(1.2.4) Tilings of the Walsh phase plane

Let $b \geq 2$ be an integer, let $k = k_0 + k_1 b + \dots + k_{m-1} b^{m-1} \in \mathbb{N}_0$ and $x = x_c b^c + x_{c-1} b^{c-1} + \dots \in \mathbb{R}$ (unique in the sense that infinitely many of the digits are different from $b-1$).

We define the k-th Walsh function by

$$wal_k(x) = \omega_b^{k_0 x_{-1} + k_1 x_{-2} + \dots + k_{m-1} x_{-m}} \mathbb{1}_{[0,1)}(x),$$

where $\omega_b = e^{2\pi i / b}$.

We define translations and dilations of wal_k by

$$W_{j,k,l}(x) = b^{-j/2} \, wal_k(b^{-j} x - l), \quad j, l \in \mathbb{Z}, \; k \in \mathbb{N}_0.$$

The support of the function $W_{j,k,l}$ is $[b^{j} l, b^{j}(l+1)]$.

The system $\{W_{j,k,l} : k \in \mathbb{N}_0, \; j, l \in \mathbb{Z}\}$ is overdetermined in $L_2(\mathbb{R})$. However we can identify a subset which is a complete orthonormal system in $L_2(\mathbb{R})$.

To each function $w_{j,k,l}$ we associate a corresponding tile $T_{j,k,l}$

$$T_{j,k,l} = \underbrace{[b^{j} l, b^{j}(l+1))}_{\text{"support"}} \times \underbrace{[b^{-j} k, b^{-j}(k+1))}_{\text{"frequency"}} .$$

Lemma: Let j, j', k, k', l, l' be integers such that $k, k' \geq 0$. Then

$$\int_{\mathbb{R}} w_{j,k,l}(x)\, w_{j',k',l'}(x)\, dx = 0 \iff T_{j,k,l} \cap T_{j',k',l'} = \emptyset .$$

Proof idea: If the supports don't intersect then integral is 0, and if the supports intersect but the frequencies are different, then the integral is 0. Note $\mathrm{wal}_k(\{b^a x\}) = \mathrm{wal}_{k b^a}(x)$

Lemma: Let $\mathcal{T}$ and $\mathcal{T}'$ be two finite set of tiles, such that all pairs of tiles in $\mathcal{T}$ are disjoint and also all pairs of tiles in $\mathcal{T}'$ are disjoint. Let W and W' be the corresponding Walsh functions. Then

$$\bigcup_{T \in \mathcal{T}} T = \bigcup_{T' \in \mathcal{T}'} T' \iff \operatorname{span} W = \operatorname{span} W' .$$

Lemma: Let $\mathcal{T}$ be a set of Tiles such that

(i) $\forall T, T' \in \mathcal{T} : T \cap T' = \phi$,

(ii) $$\bigcup_{T \in \mathcal{T}} T = \mathbb{R} \times \mathbb{R}_0^+ .$$

Then the corresponding system W of Walsh functions is a complete orthonormal system of $L_2(\mathbb{R})$.

Parseval's identity implies that if we have two disjoint tilings which cover the same are, the sums of the correspondings squares of the Walsh coefficients coincide.

(1.3) Numerical integration

Let $f:[0,1)^s \to \mathbb{R}$ be in the reproducing kernel Hilbert space $\mathcal{H}_\alpha$ with kernel

$$K_\alpha(\underline{x},\underline{y}) = \prod_{j=1}^{s} K_\alpha(x_j, y_j), \qquad (1.3.1a)$$

and

$$K_\alpha(x_j, y_j) = 1 + \frac{B_1(x)}{1!}\frac{B_1(y)}{1!} + \cdots + \frac{B_\alpha(x)}{\alpha!}\frac{B_\alpha(y)}{\alpha!} + (-1)^{\alpha+1}\frac{B_{2\alpha}(|x-y|)}{(2\alpha)!}.$$

We can represent $f \in \mathcal{H}_\alpha$ by its Walsh series

$$f(\underline{x}) = \sum_{\underline{k}\in\mathbb{N}_0^s} \hat{f}(\underline{k})\, \mathrm{wal}_{\underline{k}}(\underline{x}).$$

Then

$$e_P(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(\underline{x}_n) - \underbrace{\int_{[0,1)^s} f(\underline{x})\, d\underline{x}}_{=\hat{f}(\underline{0})} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{\underline{k}\in\mathbb{N}_0^s} \hat{f}(\underline{k})\,\mathrm{wal}_{\underline{k}}(\underline{x}) - \hat{f}(\underline{0})$$

$$= \sum_{\underline{k}\in\mathbb{N}_0^s\setminus\{\underline{0}\}} \hat{f}(\underline{k}) \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n),$$

where $P = \{\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{N-1}\} \subseteq [0,1)^s$.

Hence

$$|e(f)| \leq \sum_{\underline{k} \in \mathbb{N}_0^s \setminus \{\underline{0}\}} |\hat{f}(\underline{k})| \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right| .$$

Using a bound on the Walsh coefficients

~~we obtain~~

$$|\hat{f}(\underline{k})| \leq C_{\alpha,s}\, b^{-\mu_\alpha(\underline{k})} \|f\|_{\mathcal{H}_\alpha}$$

we obtain

$$|e(f)| \leq C_{\alpha,s} \|f\|_{\mathcal{H}_\alpha} \sum_{\underline{k} \in \mathbb{N}_0^s \setminus \{\underline{0}\}} b^{-\mu_\alpha(\underline{k})} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right|$$

Thus we want to construct point sets for which

$$\sum_{\underline{k} \in \mathbb{N}_0^s \setminus \{\underline{0}\}} b^{-\mu_\alpha(\underline{k})} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right|$$

converges rapidly to 0.

