Asymmetrizing cost and density of vertex-transitive cubic graphs

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Abstract

A set S of vertices in a graph G with nontrivial automorphism group is *asymmetrizing* if the identity mapping is the only automorphism of G that preserves S as a set. If such sets exist, then their minimum cardinality is the *asymmetrizing cost* $\rho(G)$ of G. For finite graphs the *asymmetrizing density* $\delta(G)$ of G is the quotient of the size of S by the order of G. For infinite graphs $\delta(G)$ is defined by a limit process.

Many classes of infinite graphs with $\delta(G) = 0$ are known, but seemingly no infinite vertex transitive graphs with $\delta(G) > 0$. Here, we construct connected, infinite vertex transitive cubic graphs of asymmetrizing density $\delta(G) = \frac{1}{n2^{n+1}}$ for each $n \geq 1$.

We also construct finite vertex transitive cubic graphs of arbitrarily large asymmetrizing cost. The examples are Split Praeger–Xu graphs, for which we provide another characterization.

This contrasts with our results for vertex transitive cubic graphs that have one arc orbit or are so-called synchronously connected graphs with two arc orbits. For them we show that $\rho(G)$ is either ≤ 5 or infinite. In the latter case $\delta(G) = 0$.

1 Introduction

A vertex coloring of a graph G is *asymmetric* if the identity is the only automorphism of G that preserves it. The smallest number of colors needed is the *asymmetrizing number* or *distinguishing number* $\mathrm{D}(G)$ of G. One says such a coloring *breaks* the automorphisms of G. When $\mathrm{D}(G) = 2$ each of the two colors induces a set of vertices which is preserved only by the identity automorphism. Such sets are called *asymmetrizing.*

Asymmetric colorings go back at least to 1977, when Babai [2] showed that every k-regular tree, where $k \geq 2$ is an arbitrary cardinal, has an asymmetric 2-coloring. Independently Albertson and Collins [1] introduced the term *distinguishing coloring* for an asymmetrizing coloring. Their paper spawned numerous other publications on the subject, for example [4, 8, 11, 12].

For connected, finite or infinite graphs G of maximal degree 3 the asymmetrizing coloring number $D(G)$ is at most 3, as was shown in [7]. It was also shown that the graphs in the family with asymmetrizing coloring number 3 consist of five infinite classes of graphs that are not vertex-transitive and four finite vertex-transitive graphs, namely K_4, $K_{3,3}$, the cube and the Petersen graph. Hence all but four connected, cubic vertex-transitive graphs have asymmetrizing coloring number 2.

This is the class of graphs we investigate. Given such a graph G we are interested in the smallest size of its asymmetrizing sets, that is, in the asymmetrizing cost $\rho(G)$. For the case when all asymmetrizing sets are infinite, we introduce the density of subsets of $V(G)$, and call the minimum density of asymmetrizing sets the asymmetrizing density $\delta(G)$ of G.

For graphs with one arc orbit we prove that $\rho(G)$ is either at most 5 or infinite. If $\rho(G)$ is infinite, then $\delta(G) = 0$. For so called synchronously connected vertex transitive cubic graphs with two arc orbits we show that $\rho(G) \leq 3$.

For connected, vertex transitive cubic graphs with two arc orbits that are not synchronously connected these bounds do not hold in general. In the finite case we show this by construction of connected, vertex transitive cubic finite graphs of arbitrarily large asymmetrizing cost. It turns out that our examples are Split Praeger–Xu graphs [13], for which we provide a new, straightforward characterization.

In the infinite case we construct connected, vertex transitive cubic infinite graphs with asymmetrizing density $\delta(G) = \frac{1}{n2^{n+1}}$ for each $n \geq 1$. They are not synchronously connected. Despite the fact that many classes of infinite graphs with asymmetrizing density 0 are known, and that one can easily construct graphs with positive asymmetrizing density that are not vertex transitive, see [10], these seem to be the first examples of vertex transitive graphs with positive asymmetrizing density.

