes

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- $N_p(\epsilon) := \min\{N : \exists x_1, \dots, x_N \ s.t. \ \mathrm{M}_p \subset \cup_{k=0}^{N}(x_k + \epsilon B_\infty)\}$ – least number of L^∞ balls of radius ϵ needed to cover M_p

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Theorem (Kuelbs, Li)

$$-\log \mathbb{P}(\|B\|_{C[0,1]^d} < \epsilon) \approx \epsilon^{-2} \left(\log \tfrac{1}{\epsilon}\right)^{\beta} \quad iff$$
$$\log N_2(\epsilon) \approx \epsilon^{-1} \left(\log \tfrac{1}{\epsilon}\right)^{\beta/2}$$

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Conjecture

For $d \geq 2$, one has the estimate $\log N_2(\epsilon) \gtrsim \frac{1}{\epsilon} \left(\log \frac{1}{\epsilon}\right)^{d-1/2}$

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- Then

$$\|F_\sigma - F_{\sigma'}\|_\infty \gtrsim n^{-d+3/2} 2^{-2n} \sum_{|R|=2^{-n}} |\sigma_R - \sigma'_R|$$

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- One can choose *many* σ's for which this sum is *large* (Varshamov-Gilbert bound)

Connections between problems

Discrepancy estimates

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

Small Ball Conjecture

$$n^{\frac{d-2}{2}} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{R:\,|R|=2^{-n}} |\alpha_R|.$$

lower bound

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Small deviations for the Brownian sheet

$$-\log \mathbb{P}(\|B\|_{C([0,1]^d)} < \varepsilon) \simeq \varepsilon^{-2}(\log 1/\varepsilon)^{2d-1}$$

Metric entropy of MW^2

$$\log N(\varepsilon, 2, d) \simeq \varepsilon^{-1}(\log 1/\varepsilon)^{d-1/2}$$

Kuelbs, Li, '93

Near L^∞: BMO and $\exp(L^\alpha)$ estimates

Theorem (DB, Lacey, Parissis, Vagharshakyan, 2009)

- *For any N-point set $\mathcal{P}_N \subset [0,1]^2$ we have*

$$\|D_N\|_{\mathrm{BMO}} \gtrsim \sqrt{\log N}$$

- *The van der Corput set satisfies*

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$$\|D_N\|_{\exp(L^\alpha)} \gtrsim (\log N)^{1-1/\alpha}, \qquad 2 \le \alpha < \infty.$$

- *The digit-scrambled van der Corput set satisfies*

$$\|D_N\|_{\exp(L^\alpha)} \lesssim (\log N)^{1-1/\alpha}, \qquad 2 \le \alpha < \infty.$$

Exponential estimates in higher dimensions

Theorem (DB, Markhasin (2014))

There exist sets $\mathcal{P}_N \subset [0,1]^d$ (averages of "linear digital nets") for which

$$\|D_N\|_{\exp\left(L^{\frac{2}{d-1}}\right)} \lesssim (\log N)^{\frac{d-1}{2}}$$

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- This would imply that

$$\mu\{x:\, D_N(x) \geq (\log N)^{d/2}\} \lesssim N^{-c}.$$

Estimates "near" L^1

Theorem (Lacey, 2010)

$$\|D_N\|_{L(\log L)^{\frac{d-2}{2}}} \gtrsim (\log N)^{\frac{d-1}{2}}.$$

- $L(\log L)^{\frac{d-1}{2}}$ is "easy"

Theorem (Lacey, 2010)

For $0 < p \leq 1$ we have the estimate in the (dyadic) d-parameter Hardy space

$$\|D_N\|_{H^p} \gtrsim (\log N)^{\frac{d-1}{2}}.$$

Other endpoint: L^1

Theorem (Halász, 1981)

In dimension $d = 2$ for any collection of N points $\mathcal{P}_N \subset [0,1]^2$

$$\|D_N\|_1 \gtrsim \sqrt{\log N}.$$

- $C_1 \geq 0.00854...$ (Vagharshakyan, 2013)
- This continues to hold for $d \geq 3$: $\|D_N\|_1 \gtrsim \sqrt{\log N}$
- ... nothing better is known in higher dimensions!
- Conjecture:

$$\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$$

- Known:
  $L(\log L)^{\frac{d-2}{2}}$ and H^p, $0 < p \leq 1$, norms satisfy this estimate

L^1 "dichotomy" results

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Theorem (Amirkhanyan, DB, Lacey, 2013)

- *If* $\mathcal{P}_N \subset [0,1]^d$ *satisfies* $\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$ *for some* $1 < p < \infty$, *then*

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$$\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$$

- *Every* $\mathcal{P}_N \subset [0,1]^d$ *satisfies either*

$$\|D_N\|_1 \gtrsim (\log N)^{(d-1)/2-\epsilon} \quad \text{or} \quad \|D_N\|_2 \gtrsim \exp(c(\log N)^{\epsilon})\,.$$

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$$\|D_N\|_1 \gtrsim (\log N)^{(d-1)/2-\epsilon} \quad \text{or} \quad \|D_N\|_2 \gtrsim \exp\left(c(\log N)^{\epsilon}\right).$$
- *For* $d \geq 3$, *if* $\mathcal{P}_N \subset [0,1]^d$ *satisfies* $\|D_N\|_1 \lesssim \sqrt{\log N}$, *then*
$$\|D_N\|_2 \gtrsim N^C.$$

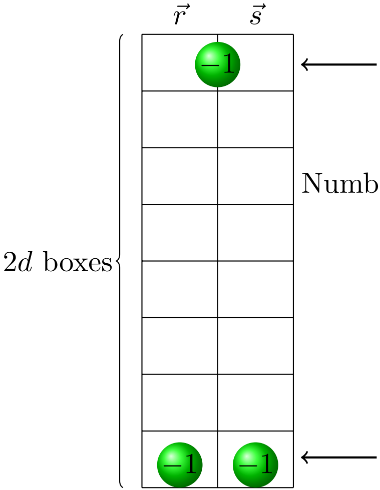
"Beck Gain" lemma: preservation of orthogonality

Lemma

Beck Gain: We have the estimate

$$\Big\| \sum_{\substack{\vec{r} \neq \vec{s} \in \mathbb{H}_n^d \\ r_1 = s_1}} f_{\vec{r}} \cdot f_{\vec{s}} \Big\|_p \lesssim p^{(2d-1)/2} n^{(2d-3)/2}$$

Number of parameters



a coincidence

Number of parameters

$$= 2d - 1 - 1 - 1$$

hyperbolicity