Another approach is the following. Consider the reproducing kernel K_α. The worst-case error is given by

$$e^2(P,\mathcal{H}_\alpha) = \int_{[0,1]^s}\int_{[0,1]^s} K_\alpha(\underline{x},\underline{y})\, d\underline{x}\, d\underline{y} - \frac{2}{N}\sum_{n=0}^{N-1}\int_{[0,1]^s} K_\alpha(\underline{x}_n,\underline{y})\, d\underline{y} + \frac{1}{N^2}\sum_{n,m=0}^{N-1} K(\underline{x}_n,\underline{x}_m)$$

Now we use the Walsh series expansions of K_α

$$K_\alpha(\underline{x},\underline{y}) = \sum_{\underline{k},\underline{\ell}\in\mathbb{N}_0^s} \hat{K}_\alpha(\underline{k},\underline{\ell})\, \mathrm{wal}_{\underline{k}}(\underline{x})\, \overline{\mathrm{wal}_{\underline{\ell}}(\underline{y})},$$

where

$$\hat{K}_\alpha(\underline{k},\underline{\ell}) = \int_{[0,1]^s}\int_{[0,1]^s} K_\alpha(\underline{x},\underline{y})\, \overline{\mathrm{wal}_{\underline{k}}(\underline{x})}\, \mathrm{wal}_{\underline{\ell}}(\underline{y})\, d\underline{x}\, d\underline{y},$$

to obtain

$$e^2(P,\mathcal{H}_\alpha) = \sum_{\underline{k},\underline{\ell}\in\mathbb{N}_0^s} \hat{K}_\alpha(\underline{k},\underline{\ell}) \int_{[0,1]^s}\int_{[0,1]^s} \mathrm{wal}_{\underline{k}}(\underline{x})\, d\underline{x}\, \overline{\mathrm{wal}_{\underline{\ell}}(\underline{y})}\, d\underline{y}$$

$$- \frac{2}{N}\sum_{n=0}^{N-1}\sum_{\underline{k},\underline{\ell}\in\mathbb{N}_0^s} \hat{K}_\alpha(\underline{k},\underline{\ell})\, \mathrm{wal}_{\underline{k}}(\underline{x}_n) \int_{[0,1]^s} \overline{\mathrm{wal}_{\underline{\ell}}(\underline{y})}\, d\underline{y}$$

$$+ \frac{1}{N^2}\sum_{n,m=0}^{N-1}\sum_{\underline{k},\underline{\ell}\in\mathbb{N}_0^s} \hat{K}_\alpha(\underline{k},\underline{\ell})\, \mathrm{wal}_{\underline{k}}(\underline{x}_n)\, \overline{\mathrm{wal}_{\underline{\ell}}(\underline{x}_m)}.$$

If we use the space with reproducing kernel K_α given by (B.1a), then

$$\int_{[0,1]^s} K(\underline{x}, \underline{y}) \, d\underline{y} = 1,$$

and therefore

$$e^2(P, \mathcal{H}_\alpha) = -1 + \frac{1}{N^2} \sum_{n,m=0}^{N-1} K_\alpha(\underline{x}_n, \underline{x}_m).$$

Using the Walsh series expansion we have

$$e^2(P, \mathcal{H}_\alpha) = \sum_{\substack{\underline{k}, \underline{\ell} \in \mathbb{N}_0^s \\ (\underline{k}, \underline{\ell}) \neq (\underline{0}, \underline{0})}} \hat{K}_\alpha(\underline{k}, \underline{\ell}) \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \frac{1}{N} \sum_{m=0}^{N-1} \overline{\mathrm{wal}_{\underline{\ell}}(\underline{x}_m)}.$$

Bounds on the Walsh coefficients $\hat{K}(\underline{k}, \underline{\ell})$ can be obtained as before.

The following inequality

$$|\hat{K}_\alpha(\underline{k}, \underline{\ell})|^2 \leq |\hat{K}_\alpha(\underline{k}, \underline{k})| \, |\hat{K}_\alpha(\underline{\ell}, \underline{\ell})|$$

yields

$$e^2(P, \mathcal{H}_\alpha) \leq \sum_{\underline{k}, \underline{\ell} \in \mathbb{N}_0^s} \sqrt{|\hat{K}_\alpha(\underline{k}, \underline{k})|} \sqrt{|\hat{K}_\alpha(\underline{\ell}, \underline{\ell})|} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \frac{1}{N} \sum_{m=0}^{N-1} \overline{\mathrm{wal}_{\underline{\ell}}(\underline{x}_m)}$$

$$= \left| \sum_{\underline{k} \in \mathbb{N}_0^s} \sqrt{|\hat{K}_\alpha(\underline{k}, \underline{k})|} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right|^2.$$

Thus

$$e(P, \mathcal{H}_\alpha) \le \sum_{\underline{k} \in \mathbb{N}_0^s} \sqrt{\hat{K}_\alpha(\underline{k}, \underline{k})} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right| .$$

This simplifies the analysis but yields error bounds where the exponent of $\log N$ is not optimal. It only works for $\alpha \geq 2$, otherwise the sum does not converge.