2 Preliminaries

If a graph G has asymmetrizing number 2, then its set of vertices can be partitioned into two sets $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, such that the stabilizer of either one is the trivial automorphism. In other words, if $\alpha \in \mathrm{Aut}(G)$ and $\alpha(V_i) = V_i$ for $i = 1$ or $i = 2$, then $\alpha = \mathrm{id}$. Either of the sets V_1 or V_2 is an asymmetrizing set in the sense that the identity is the only automorphism that preserves it as a set. The smallest possible size of such a set is the *asymmetrizing cost* of G. It was introduced in [3] as *2-distinguishing cost* and is also called *Boutin-Imrich cost.* We denote it by $\rho(G)$.

If we use only the colors black and white, and always black for a minimum asymmetrizing set, then $\rho(G)$ is the minimal number of black vertices needed to break all automorphisms.

Note that $\rho(G)$ can be finite for infinite graphs. In fact, in [3] it was shown that $\rho(G)$ is finite for connected, locally finite infinite graphs G if and only if $\mathrm{Aut}(G)$ is countable. An example for an infinite graph with countable automorphism group and finite cost is the infinite ladder of Figure 1. It has asymmetrization cost 3, as is easily verified.

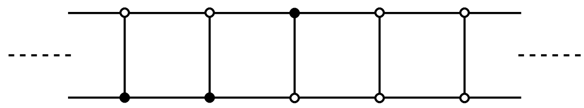

Figure 1: The infinite ladder with an asymmetrizing coloring.

On the other hand, the chain of quadrangles of Figure 2 has uncountable automorphism group, and hence infinite asymmetrization cost by [3]. To see this, observe that the horizontal edges in the figure are matching edges and that they come in disjoint pairs connecting adjacent quadrangles. One can interchange the edges in any such pair without moving any other vertex of the graph. Hence the automorphism group of the chain of quadrangles is uncountable.

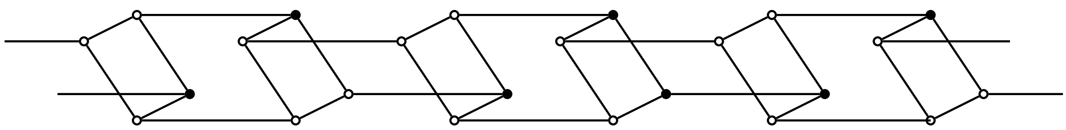

Figure 2: A chain of quadrangles with an asymmetrizing coloring.

Let G denote the chain of quadrangles in Figure 2. We show now that the coloring in the figure is asymmetrizing. There is only one matching edge with two black endpoints, and only one of these endpoints has a black neighbor. Hence, this edge is fixed pointwise.

If we contract each quadrangle to a single vertex, we obtain a two sided infinite path G', on which $\mathrm{Aut}(G)$ acts as the infinite dihedral group. As the endpoints of one edge in G' are fixed, G' is fixed pointwise. This means that all pairs of matching edges between two adjacent quadrangles are fixed setwise, but as one edge of each pair has no black endpoints, and the other at least one, they have to be fixed individually.

2.1 Density

If ρ is infinite we try to find sparse asymmetrizing sets. This leads to the concept of the density of sets of vertices. It was first introduced in [10].

Let S be a set of vertices of a graph G, $v \in G$, and let $B(v,n) = \{w \in G : d(v,w) \leq n\}$ denote the ball of radius n with center v. Then

$$\delta_v(S) := \limsup_{n\to\infty} \frac{\mid B(v,n) \cap S \mid}{\mid B(v,n) \mid}$$

is the *density* of S at v. If $\delta_v(S)$ exists for all vertices, which is the case for locally finite graphs, then the density of S is defined as $\delta(S) = \sup\{\delta_v(S) : v \in V(G)\}$.

The infimum of $\delta(S)$ over all asymmetrizing sets S is then the *asymmetrizing density* $\delta(G)$ of G. Clearly the definition extends to graphs with finite asymmetrization cost. In this case $\delta(G) = \rho(G)/|V(G)|$ for finite graphs and zero if $|V(G)|$ is infinite.

In order to prove that a locally finite graph of bounded degree has asymmetrizing density zero it suffices to find an asymmetrizing set S with $\delta_w(S) = 0$ at some vertex w.

