Halton sequence

Let $b \geq 2$. For $n \in \mathbb{N}_0$

$$n = n_0 + n_1 b + n_2 b^2 + \dots$$

$$\varphi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \dots \in [0,1)$$

$$V_b = (x_n)_{n \geq 0} \qquad x_n = \varphi_b(n).$$

$$D_N^*(V_b) \asymp \frac{\log N}{N}$$

$$c\frac{\log N}{N} \underset{\text{infinitely often}}{\leq} D_N^*(V_b) \underset{\forall N \geq 2}{\leq} C \frac{\log N}{N}$$

$$L_2(V_b) \asymp \frac{\log N}{N}$$

Halton sequ. ($s = 2$, $b = 2, 3$)

$$\underline{x}_n = (\varphi_2(n), \varphi_3(n)) \qquad H_{2,3}$$

$$D_N^*(H_{2,3}) \asymp \frac{(\log N)^2}{N}$$

$$L_2(H_{2,3}) \asymp ?$$

Let $\Delta_N(x,y) = \frac{1}{N}\sum_{n=0}^{N-1} 1_{[0,x)\times[0,y)}(t_n, s_n) - xy$

$\underline{x}_n = (t_n, s_n)$

$L_2 \geq \underline{\underline{|\hat{\Delta}_N(0,0)|}}$

$$\hat{\Delta}_N(0,0) = \int_0^1\int_0^1 \Delta_N$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left((1-t_n)(1-s_n) - \frac{1}{4}\right)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2} - t_n\right) + \frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2} - s_n\right)$$

$$+ \frac{1}{N}\sum_{n=0}^{N-1}\left(t_n s_n - \frac{1}{4}\right)$$

$$\frac{1}{N}\sum_{n=0}^{N-1}\left(\frac{1}{2} - t_n\right) = O\left(\frac{\log N}{N}\right)$$

$$\hat{\Delta}_N(0,0) = \frac{1}{N}\sum_{n=0}^{N-1}\left(t_n s_n - \frac{1}{4}\right) + O\left(\frac{\log N}{N}\right)$$

Q: $\left|\sum_{n=0}^{N-1}\left(t_n s_n - \frac{1}{4}\right)\right| \gg (\log N)^2$