To get better bounds one can study the Walsh coefficients of $\hat{K}_\alpha(\underline{k}, \underline{l})$ directly. Note that since $K_\alpha(\underline{x}, \underline{y}) = \prod_{j=1}^{s} K_\alpha(x_j, y_j)$ it is enough to study the one-dimensional case, i.e.,

$$\hat{K}_\alpha(\underline{k}, \underline{l}) = \prod_{j=1}^{s} \hat{K}_\alpha(k_j, l_j).$$

Thus one needs to study

$$\hat{K}_\alpha(k, l) = \int_0^1 \int_0^1 \left[1 + \frac{B_1(x)}{1!} \frac{B_1(y)}{1!} + \dots + \frac{B_\alpha(x)}{\alpha!} \frac{B_\alpha(y)}{\alpha!} + (-1)^{\alpha+1} \frac{B_{2\alpha}(|x-y|)}{(2\alpha)!} \right] \overline{\mathrm{wal}_k(x)\, \mathrm{wal}_l(y)} \, dx \, dy. \quad (1.3.a)$$

The Bernoulli polynomials are

$B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$, $B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x$,

$B_6(x) = x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}$, ...

To calculate the Walsh coefficients of K_α, we need to compute $\int_0^1 B_r(x)\,\overline{\mathrm{wal}_k(x)}\,dx$. This is just the Walsh coefficient of a polynomial which can be done exactly (for small values of α). Further we need

$$\int_0^1\int_0^1 B_{2\alpha}(|x-y|)\,\overline{\mathrm{wal}_k(x)}\,\mathrm{wal}_\ell(y)\,dx\,dy.$$

The Bernoulli polynomial $B_{2\alpha}$ has the following form

$$B_{2\alpha}(|x-y|) = (x-y)^{2\alpha} - \alpha|x-y|^{2\alpha-1} + c_{2\alpha-2}^{(2\alpha)}(x-y)^{2\alpha-2}$$
$$+ c_{2\alpha-4}^{(2\alpha)}(x-y)^{2\alpha-4} + \dots + c_2^{(2\alpha)}(x-y)^2 + c_0^{(2\alpha)}.$$

The only term which is not a polynomial is $|x-y|^{2\alpha-1}$. We can write

$$|x-y|^{2\alpha-1} = (x-y)^{2\alpha-2}\,|x-y|.$$

So one only needs to study

$$\int_0^1\int_0^1 |x-y|\,\overline{\mathrm{wal}_k(x)}\,\mathrm{wal}_\ell(y)\,dx\,dy,$$

which can be done as before.

(1.4) Construction of point sets

(1.4.1) Digital nets and sequences

Let b be a prime number and $\mathbb{Z}_b = \{0, 1, \dots, b-1\}$ equipped with addition and multiplication modulo b. For instance for $b=2$, we have $0+0=1+1=0$, $0+1=1+0=1$, $0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$, $1 \cdot 1 = 1$.

Let $C_j \in \mathbb{Z}_b^{m \times m}$ for $j = 1, 2, \dots, s$ be $m \times m$ matrices.

We want to construct b^m points in $[0,1)^s$. Let $n \in \{0, 1, \dots, b^m - 1\}$ be given by its base b expansion

$$n = n_0 + n_1 b + \dots + n_{m-1} b^{m-1}, \qquad n_0, n_1, \dots, n_{m-1} \in \mathbb{Z}_b.$$

Form the vector $\vec{n} = \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_{m-1} \end{pmatrix}$ and compute

$$\begin{pmatrix} y_{j,n,1} \\ y_{j,n,2} \\ \vdots \\ y_{j,n,m} \end{pmatrix} = \vec{y}_{j,n} = C_j \vec{n}, \qquad 1 \le j \le s.$$

Then set $x_{j,n} = \frac{y_{j,n,1}}{b} + \frac{y_{j,n,2}}{b^2} + \dots + \frac{y_{j,n,m}}{b^m}$ for $j = 1, \dots, s$. and $\underline{x}_n = (x_{1,n}, x_{2,n}, \dots, x_{s,n}) \in [0,1)^s$. The point set $\{\underline{x}_0, \underline{x}_1, \dots, x_{b^m-1}\}$ is called a digital net.

We can also construct infinite sequences of points.
Let $C_j \in \mathbb{Z}_b^{\mathbb{N}\times\mathbb{N}}$, that is

$$C_j = (c_{j,k,\ell})_{k,\ell\in\mathbb{N}} = \begin{pmatrix} c_{j,1,1} & c_{j,1,2} & c_{j,1,3} & \cdots \\ c_{j,2,1} & c_{j,2,2} & c_{j,2,3} & \cdots \\ c_{j,3,1} & c_{j,3,2} & c_{j,3,3} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Write $n \in \mathbb{N}_0$ in its base b expansion

$$n = n_0 + n_1 b + n_2 b^2 + \cdots, \quad n_0, n_1, \ldots \in \mathbb{Z}_b.$$

Then compute

$$\vec{y}_{j,n} = C_j \vec{n} = \begin{pmatrix} n_0 c_{j,1,1} + n_1 c_{j,1,2} + \cdots \\ n_0 c_{j,2,1} + n_1 c_{j,2,2} + \cdots \\ n_0 c_{j,3,1} + n_1 c_{j,3,2} + \cdots \\ \vdots \end{pmatrix}, \quad 1 \le j \le s.$$

Note that the digit expansion of n is finite, hence $n_0 c_{j,k,1} + n_1 c_{j,k,2} + \cdots$ is only a finite sum.