Lemma 2.1. *Let G be an infinite, connected graph of bounded degree. Then $\delta(G) = 0$ if there exists an asymmetrizing set S of density zero at some vertex w.*

Proof Let G be graph satisfying the assumptions of the lemma. Clearly

$$|B(v, n+1)| \leq c \cdot |B(v, n)| \text{ for all } n \in \mathbb{N}. \tag{1}$$

By [10, Lemma 1] this implies that a set S has zero density $\delta_v(S)$ at each vertex v, if $\delta_w(S) = 0$ for some $w \in V(G)$. □

The homogeneous tree T_3 of degree 3 is a example of a graph with asymmetrizing density 0. It is vertex transitive and has uncountable automorphism group. For the proof and for many other examples of graphs with asymmetrizing density we refer the reader to [10].

For positive asymmetrizing density a we have the following lemma.

Lemma 2.2. *Let G be an infinite, connected graph G for which there exists a constant c such that*

$$|B(v, n)| \leq |B(v, n+k)| \leq |B(v, n)| + kc \tag{2}$$

for all natural number k. Then $\delta(G) = a$ if there exists an asymmetrizing set S of density a at some vertex w.

Proof Let $\delta_w(S) = a$, $v \in G$, and $d = d(v, w)$. Clearly

$$|B(v, n) \cap S| \leq |B(w, n+d) \cap S| \leq |B(v, n+2d) \cap S|,$$

which is equivalent to

$$\frac{|B(w,n) \cap S|\,|B(w,n)|}{|B(w,n)|\,|B(v,n+2d)|} \leq \frac{|B(v,n+d) \cap S|\,|B(v,n+d)|}{|B(v,n+d)|\,|B(v,n+2d)|} \leq \frac{|B(w,n+2d) \cap S|}{|B(w,n+2d)|}.$$

By assumption the supremum of the right side of the inequality is a. This is also true of the left side, because $|B(w, n)|/|B(v, n + 2d)|$ converges to 1 as $n \to \infty$. Hence the supremum of the middle term is a. Because $\lim_{n\to\infty} |B(v, n + d)|/|B(v, n + 2d)| = 1$ this implies that

$$\limsup_{n\to\infty} \frac{|B(v, n+d) \cap S|}{|B(v, n+d|} = a,$$

which is equivalent to $\delta_v(S) = a$. □

An example for a graph with positive asymmetrization density is the chain of quadrangles of Figure 2. To see this note that the coloring in the figure has density 1/4 at the black vertex adjacent to the matching edge with two black vertices. Let v be this vertex. Because

$$|B(v, n+k)| = |B(v, n)| + 4k$$

for all natural number $k > 1$, the chain of quadrangles has density 1/4 by Lemma 2.2.

We wish to add that it is relatively easy to construct graphs with nonzero asymmetrizing density that are not vertex-transitive, see [10]. The present article seems to be the first that exhibits vertex-transitive graphs with positive density.

2.2 Upper bounds for the density

We already mentioned that the definition of density extends to finite graphs and that it is the quotient $\rho(G)/|V(G)|$ of the asymmetrizing cost by the order of the graph. Clearly $\rho(G) \leq 1/2$ for finite graphs, and the bound is sharp.

For example, consider the asymmetric tree of order 7, split each vertex v into v', v'', and connect each of the vertices v', v'' with the vertices u',u'' arising from u if u and v are adjacent. Clearly the new graph can be asymmetrized only by assigning different colors to the elements of each pair v', v''.

However, the only finite vertex-transitive cubic graph of density 1/2 we know of is the prism over a triangle. All other vertex-transitive finite cubic graphs seem to have density $\leq 1/4$, and this also appears to be true for infinite vertex-transitive graphs.

2.3 Arc orbits

We shall classify vertex-transitive cubic graphs with respect to the number of arc orbits and divide the graphs with two arc orbits into graphs with synchronous, respectively asynchronous connection.

The orbit of an arc vw with $v, w \in G$ under the action of $\mathrm{Aut}(G)$ is the set

$$O(vw) = \{xy \mid x = \alpha(v), y = \alpha(w), \alpha \in \mathrm{Aut}(G)\}.$$

By vertex-transitivity every vertex has to be incident to at least one edge from every arc orbit, hence the number of arc orbits in a vertex-transitive cubic graph is 1,2 or 3.

If it is 3, and if we fix a vertex v, then all neighbors of v are also fixed. For a connected graph this implies that all vertices are fixed if one is fixed. If we color one vertex of such a graph black and leave all others white, then this is an asymmetrizing coloring.

