Sets of Bounded Remainder for the Billiard on a Square

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October 18, 2016

Abstract

We study sets of bounded remainder for the billiard on the unit square. In particular, we note that every convex set S whose boundary is twice continuously differentiable with positive curvature at every point, is a bounded remainder set for almost all starting angles α and every starting point $\mathbf{x}$. We show that this assertion for a large class of sets does not hold for *all* irrational starting angles α.

Keywords: Bounded remainder set, billiard path, discrepancy, distribution modulo 1, unfolding-technique.

1 Introduction and Statement of results

In this paper we will be concerned with bounded remainder sets for the two-dimensional billiard on the unit-square $I^2 = [0,1)^2$.

Definition 1. *Let $\mathbf{x} = (x_1, x_2) \in I^2$ and let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We say that the function $Y : [0,\infty) \to I^2$ defined by*

$$Y(t) = (2 \cdot \| \frac{x_1 + t}{2} \|, 2 \cdot \| \frac{x_2 + \alpha t}{2} \|) \qquad 0 \leq t < \infty, \tag{1}$$

where $\|z\| := \min_{a \in \mathbb{Z}} |z - a|$, is the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.

*The first author is supported by CERN, European Organization for Nuclear Research, Geneva, as part of the Austrian Doctoral Student Programme.

**The second author is supported by the Austrian Science Fund (FWF): Project F5507-N26, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

It is easily checked that this definition indeed coincides with our image of a real billiard-path in the unit interval.

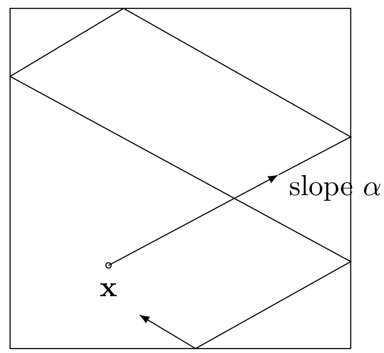

Figure 1

Definition 2. *Let $S \subset I^2$ be an arbitrary measurable subset of the unit square with Lebesgue measure $\lambda(S)$. We say that S is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x} = (x_1, x_2) \in I^2$ if the distribution error*

$$\Delta_T^Y(S, \alpha, \mathbf{x}) = \int_0^T \chi_S(Y(t))\, dt \quad - \quad T\lambda(S) \tag{2}$$

is uniformly bounded for all $T > 0$. Here, χ_S denotes the characteristic function for the set S.

Distribution properties for continuous motions in an s-dimensional unit cube were studied for example by Drmota in [3] (see also [4] or [7]) and quite recently by Beck [1]. Beck ([1], [2]) especially studied continuous irrational rotations and billiard paths.
For the two-dimensional billiard path for example he showed the following surprising result:

Theorem. *(Beck) Let $S \subseteq I^2$ be an arbitrary Lebesgue measurable set in the unit square with positive measure. Then for every $\epsilon > 0$, almost all $\alpha > 0$ and every starting point $\mathbf{x} = (x_1, x_2) \in I^2$ we have*

$$\Delta_T^Y(S, \alpha, \mathbf{x}) = \mathcal{O}\Big((logT)^{3+\epsilon}\Big). \tag{3}$$

As pointed out by Beck, the poly-logarithmic error term is shockingly small compared to the linear term $T\lambda(S)$. Moreover, it holds for *all* measurable sets S.

It is thus natural to ask if imposing certain regularity conditions on S could give an even lower bound on the error term.
We will show in the following that the estimate of Beck indeed can be significantly improved for a large collection of sets S. We show:

Theorem 1.

a) For almost all $\alpha > 0$ and every $\mathbf{x} \in I^2$, every polygon $S \subset I^2$ with no edge of slope α or $-\alpha$ is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.

b) For almost all $\alpha > 0$ and every $\mathbf{x} = (x_1, x_2) \in I^2$, every convex set $S \subset I^2$ whose boundary ∂S is a twice continuously differentiable curve with positive curvature at every point is a bounded remainder set for the two-dimensional billiard with starting slope α and starting point $\mathbf{x}$.