(1.4.2) Geometric Properties

A point set $P = \{\underline{x}_0, \underline{x}_1, \dots, \underline{x}_{b^m-1}\}$ is called a (t,m,s)-net in base b if for every elementary interval

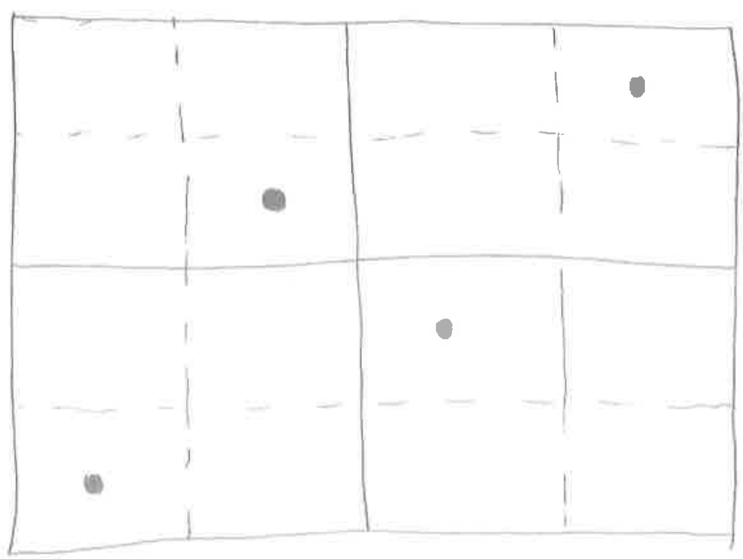
$$J = \prod_{j\in I}^{s} \left[\frac{a_j}{b^{d_j}}, \frac{a_j+1}{b^{d_j}}\right), \quad 0 \le a_j < b^{d_j},$$

$d_1, \dots, d_s \ge 0$ and $d_1 + d_2 + \dots + d_s = m - t$, we have

$$|P \cap J| = b^t.$$

This implies that

$$\frac{|P\cap J|}{|P|} = \mathrm{Vol}(J).$$



A sequence $\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots$ is called a (t,s)-sequence in base b is for all $m \ge t$ and all $k \in \mathbb{N}$ we have that the point set

$$\underline{x}_{(k-1)b^m}, \underline{x}_{(k-1)b^m+1}, \dots, \underline{x}_{kb^m-1}$$

is a (t,m,s)-net.

<u>Lemma</u>: A digital net $\{\underline{x}_0, \underline{x}_1, \dots, \underline{x}_{b^m-1}\}$ with generating matrices $C_1, \dots, C_s \in \mathbb{Z}_b^{m \times m}$ is a (t,m,s)-net (in base b prime) if

$$\underline{c}_{1,1}, \underline{c}_{1,2}, \dots, \underline{c}_{1,d_1}, \dots, \underline{c}_{s,1}, \underline{c}_{s,2}, \dots, \underline{c}_{s,d_s},$$

where $\underline{c}_{j,k}$ is the k-th row of $C_j = \begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jm} \end{pmatrix}$, are linerly independent over $\mathbb{Z}_b$ for all $d_1, \dots, d_s \geq 0$ such that $d_1 + d_2 + \dots + d_s = m - t$.

<u>Proof</u>: Let an elementary interval J be given by

$$J = \prod_{j=1}^{s} \left[\frac{a_{j,1}}{b} + \frac{a_{j,2}}{b^2} + \dots + \frac{a_{j,d_j}}{b^{d_j}}, \frac{a_{j1}}{b} + \frac{a_{j2}}{b^2} + \dots + \frac{a_{jd_j}}{b^{d_j}} + \frac{1}{b^{d_j}} \right).$$

Then $\underline{x}_n \in J$ if and only if $x_{n,j} \in \left[\frac{a_{j1}}{b^1} + \dots + \frac{a_{jd_j}}{b^{d_j}}, \frac{a_{j1}}{b^1} + \dots + \frac{a_{jd_j}}{b^{d_j}} + \frac{1}{b^{d_j}} \right)$ for $j = 1, \dots, s$. The latter is true if and only if

$$x_{j,n} = \frac{a_{j1}}{b} + \frac{a_{j2}}{b^2} + \dots + \frac{a_{jd_j}}{b^{d_j}} + \frac{x_{j,n,d_j+1}}{b^{d_j+1}} + \dots + \frac{x_{j,n,m}}{b^m}, \quad 1 \leq j \leq s$$

Thus the number of points of the digital net in the elementary interval J is given by the number of solutions $\vec{n}$ of

$$\underbrace{\begin{pmatrix} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,d_1} \\ \vdots \\ c_{s,1} \\ \vdots \\ c_{s,d_s} \end{pmatrix}}_{(m-t) \times m} \vec{n} = \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1d_1} \\ \vdots \\ a_{s,1} \\ \vdots \\ a_{sd_s} \end{pmatrix} \qquad (1.4.2\,a)$$

The number of solutions (1.4.2a) is b^t for choices of $\begin{pmatrix} a_{1,1} \\ \vdots \\ a_{1,d_1} \\ \vdots \\ a_{s1} \\ \vdots \\ a_{sd_s} \end{pmatrix}$ if and only if the matrix

$$\begin{pmatrix} c_{1,1} \\ c_{1,2} \\ \vdots \\ c_{1,d_1} \\ \vdots \\ c_{s,1} \\ \vdots \\ c_{sd_s} \end{pmatrix}$$

has rank $m-t$, that is, the rows are linearly independent. $\square$

We call a digital net P which is also a (t,m,s)-net a digital (t,m,s)-net.
Similarly, a digital sequence S, which is a (t,s)-sequence is called a digital (t,s)-sequence.