If the number of arc orbits is 2, then one orbit consists of isolated edges that meet every vertex, and thus form a so-called *matching*, whereas the edges of the other orbit form a subgraph where every vertex has degree two. By vertex-transitivity such an orbit consists of cycles of the same lengths or of two-sided infinite pathes, also called double rays.

Examples of such graphs are the infinite ladder and the chain of quadrangles, see Figures 1 and 2. The figures also depict asymmetrizing colorings. As we have seen, the asymmetrizing cost of the ladder is 3, but the asymmetrization cost of the chain of quadrangles is infinite.

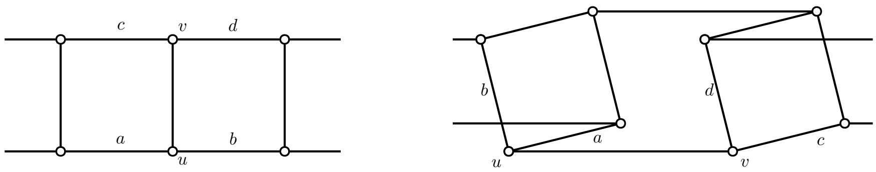

Figure 3: Synchronous and asynchronous connection

The infinite ladder and the chain of quadrangles are also examples of graphs with fundamentally different types of automorphisms. Consider a matching edge $e = uv$ in the ladder, where a, b are the edges incident with u and c, d the edges incident with v. Then any automorphism of the ladder that fixes u and v and swaps a with b also swaps c and d; compare the left side of Figure 3.

By contrast, let $e = uv$ be a matching edge in the chain of quadrangles, where a, b are the edges incident with u and c, d the edges incident with v. Then the edges a,b and the edges c,d can be swapped independently of each other, while u and v remain fixed.

To distinguish these types of graphs we introduce the concepts of *synchronous* and *asynchronous* connection.

Definition 1. *Let G be a cubic graph with two arc orbits, $e = uv$ a matching edge, a, b the edges incident with u and c, d the edges incident with v. If G has an automorphism that swaps a, b, but fixes c and d, then we say that G is asynchronously connected. Otherwise we say the connection is synchronous.*

Note that the existence of one matching edge uv where the edges incident to one endpoint can be swapped while the edges incident to the other remain fixed implies that this is the case for all matching edges because $\mathrm{Aut}(G)$ acts arc transitively on the matching edges.

Clearly the infinite ladder is synchronously connected, but not the infinite chain of quadrangles.

3 Graphs with one arc orbit

If there is only one arc orbit then there exists an automorphism φ to any two arbitrarily chosen edges uv and xy such that $\varphi(u) = x$ and $\varphi(v) = y$. Such graphs are called *arc-transitive.* We subdivide them according to their girth, where the *girth* of a graph G is minimum length of the cycles of G. We denote it by $g(G)$.

We shall prove the following theorem.

Theorem 3.1. *Let G be an arc-transitive cubic graph different from K_4, $K_{3,3}$, the cube and the Petersen graph. If it has finite girth, then $\rho(G) \leq 5$, otherwise it is the T_3, which has infinite asymmetrizing cost and asymmetrizing density 0.*

If a cubic graph has no cycles, then it is the infinite tree T_3 with infinite cost and density zero, see [10].

It remains to prove the theorem for graphs with finite girth. In order to do this we divide the theorem into Lemma 3.1 for girth at most 6 and Lemma 3.2 for girth at least 7. The methods of proof are entirely different.

Lemma 3.1. *Let G be an arc-transitive cubic graph of girth at most 6 different from K_4, $K_{3,3}$, the cube and the Petersen graph. Then $\rho(G) \leq 5$.*

Proof Because we forbid multiple edges the smallest girth is 3. If a symmetric graph G has a triangle and $v \in V(G)$ with neighbors x, y, z, then there must be an edge between any two of them and G is the K_4, which has no asymmetric 2-coloring.

For girth 4, let G be a symmetric graph of girth 4 and $v \in V(G)$ with the neighbors x, y, z. Clearly, any two of the edges vx, vy, vz must span a square. Let the squares be $vxay$, $vybz$ and $vzcx$. If, say, $a = b$, then x is in the two squares $vxay$ and $vxaz$. Let u be the third neighbor of y.