We will see in the proofs of these results that this Theorem easily follows from an analogous result shown in [5] for the continuous irrational rotation $X(t) := (\{x_1 + t\}, \{x_2 + \alpha t\})_{t \geq 0}$, and by the "unfolding-technique" suggested by Beck in [2].
It is obvious that the results given in Theorem 1 do not hold for a rational slope α.
However one could ask whether the results can be improved first by omitting the condition on the slopes of the edges of the polygon S in part a) of the Theorem and, second, whether both results maybe are valid even for *all* irrational slopes α.
We will give an easy argument that indeed the condition on the slopes of the edges cannot be omitted in general. Moreover, we will prove - and this will be the main effort in this paper - that the results of Theorem 1a and 1b in general do not hold for *all* irrational α.

I.e., we will show:

Theorem 2.

a) For every $\alpha > 0$ there is a polygon S with an edge of slope α or $-\alpha$ such that S is not a bounded remainder set for the billiard with starting-slope α and for any starting point $\mathbf{x}$.

b) For every $\boldsymbol{m} \in [0, 1)^2$ there are uncountably many radii r, dense in an interval of positive length, such that there is a slope α and a starting point $\boldsymbol{x}$ such that the disk with midpoint $\boldsymbol{m}$ and radius r is not a set *of bounded remainder with respect to the billiard with starting slope α and starting point $\boldsymbol{x}$.*

In Chapter 2 we prove Theorem 1 and Theorem 2a. In Chapter 3 we carry out the main work, namely the proof of Theorem 2b.

2 Proofs of Theorem 1 and of Theorem 2a

The proofs of these two results can be traced back to the results given in [5] via the technique of unfolding.
As was pointed out in detail for example by Beck in [2] the technique of "unfolding" a billiard path (see Figure 2)

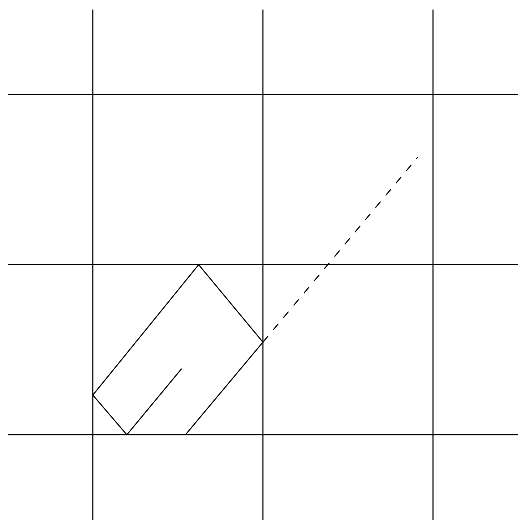

Figure 2

shows that the problem of uniformity of a billiard path in the unit square with respect to a given test set S is equivalent to the problem of uniformity of the corresponding continuous rotation in the 2×2 square, where each one of the four unit-subsquares contains a reflected copy of the given test set (see S_1, S_2, S_3, S_4 in Figure 3) Of course this again can be reduced to the problem of studying continuous irrational rotation in $[0,1)^2$ with respect to a factor $1/2$ reduced versions of S_1, S_2, S_3, S_4.

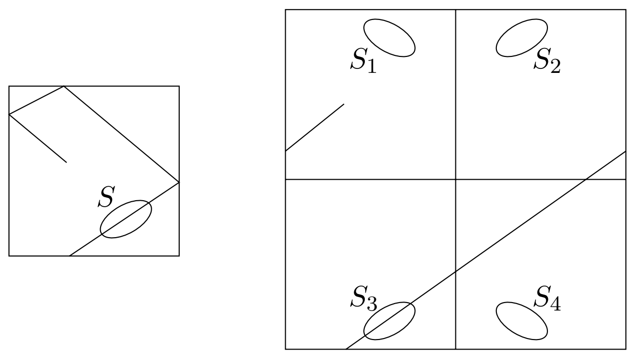

Figure 3

So in all the following, when studying the distribution error $\Delta_T^Y(S,\alpha,\mathbf{x})$ for the two-dimensional billiard this task can be traced back to the investigation of the distribution error $\Delta_T^X(\tilde{S},\alpha,\mathbf{x})$ for the continuous irrational rotation where $\tilde{S}$ consists of four mirrored and by a factor $1/2$ reduced copies of S lying symmetrical to $(1/2, 1/2)$.