(1.4.3) Explicit constructions

Niederreiter sequence.

Let b be a prime number, $s \in \mathbb{N}$ and let $p_1, p_2, \dots, p_s \in \mathbb{Z}_b[x]$ be distinct monic irreducible polynomials over $\mathbb{Z}_b$. Let $e_i = \deg(p_i)$. For $j \geq 1$ and $0 \leq k < e_i$ consider the expansion

$$\frac{x^{e_i - k - 1}}{p_i(x)^j} = \sum_{r=0}^{\infty} a^{(i)}(j,k,r)\, x^{-r-1}$$

as a formal expansion (formal Laurent series $\mathbb{Z}_b((x^{-1}))$).

Then we define the matrix $C_i = \left(c^{(i)}_{j,r} \right)_{\substack{j \geq 1 \\ r \geq 0}}$ by

$$c^{(i)}_{j,r} = a^{(i)}(Q+1, k, r) \in \mathbb{Z}_b \quad \text{for } 1 \leq i \leq s,\ j \geq 1,\ r \geq 0,$$

where $j - 1 = Q e_i + k$ with integers $Q = Q(i,j)$ and $k = k(i,j)$ satisfying $0 \leq k < e_i$.

Def.: A digital sequence over $\mathbb{Z}_b$ generated by $C_1, \dots, C_s \in \mathbb{Z}_b^{\mathbb{N} \times \mathbb{N}}$ defined above is called a Niederreiter sequence.

Consider the first few rows of the matrix i.
To simplify the notation we ignore the index i for the moment. We have

$$\frac{x^{e-1}}{p(x)} = a(\phi,0,0)x^{-1} + a(\phi,0,1)x^{-2} + a(\phi,0,3)x^{-3} + \dots$$

$$\frac{x^{e-2}}{p(x)} = 0\,x^{-1} + a(\phi,0,0)x^{-2} + a(\phi,0,1)x^{-3} + \dots$$

$$\vdots$$

$$\frac{x^{0}}{p(x)} = 0\,x^{-1} + \quad +0x^{-e+1} + a(\phi,0,0)\,x^{-e} + a(\phi,0,1)x^{-e-1} + \dots$$

$$\frac{x^{e-1}}{(p(x))^2} = 0x^{-1} + \dots + 0x^{-e} + a(2,0,e)x^{-e-1} + a(2,0,e+1)x^{-e-2} + \dots$$

$$\vdots$$

$$\frac{x^{0}}{(p(x))^2} = 0x^{-1} + \dots + 0x^{-2e+1} + a(2,0,e)\,x^{-2e} + a(2,0,e+1)\,x^{-2e-1} + \dots$$

$$C = \begin{pmatrix} a(1,0,0) & a(1,0,1) & \dots & & & \\ 0 & a(1,0,0) & \dots & & & \\ \vdots & & & & & \\ 0 & \dots & 0 & a(1,0,0) & a(1,0,1) & \dots \\ 0 & \dots & & 0 & a(2,0,e) & \dots \\ & & & & & \end{pmatrix}$$

Theorem The Niederreiter sequence with generating matrices defined as above is a digital (t,s)-sequence with

$$t = \sum_{i=1}^{s} (e_i - 1).$$

Proof: We need to show that for all integers $m > t$ and all $d_1, \dots, d_s \in \mathbb{N}_0$ with $1 \le d_1 + \dots + d_s \le m - t$, the vectors

$$\underline{c}_j^{(i)} = (c_{j,0}^{(i)}, \dots, c_{j,m-1}^{(i)}) \in \mathbb{Z}_b^m \quad \text{for } 1 \le i \le s,\ 1 \le j \le d_i$$

are linearly independent over $\mathbb{Z}_b^m$.

Suppose to the contrary that

$$\sum_{i=1}^{s} \sum_{j=1}^{d_i} f_j^{(i)} c_j^{(i)} = \underline{0} \in \mathbb{Z}_b^m. \qquad (1.4.3\,a)$$

Without loss of generality we may assume that $d_i \ge 1$ for all $1 \le i \le s$. By considering the components of (1.4.3.a) we obtain

$$\sum_{i=1}^{s} \sum_{j=1}^{d_i} f_j^{(i)} c_{jr}^{(i)} = 0 \quad \text{for } 0 \le r < m. \qquad (1.4.3b)$$

Consider the rational function

$$L = \sum_{i=1}^{s} \sum_{j=1}^{d_i} f_j^{(i)} \frac{x^{e_i - k(i,j)-1}}{(p_i(x))^{Q(i,j)+1}} \qquad (1.4.3c)$$

$$= \sum_{r=0}^{\infty} \left(\sum_{i=1}^{s} \sum_{j=1}^{d_i} f_j^{(i)} c_{j,r}^{(i)} \right) x^{-r-1} .$$

From (1.4.3b) we obtain that

$$L = \sum_{r=m}^{\infty} \left(\sum_{i=1}^{s} \sum_{j=1}^{d_i} f_j^{(i)} c_{j,r}^{(i)} \right) x^{-r-1} .$$