If $u = c$, then every vertex is in three squares, has three neighbors, and G is the $K_{3,3}$ spanned by $\{x, y, z, v, a, c\}$ and has no asymmetric 2-coloring.

If $u \neq c$, then it cannot be in a square with v, because the other 2 neighbors of y, namely v, a, have degree 3. Hence we can assume that a,b,c are pairwise distinct.

We know that y has to be in three squares. By construction it is in the squares $vxay$ and $vybz$, hence the edges ya and yb must be in a square. Let w be the third neighbor to a. The third square containing y clearly must contain the edges by, ya and aw. Thus w is adjacent to b. By the same argument w is also adjacent to c. Therefore G is the cube, which has no asymmetric 2-coloring.

For girth 5 we invoke a result of Glover and Marušić [6], who showed that there are only two edge-transitive cubic graphs of girth 5, namely the Petersen graph and the pentagon dodecahedron. The Petersen graph does not have an asymmetric 2-coloring, and the asymmetrizing cost for the pentagon dodecahedron is 3, as is easily seen.

We conclude the proof with the remark that the Heawood graph is the only finite or infinite cubic graph of girth 6, see [7, Theorem 27]. It is the dual of the triangulation of the torus with underlying graph K_7. As shown in [7] its asymmetrizing cost is 5. $\square$

The Heawood graph is also known as Tutte's 6-cage [14]. Tutte showed that it is the only finite 4-arc regular graph of girth 6. The cited paper is the first on arc-transitive cubic graphs.

For girth ≥ 7 we will heavily rely on Tutte's results in [14, 15], as well as on those of Djokovic and Miller [5], who extended them to infinite graphs.

Following Tutte [14] we call a sequence of vertices $v_0, v_1, \ldots, v_s \in V(G)$ an *s-arc* if $v_i v_{i-1} \in E(G)$ for $1 \leq i \leq s$, but $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$. Then G is *s-arc-transitive* if Aut(G) is transitive on the set of all s-arcs on G. A 1-arc-transitive graph is also called *symmetric*. Moreover, we call G *s-arc-regular* if for any two s-arcs $v_0v_1\dots v_s$ and $w_0w_1\dots w_s$ there is a unique automorphism φ which maps $v_0v_1\dots v_s$ into $w_0w_1\dots w_s$, respecting the order of the vertices.

For symmetric cubic finite graphs Tutte [15] proved the following theorem.

Theorem 3.2 (Tutte 1959)**.** *Let G be a finite connected, symmetric cubic graph. Then G is s-arc-regular for some $s \le 5$.*

Djokovic and Miller [5] extended it to infinite graphs.

Theorem 3.3 (Djokovic and Miller 1980)**.** *Every infinite connected symmetric cubic graph is s-arc-regular for some $s \le 5$ with the exception of the infinite cubic tree.*

For the girth of s-arc-regular cubic graphs we will use the bound

$$2s \le g(G) + 2 \tag{3}$$

from [14].

Lemma 3.2. *Let G be an arc-transitive cubic graph of girth at least 7. Then $\rho(G) \le 4$.*

Proof By symmetry we can invoke Theorems 3.2 and 3.3. They imply that our graphs are s-arc-regular for some $s \le 5$.

If $s = 0$, then G is vertex-regular. It therefore suffices to color exactly one vertex black to break all automorphisms, and thus $\rho(G) = 1$.

If $s = 1, 2$ or 3 we choose a path $uxvw$ in G. This is possible because the girth is > 6. We color u, v, w black as visualized in Figure 4. Each color preserving automorphism φ fixes u because it is the only black vertex without black neighbors. As v and w have different distances from u, they are also fixed. Hence φ fixes the s-arcs ux, uxv and $uxvw$, where $s = 1, 2, 3$, respectively. By s-arc regularity φ is the identity.

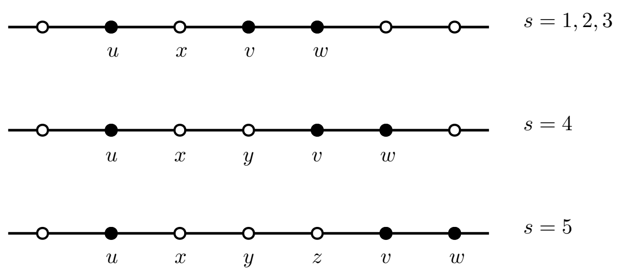

Figure 4: Colorings of s-arcs.