Proof of Theorem 1.
Theorem 1 follows immediately from the above considerations on the unfolding technique and from Theorem 1 and Theorem 2 in [5]. □

Proof of Theorem 2a.
Let $\alpha > 0$ be given. Consider first a triangle S (see Figure 4) with corners in $(a,1)$, $(1,1-a\alpha)$, $(1,1)$, where a is such that $0 < a\alpha < 1$ and such that $a \neq \frac{1}{\alpha}\{k\alpha\}$ for all $k \in \mathbb{N}$.

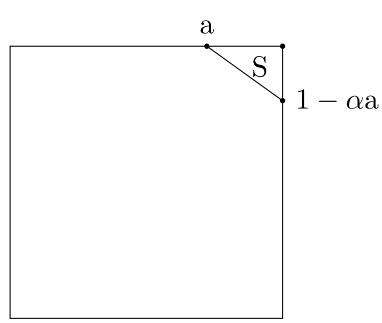

Figure 4

So one side of S has slope $-\alpha$. Unfolding leads to the investigation of the continuous irrational rotation with slope α with respect to the parallelogram $\tilde{S}$ (see Figure 5) with corners in

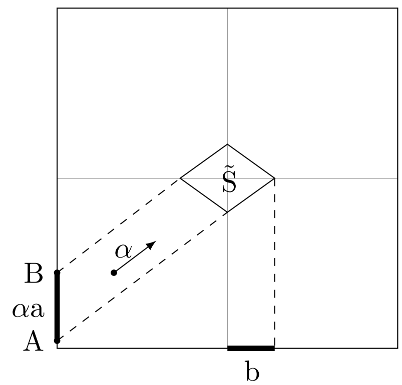

Figure 5

$(\frac{1}{2}\ ,\ \frac{a}{2})\ ,\ (\frac{1}{2}\ ,\ \frac{1+\alpha a}{2})\ ,\ (\frac{1}{2}+\frac{a}{2}\ ,\ \frac{1}{2})\ ,\ (\frac{1}{2}\ ,\ \frac{1-\alpha a}{2})$.

Note that two sides of $\tilde{S}$ have slope α, the other two have slope $-\alpha$. The length of the interval $[A, B]$ is $a\alpha \neq \{k\alpha\}$ for all $k \in \mathbb{N}$. We show that for no starting point $\mathbf{x}$ the set $\tilde{S}$ is of bounded remainder for the irrational continuous rotation with slope α. Hence the set S is not of bounded remainder for the billiard with starting slope α (and any starting point $\mathbf{x}$).
Indeed, it is easy to see that for this set $\tilde{S}$ we have

$$\Big|\int_0^T \chi_{\tilde{S}}(\{t\},\{\alpha t\})dt - b\cdot\sum_{n=1}^{[T]}\chi_{[A,B]}(\{n\alpha\})\Big| \quad \leq \quad 1.$$

It was shown by Kesten in [6] that

$$\Big|\sum_{n=1}^{[T]}\chi_{[A,B]}(\{n\alpha\}) - a\alpha\cdot[T]\Big|$$

is unbounded since $B - A = a\alpha \neq \{k\alpha\}$ for all $k \in \mathbb{N}$. Hence (note that $\lambda(\tilde{S}) = a\alpha\cdot b$)

$$\Big|\int_0^T \chi_{\tilde{S}}(\{t\},\{\alpha t\})dt - T\cdot\lambda(\tilde{S})\Big|$$

is unbounded. □

3 Proof of Theorem 2b

The proof of Theorem 2b will need the most work. We start with some auxiliary results. Especially we will have to deal with functions of the form

$$g_m(x) := \frac{1}{2m} \sum_{k=0}^{2m-1} \sqrt{1 - (1 - \frac{k}{m} - x)^2} \tag{4}$$

for $x \in [0, \frac{1}{m}]$, where m is a given positive integer. The function g_m is illustrated in Figure 6.