Put $\tilde{Q}_i = \left\lfloor \frac{d_i - 1}{e_i} \right\rfloor$ for $1 \le i \le s$ and set

$$g(x) = \prod_{j=1}^{s} p_i(x)^{\tilde{Q}_i + 1} .$$

Then Lg is a polynomial, since the largest value of $Q(i,j)$ in (1.4.3c) is

$$d_i - 1 = Q(i,j) e_i + k$$

$$\frac{d_i - 1 - k}{e_i} = Q(i,j) \Rightarrow Q(i,j) \le \left\lfloor \frac{d_i - 1}{e_i} \right\rfloor .$$

On the other hand

$$\deg(Lg) \le -1-m+\deg(g) = -1-m+\sum_{i=1}^{s}(\tilde{Q}_i+1)e_i$$

$$\le -1-m+\sum_{i=1}^{s}(d_i-1+e_i)$$

$$= -1-m+\sum_{i=1}^{s}d_i+\sum_{i=1}^{s}(e_i-1)$$

$$\le -1-m+m-t+t \le -1 .$$

Thus $Lg=0$ and therefore $L=0$, which implies that

$$\sum_{i=1}^{s}\sum_{j=1}^{d_i} f_j^{(i)} \frac{x^{k(i,j)}}{(p_i(x))^{Q(i,j)+1}} = 0 .$$

The left-hand side is a partial fraction decomposition of a rational function and hence the uniqueness of the partial fraction decomposition implies that $f_j^{(i)}=0$ for all $1\le j\le d_i$, $1\le i\le s$.

$\square$

(1.4.4) Polynomial lattice (rules) point sets

Def.: Let b be a prime number and $m, s \in \mathbb{N}$. Choose an (irreducible) polynomial $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m \geq 1$.
Let $\underline{q} = (q_1, q_2, \dots, q_s) \in (\mathbb{Z}_b[x])^s$ with $\deg(q_i) < m$.
For $1 \leq i \leq s$ consider the expansion

$$\frac{q_i(x)}{p(x)} = \sum_{l=1}^{\infty} u_l^{(i)} x^{-l} \quad \in \mathbb{Z}_b((x^{-1}))$$

Then define the matrix $C_i = (c_{j,r+1}^{(i)})$ by

$$c_{j,r+1}^{(i)} = u_{r+j}^{(i)} \quad \in \mathbb{Z}_b. \quad 1 \leq i \leq s, 1 \leq j \leq m, 0 \leq r < m.$$

Then $C_1, \dots, C_s$ are the generating matrices of a digital net $P(\underline{q}, p)$. We call $P(\underline{q}, p)$ a polynomial lattice point set (with generating vector $\underline{q}$ and modulus p).

Note that

$$C_i = \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & \cdots & u_m^{(i)} \\ u_2^{(i)} & u_3^{(i)} & \cdots & u_{m+1}^{(i)} \\ \vdots & \vdots & & \vdots \\ u_m^{(i)} & u_{m+1}^{(i)} & \cdots & u_{2m-1}^{(i)} \end{pmatrix}$$

is a Hankel matrix associated with the linear recurring sequence $(u_1^{(i)}, u_2^{(i)}, u_3^{(i)}, \dots)$

Define the map

$$V_m : \mathbb{Z}_b((x^{-1})) \to [0,1)$$

by
$$V_m\left(\sum_{l=w}^{\infty} t_l x^{-l}\right) = \sum_{l=\max(1,w)}^{\infty} t_l b^{-l}$$

For an integer $k \geq 0$ we define the polynomial

$$k(x) = K_0 + K_1 x + \dots + K_a x^a \in \mathbb{Z}_b[x]$$

for k with b-adic expansion $k = K_0 + K_1 b + \dots + K_a b^a$.

Theorem: Let b be prime and $m, s \in \mathbb{N}$. For $p \in \mathbb{Z}_b[x]$ with $\deg(p) = m$ and $\underline{q} = (q_1, \dots, q_s) \in (\mathbb{Z}_b[x])^s$, the polynomial lattice point set $P(\underline{q}, p)$ consists of the points

$$\underline{x}_h = \left(V_m\left(\frac{h(x)\, q_1(x)}{p(x)}\right), \dots, V_m\left(\frac{h(x)\, q_s(x)}{p(x)}\right)\right) \in [0,1)^s$$

for $h = 0, 1, \dots, b^m - 1$.

(1.4.5) Group structure and characters

Consider a digital net P with generating matrices $C_1,\dots,C_s \in \mathbb{Z}_b^{m\times m}$, b prime. The points of P are given by

$$\vec{y}_{j,n} = C_j \vec{n}, \quad x_{j,n} = \frac{y_{j,n,1}}{b}+\dots+\frac{y_{j,n,m}}{b^m}, \quad \underline{x}_n = (x_1,\dots,x_s).$$

Let

$$\underline{x}_n \oplus \underline{x}_\ell = \underline{z}, \text{ where } \underline{z} = (z_1,\dots,z_s),\ z_j = \frac{z_{j,1}}{b}+\dots+\frac{z_{j,m}}{b^m},$$

$$z_{j,k} = y_{j,n,k} + y_{j,\ell,k} \pmod{b}.$$

Then

$$\underline{x}_n \oplus \underline{x}_\ell = \underline{x}_{n\oplus\ell},$$

where for $n = n_0+n_1 b+\dots+n_{m-1}b^{m-1}$ and $\ell = \ell_0+\ell_1 b+\dots+\ell_{m-1}b^{m-1}$ we set

$$n\oplus\ell = u = \frac{u_1}{b}+\frac{u_2}{b}+\dots+\frac{u_{m-1}}{b^{m-1}},$$

where $u_k = n_k + \ell_k \pmod{b}$.