Now, let $s = 4$. For girth $g > 7$ we choose a path $uxyvw$ and color u, v and w black as in Figure 4. Then we argue as before to prove that the 4-arc $uxyvw$ is fixed by all color preserving automorphisms. If the girth is 7 this coloring allows that both v and w have distance 3 from u. In this case it suffices to color y black to fix the 4-arc $uxyvw$ by all color preserving automorphisms.

For $s = 5$ we first observe that the girth is at least 8 by Equation 3. We choose a path $uxyzvw$ of length 5 and color u, v and w black. If the girth is different from 9 this coloring fixes the 5-arc $uxyzvw$. If the girth is 9, then v, w could be interchanged by color preserving automorphisms. To avoid this we also color z black. This fixes $uxyzvw$ by the same arguments as before. By 5-arc regularity this is an asymmetrizing coloring.

Clearly the cost of our colorings is at most 4, which proves the lemma. Together with Theorem 3.3 and Lemma 3.1 this completes the proof of Theorem 3.1. □

4 Graphs with two arc orbits

Now we turn to vertex-transitive cubic graphs G with two arc orbits. Here every vertex of G has two incident edges in an orbit C and one edge in the other orbit D, whose edges that form a complete matching. C consists of cycles or two-sided infinite paths. It is important to note that the group of automorphisms of G acts on the elements of C by reflections and vertex-transitively, that is, as a dihedral group.

We first consider the case when C consists of two-sided infinite paths. It is possible that there is only one edge between any two adjacent paths P_1, P_2. Then G is T_3 and symmetric.

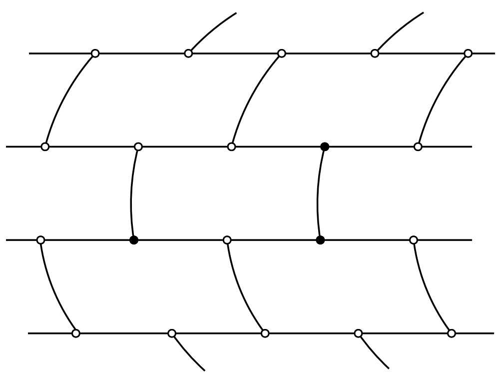

Figure 5: Part of a brick cylinder.

If there are at least two, then there are infinitely many, and it easy to see that either all vertices of P_1 are adjacent to P_2, or every other vertex. In the first case G is the infinite ladder. In the other G looks like a brick cylinder, see Figure 5. In either case $\rho(G) = 3$ and the connection is synchronous. (If C contains infinitely elements, then the result is a tiling of the plane with hexagons, which has only one arc-orbit.)

Hence, we can assume now that the length l of the cycles is finite. Let k be the number of edges between two adjacent cycles. If $k = l$, then G is a prism. If $2 < k < l$, then $k = 1/2$, $l \geq 6$, and G cylinder-like; compare the left two graphs in Figure 6 for $l = 6$. In

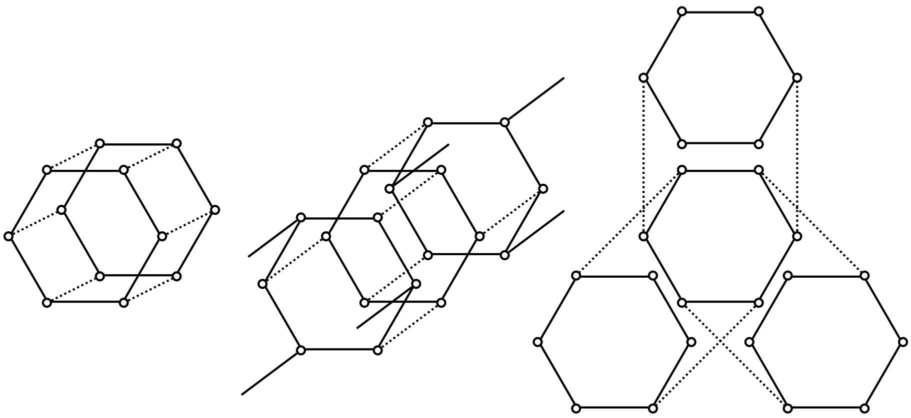

Figure 6: Possible connections of hexagons for $k > 1$.

both cases the connection is synchronous. For the prism $\rho(G) = 3$, and for the graph in the middle it holds $\rho(G) = 2$, see Theorem 4.1.