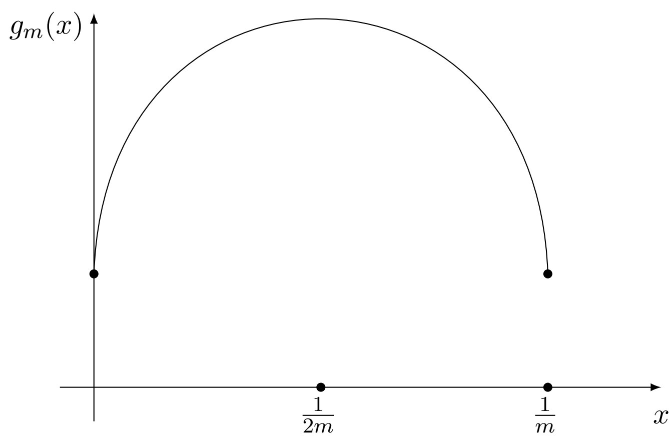

Figure 6

Let now m be fixed, and $h_m(x) := g_m(x) - g_m(0)$. It is easy to see that $h_m(0) = h_m(\frac{1}{m}) = 0$, h_m is arbitrarily often differentiable on $(0, \frac{1}{m})$, continuous on $[0, \frac{1}{m}]$, symmetric around $\frac{1}{2m}$ and strictly convex on $[0, \frac{1}{m}]$.

Further we have

Lemma 1.

a) There exist c', c'' *with* $0 < c' < c''$ *such that for all* m *large enough we have*

$$c' \cdot \frac{1}{m^{3/2}} \quad < \quad h_m(\frac{1}{2m}) \quad < \quad c'' \cdot \frac{1}{m^{3/2}}. \tag{5}$$

b) There exists $c''' > 0$ *such that*

$$h_m(x) \quad \geq \quad c''' \cdot \frac{1}{m}\sqrt{x} \tag{6}$$

for all $x \in [0, \frac{1}{2m}]$.

Proof. This is shown by some tedious but elementary analysis. We do not give all details but just give two hints:

To prepare part a) note that

$$\begin{aligned} h_m(x) \quad &= \quad g_m(x) - g_m(0) \quad = \\ &= \frac{1}{2m} \sum_{k=0}^{2m-1} \left(\sqrt{1 - \left(1 - \frac{k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{k}{m}\right)^2} \right) \\ &= \frac{1}{2m} \sum_{k=0}^{m} \left(\sqrt{1 - \left(1 - \frac{k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{k}{m}\right)^2} \right. \\ &\qquad \left. + \quad \sqrt{1 - \left(1 - \frac{2m-1-k}{m} - x\right)^2} \quad - \quad \sqrt{1 - \left(1 - \frac{2m-1-k}{m}\right)^2} \right) \\ &=: \frac{1}{2m} \sum_{k=0}^{m} w_m(k,x) \end{aligned}$$

and note that simple calculation shows that for $k \geq 0$ there are absolute constants $c_1', c_2' > 0$ such that

$$c_1' \cdot \frac{1}{m^{1/2} \max(k,1)^{3/2}} \quad \leq \quad w_m(k, \frac{1}{2m}) \quad \leq \quad c_2' \cdot \frac{1}{m^{1/2} \max(k,1)^{3/2}}$$

always.

To show part b, note that for x small, only the first summand of h_m, i.e., the summand for k=0,

$$\frac{1}{m}\sqrt{1-(1-x)^2} \quad \sim \quad \frac{1}{m}\sqrt{x}$$

is of relevance.