This follows from

$$\vec{y}_{j,n} + \vec{y}_{j,\ell} = C_j\vec{n} + C_j\vec{\ell} = C_j(\vec{n}+\vec{\ell}) \in \mathbb{Z}_b^m$$

Thus $(P,\oplus)$ is an additive group.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the multiplicative group of complex numbers of absolute value one.

We consider the set of homomorphisms from $(P, \oplus)$ to $\mathbb{T}$, $h : (P, \oplus) \to \mathbb{T}$

$$h(\underline{x} \oplus \underline{y}) = h(\underline{x}) \cdot h(\underline{y}) \quad , \forall \underline{x}, \underline{y} \in P.$$

For $\underline{x}_n, \underline{x}_m \in P$ we have for $\underline{k} \in \{0, 1, \ldots, b^m - 1\}^s$:

$$\mathrm{wal}_{\underline{k}}(\underline{x}_n \,\cancel{\oplus\, x_m}) = \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_{jn}) = \prod_{j=1}^{s} \omega_b^{\kappa_0 x_{jn1} + \kappa_1 x_{jn2} + \cdots + \kappa_{m-1} x_{jnm}}$$

$$= \prod_{j=1}^{s} \omega_b^{\vec{k} \cdot \vec{y}_{jn}} = \prod_{j=1}^{s} \omega_b^{\vec{k} \cdot C_j \vec{n}} = \prod_{j=1}^{s} \omega_b^{\vec{k}^T C_j \vec{n}}$$

and hence

$$\mathrm{wal}_{\underline{k}}(\underline{x}_n \oplus \underline{x}_l) = \prod_{j=1}^{s} \omega_b^{\vec{k}^T C_j \vec{n} + \vec{k}^T C_j \vec{l}}$$

$$= \prod_{j=1}^{s} \omega_b^{\vec{k}^T C_j (\vec{n} + \vec{l})} = \mathrm{wal}_{\underline{k}}(\underline{x}_{n \oplus l}).$$

In particular, for a digital net P we have

$$\frac{1}{b^m} \sum_{n=0}^{b^m - 1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) = \frac{1}{b^m} \sum_{n=0}^{b^m - 1} \prod_{j=1}^{s} \omega_b^{\vec{k}_j C_j \vec{n}} = \frac{1}{b^m} \sum_{n=0}^{b^m - 1} \omega_b^{(\vec{k}_1^T C_1 + \cdots + \vec{k}_s^T C_s)\vec{n}}$$

If $\vec{k}_1^T C_1 + \cdots + \vec{k}_s^T C_s = 0 \in \mathbb{Z}_b^m$, then

$$\frac{1}{b^m} \sum_{n=0}^{b^m - 1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) = 1.$$

If $\vec{k}_1^T C_1 + \dots + \vec{k}_s^T C_s \neq \underline{0} \in \mathbb{Z}_b^m$, then let $\vec{k}_1 C_1 + \dots + \vec{k}_s C_s = \vec{z}$

$$\frac{1}{b^m} \sum_{n=0}^{b^m-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \omega_b^{\vec{z} \cdot \vec{n}} = \prod_{u=1}^{m} \underbrace{\frac{1}{b} \sum_{n_u=0}^{b-1} \omega_b^{z_u \cdot n_{u-1}}}_{= \begin{cases} 0 \text{ if } z_u \neq 0 \\ 1 \text{ if } z_u = 0 \end{cases}} = 0$$

We define the dual net by

$$\mathcal{D}^\perp = \left\{ \underline{k} \in \mathbb{N}_0^s : C_1^T \mathrm{tr}(\vec{k}_1) + \dots + C_s^T \mathrm{tr}(\vec{k}_s) = \vec{0} \in \mathbb{Z}_b^m \right\}.^*$$

With this notation we have

$$\frac{1}{b^m} \sum_{n=0}^{b^m-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) = \begin{cases} 1 \text{ if } \underline{k} \in \mathcal{D}^\perp, \\ 0 \text{ otherwise.} \end{cases}$$

(*) where

$$\mathrm{tr}(\vec{k}_j) = \begin{pmatrix} k_{j0} \\ k_{j1} \\ \vdots \\ k_{jm-1} \end{pmatrix} \text{ for } k_j = k_{j0} + k_{j1} b + \dots$$

(1.4.6) Error bounds for digital nets

To make the integration error small, we need to make

$$E_\alpha(P) = \sum_{\underline{k} \in \mathbb{N}_0^s \setminus \{0\}} b^{-\mu_\alpha(\underline{k})} \left| \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{wal}_{\underline{k}}(\underline{x}_n) \right|$$

small. For digital nets P we have

$$E_\alpha(P) = \sum_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} b^{-\mu_\alpha(\underline{k})} .$$

Then

$$|e(f)| \lesssim \|f\|_\alpha \, E_\alpha(P) .$$

Consider now

$$\cancel{\rho_\alpha(P) =} \max_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} b^{-\mu_\alpha(\underline{k})}$$

It can be shown that

$$E_\alpha(P) \lesssim \max_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} b^{-\mu_\alpha(\underline{k})} \quad (\log N)^{(\alpha s)}$$

Let

$$\rho_\alpha(P) = \min_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} \mu_\alpha(\underline{k}) .$$