□

Lemma 2.

a) There exists $c_1 > 0$ such that

$$\left|h'_m(x)\right| \quad \leq \quad c_1 \cdot \frac{1}{\sqrt{m}} \tag{7}$$

for all $x \in [\frac{1}{10m}, \frac{9}{10m}]$ and all m large enough.

Especially it holds that

$$h'_m(x) \quad \geq \quad -c_1 \cdot \frac{1}{\sqrt{m}} \tag{8}$$

for all $x \in [0, \frac{9}{10m}]$.

b) There exists $c_2 > 0$ such that for all m large enough we have

$$h'_m(x) \quad \geq \quad 5c_1 \frac{1}{\sqrt{m}} \tag{9}$$

for all $x \in \left(0, c_2\frac{1}{m}\right)$.

(Here c_1 is the constant from part a).)

Proof. This follows immediately from Lemma 1 and the convexity of h_m. □

Let m large enough be fixed. Moreover in the following let a, b, c be given reals with $0 \leq a \leq b \leq c < \frac{1}{m}$, and

$$\begin{aligned} G_m(x) \quad := \quad & g_m\Big(x\Big) + g_m\Big((x+a) \bmod \frac{1}{m}\Big) \\ + \quad & g_m\Big((x+b) \bmod \frac{1}{m}\Big) + g_m\Big((x+c) \bmod \frac{1}{m}\Big) \end{aligned}$$

Then we have:

Lemma 3. *There are $c_3, c_4 > 0$ such that for all m large enough and all a, b, c as above there is an $x_0 \in \{0, a, b, c\}$ such that G_m is strictly increasing on $[x_0, x_0 + c_4 \cdot \frac{1}{m}]$ and*

$$G_m(x_0 + c_4 \cdot \frac{1}{3m}) - G_m(x_0) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}} \tag{10}$$

$$G_m(x_0 + c_4 \cdot \frac{2}{3m}) - G_m(x_0 + c_2 \cdot \frac{1}{3m}) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}} \tag{11}$$

$$G_m(x_0 + c_4 \cdot \frac{1}{m}) - G_m(x_0 + c_2 \cdot \frac{2}{3m}) \quad > \quad c_3 \cdot \frac{1}{m^{3/2}} \, . \tag{12}$$

Proof. At least one of the following relations holds:

$c < \frac{4}{5m}$ or $b < c - \frac{1}{5m}$, or $a < b - \frac{1}{5m}$, or $a > \frac{1}{5m}$.

Assume for example that $c < \frac{4}{5m}$ holds (the other cases are treated quite analogously). Then set $x_0 = 0$. Let $c_4 := \min(c_2, \frac{1}{10})$ where c_2 is like in *Lemma* 2 b). Then for any $x \in \left[0, c_4 \frac{1}{m}\right)$ it holds that:

$(a + x) \bmod \frac{1}{m}$, and $(b + x) \bmod \frac{1}{m}$, and $(c + x) \bmod \frac{1}{m}$

are all in $\left[0, \frac{9}{10m}\right)$:

Hence by Lemma 2a) we have that g'_m at these places is at least $-c_1 \cdot \frac{1}{\sqrt{m}}$. By Lemma 2b) for $x \in [0, c_4 \frac{1}{m}]$ we have $g'_m(x) \geq 5c_1 \cdot \frac{1}{\sqrt{m}}$ and hence $G'_m(x) \geq 2c_1 \frac{1}{\sqrt{m}}$ for all those x.
From this the assertions of Lemma 3 immediately follow.

□

Lemma 4. *For all m large enough there is a sub-interval Λ_m of $[0, \frac{1}{m}]$ of length at least $\frac{c_4}{3} \cdot \frac{1}{m}$ such that either*

$$G_m(x) > 2 \int_0^2 \sqrt{1 - (1 - y)^2} \, dy + \frac{c_3}{2m^{3/2}} = \pi + \frac{c_3}{2m^{3/2}} \tag{13}$$

or

$$G_m(x) < 2 \int_0^2 \sqrt{1 - (1 - y)^2} \, dy - \frac{c_3}{2m^{3/2}} = \pi - \frac{c_3}{2m^{3/2}} \tag{14}$$

holds for all $x \in \Lambda_m$.

Proof. This follow immediately from Lemma 3. □

Proof of Theorem 2b. For the proof we proceed in analogy to the proof of Theorem 1.7b in [5] where the corresponding result was shown for the continuous irrational rotation, and in the following we sometimes refer to this proof.
Fix an irrational $\alpha \in (\frac{1}{8}, \frac{1}{4})$ with continued fraction expansion $\alpha = [0; a_1, a_2, \dots]$ and convergents $\frac{p_n}{q_n}$ satisfying $a_{l+1} > q_l^{100}$ and p_l even, for infinitely many l. There exist uncountably many such α. Let S be a disk with diameter $d := 2\alpha/\sqrt{1+\alpha^2}$. (Note that the set of α with the above properties is dense in $(\frac{1}{8}, \frac{1}{4})$, hence the set of diameters d is dense in $(\frac{2}{\sqrt{65}}, \frac{2}{\sqrt{17}})$.)
Studying the billiard path with respect to S means to study the continuous rotation with respect to four copies of S with diameter $\alpha/\sqrt{1+\alpha^2}$ each (see Figure 7)

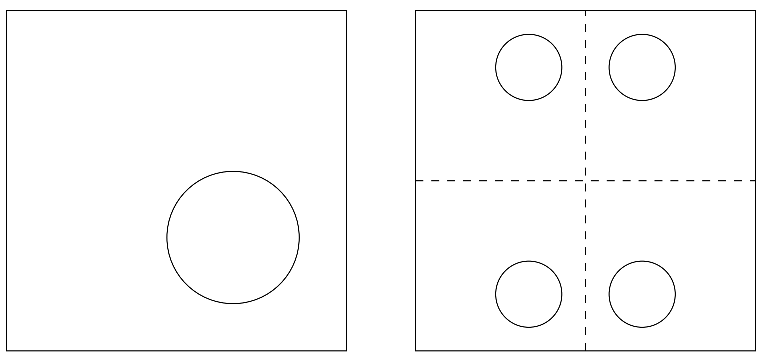

Figure 7

For such α and one copy of such disks it was shown in Theorem 1.7b in [5] that the continuous rotation is not of bounded remainder for all starting points $\mathbf{x}$.
In the proof of this Theorem the result was shown by studying the function g_m as defined in this current paper at the beginning of Section 3 (in [5] our g_m is denoted by G_m), and it was shown that the validity of the result of Theorem 1.7b in [5] is due to the fact that for every m there exists a subinterval $\Lambda_m \subseteq [0, \frac{1}{2m}]$ of length at least $\frac{1}{6m}$ such that either

$$g_m(x) > \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\,dy + \frac{\tilde{c}}{m^{3/2}} = \frac{\pi}{4} + \frac{\tilde{c}}{m^{3/2}} \tag{15}$$

or

$$g_m(x) < \frac{1}{2}\int_0^2 \sqrt{1-(1-y)^2}\,dy - \frac{\tilde{c}}{m^{3/2}} = \frac{\pi}{4} - \frac{\tilde{c}}{m^{3/2}} \tag{16}$$

holds for an absolute constant $\tilde{c} > 0$ and all $x \in \Lambda_m$.

By following the proof of Theorem 1.7b in [5] it becomes obvious that studying now four copies of disks instead of one copy means to study

$$g_m\Big(x\Big) + g_m\Big((x+a)\text{mod}\frac{1}{m}\Big) + g_m\Big((x+b)\text{mod}\frac{1}{m}\Big) + g_m\Big((x+c)\text{mod}\frac{1}{m}\Big),$$

for some a, b, c, i.e., to study the function G_m as studied in Lemma 3 and Lemma 4 of the current paper. In Lemma 4 it was shown that for G_m an analogous property (independent of the choices for a, b, and c) holds as stated above for g_m.
Again by following the proof of Theorem 1.7b in [5] it is obvious that from this property for G_m (Lemma 4) the result of our Theorem 2b follows.

□

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