Then

$$\max_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} b^{-\mu_\alpha(k)} = b^{-\rho_\alpha(P)} .$$

Consider now the case $\alpha = 1$. Let P be a digital (t,m,s)-net with generating matrices $C_1, \ldots, C_s$.
Let $C_j = \begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jm} \end{pmatrix}$. Then P is a digital (t,m,s)-net, if for all $d_1, \ldots, d_s \geq 0$ with $d_1 + \ldots + d_s = m-t$ we have that

$$c_{1,1} \ldots c_{1d_1} \ldots c_{s1} \ldots c_{sd_s} \in (\mathbb{Z}_b^T)^m$$

are linearly independent over $\mathbb{Z}_b^m$. Assume that t is the smallest integer for which this holds.
We have

$$\rho_\alpha(P) = \min_{\underline{k} \in \mathcal{D}^\perp \setminus \{0\}} \mu_1(\underline{k}) = \min_{\underline{0} \neq \underline{k} : C_1^T \vec{k}_1 + \ldots + C_s^T \vec{k}_s = 0} \mu_1(\vec{k}_1) + \ldots + \mu_1(k_s)$$

Now $C_1^T \vec{k}_1 + \ldots + C_s^T \vec{k}_s = 0$ provides a non-zero linear combination of row vectors of $C_1, \ldots, C_s$ which are equal to 0. Let $e_j = \mu_1(k_j) = \max\{r \in \{0,1,\ldots,m\} : K_r \neq 0,\ k_j = K_1 + K_2 b + \ldots + K_m b^{m-1}\}$.
Thus let $\underline{k}^*$ be such that $\rho_\alpha(P) = \mu_1(\underline{k}^*)$. Then

$$C_1^T \vec{k}_1^* + \ldots + C_s^T \vec{k}_s^* = \vec{0}$$

is a minimal linear combination of rows of $C_1, \ldots, C_s$ which are linearly dependent using e_j rows of C_j for $j = 1, \ldots, s$.

Thus
$$\rho_1(P) = m - t + 1.$$

Hence, if we use a digital (t,m,s)-net for integration we get
$$E_1(P) \lesssim b^{-m+t} (\log N)^s.$$

Assume now $\alpha \in \mathbb{N}, \alpha > 1$. Our quality criterion now is
$$\rho_\alpha(P) = \min_{\underline{k} \in \mathcal{D}^s \setminus \{0\}} \mu_\alpha(k),$$
which we want to make as large as possible.

(1.4.7) Higher order digital nets

We have

$$\rho_\alpha(P) = \min_{\underline{k} \in \mathcal{D}^\perp \setminus \{\underline{0}\}} \mu_\alpha(\underline{k})$$

$$= \min_{\underline{k} \neq 0:\ C_1^T \vec{k}_1 + \dots + C_s^T \vec{k}_s = 0} \mu_\alpha(k_1) + \dots + \mu_\alpha(k_s).$$

We have

$$\mu_\alpha(k_j) = a_{j1} + a_{j2} + \dots + a_{j\min(\nu_j, \alpha)}$$

for $k_j = \kappa_{j1} b^{a_{j1}-1} + \kappa_{j2} b^{a_{j2}-1} + \dots + \kappa_{j\nu_j} b^{a_{j\nu_j}-1}$, for

$\kappa_{j1}, \dots, \kappa_{j\nu_j} \in \{1, 2, \dots, b-1\}$ and $a_{j1} > a_{j2} > \dots > a_{j\nu_j} > 0$;

and $\mu_\alpha(0) = 0$.

The condition

$$C_1^T \vec{k}_1 + \dots + C_s^T \vec{k}_s = 0$$

is a linear dependence of rows of the matrices C_j.

We design the matrices $C_1, \dots, C_s$ such that many rows are linearly independent.

Let $C_1, \ldots, C_s \in \mathbb{Z}_b^{\alpha m \times m}$ be generating matrices of a digital net. Then we call the digital net an order α digital (t_α, m, s)-net, if for all $k_1, \ldots, k_s \in \{0, 1, \ldots, b^{\alpha m} - 1\}$ with $(k_1, \ldots, k_s) \neq \underline{0}$ and

$$C_1^T \vec{k}_1 + \ldots + C_s^T \vec{k}_s = 0$$

we have

$$\mu_\alpha(k_1) + \ldots + \mu_\alpha(k_s) \geq \alpha m - t_\alpha .$$

Hence $\rho_\alpha(P) > \alpha m - t_\alpha$.

We can write this also in terms of linear independence of rows of $C_1, \ldots, C_s$. Let $C_j = \begin{pmatrix} c_{j,1} \\ \vdots \\ c_{j,\alpha m} \end{pmatrix}$.

Then a digital net P with generating matrices $C_1, \ldots, C_s \in \mathbb{F}_b^{\alpha m \times m}$ is an order α digital (t_α, m, s)-net if for all $a_{j1} > a_{j2} > \ldots > a_{j\nu_j} > 0$ with

$$\sum_{j=1}^{s} \left(a_{j1} + \ldots + a_{j\min(\alpha,\nu_j)} \right) \leq \alpha m - t_\alpha$$

the rows

$$c_{1 a_{j1}}, c_{1 a_{j2}}, \ldots, c_{1 a_{j\nu_j}}, \ldots, c_{s a_{s1}}, \ldots, c_{s a_{s\nu_s}}$$

are linearly independent.