The graph giving rise to the configuration on the right could be synchronously or asynchronously connected. For its asymmetrizing cost or density see the remark after Theorem 4.2.

This leaves the cases when $k = 1$ or 2. Let us now consider the case when G is synchronously connected.

Theorem 4.1. *let G be a connected vertex-transitive cubic graph with two arc orbits that is synchronously connected. Then $\rho(G) \leq 3$.*

Proof Let C and D be the arc orbits, where D is the set of matching edges. We begin with the observation that it suffices to fix an element of C in order to fix all vertices of the graph. To see this, consider a matching edge uv, where u is incident with the edges a, b and v with the edges c, d, see Figure 7. If the element of C_u containing u is fixed, then also the edges a, b, and by synchronous connection the edges c, d. and thus the element of C_v of C containing v. Hence, if an element of C is fixed, then all neighboring elements of C are fixed too. Because G is connected we can fix all vertices of G by induction.

We have already seen that the $\rho(G)$ is 3 if there are more than two edges between C_u and C_v. We can therefore assume that the number of edges between C_u and C_v is 1 or 2. It can only be 2 if l is at least 4 or infinite. Moreover, if there are two edges between C_u and C_v, then they cannot form a quadrangle.

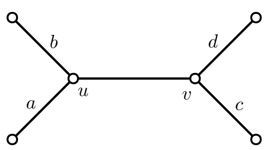

Figure 7: A matching edge with its connections.

We show now that $\rho(G) = 2$ in these cases. Just color u and a neighbor of v, say z, black. There is only one path of length 2 between u and z, because there are no quadrangles with one edge in C_u and the other in C_v. Because the edge of this path that is incident with u is a matching edge, but not the edge on this path that is incident with v, the path cannot be inverted. Hence, u, v, and z are fixed. Thus the edges c, d are also fixed, and by synchronous connection also he edges a, b, and hance C_u. $\square$

So far the only classes of graphs with synchronous connection that we have found are prisms and cylinder-like graphs. We have no general rule to determine whether a graph has synchronous connection.

4.1 Asynchronously connected graphs

We begin with graphs where C consists of triangles with at most one matching edge between any two of them. If the triangles do not form a tree we have the following theorem.

Theorem 4.2. *Let G be a connected, vertex-transitive cubic graph with two arc orbits C and D. If C consists of triangles with only one edge between adjacent triangles, and if the graph obtained from G by contracting each triangle to a single vertex is not a tree and different from K_4, $K_{3,3}$, the cube and the Petersen graph, then $\rho(G) \leq 5$.*

Proof Let G' be the graph obtained from G by contracting each triangle in G to a single vertex. Let the contraction be φ. Then $G' = \varphi(G)$. Clearly G' is a cubic, edge transitive graph. By assumption it is not a tree.

Because G' is different from K_4, $K_{3,3}$, the cube and the Petersen graph $\rho(G') \leq 5$ by Theorem 3.1. If α' is an asymmetrizing coloring of G' we extend it to an coloring α of G, by coloring one vertex in the preimage of each black vertex in G' black. Each color preserving automorphism of G stabilizes each triangle in G, and because there is just one edge between any two adjacent triangles each vertex of G is fixed. $\square$

Because there is some freedom in the choice of the black vertices in the triangles, one can show that $\rho(G) \leq 3$ and that this is also holds if G' is K_4, $K_{3,3}$, the cube and the Petersen graph. For details we refer to [9]. Thus we have the same result for synchronously and asynchronously connected graphs, which is rather unusual.

Now to the cost and density of the graph on the right of 6. Let H be such a graph. It can be contracted to a graph G with two arc orbits, one of which consists of triangles. If the graph G' arising from G by further contraction of the triangles to single vertices is not a tree, then the G has asymmetrization cost 3, and it is easy to see that this is also the case for H. If G' is a tree we can apply Theorem 4.3 to see that G and also H have infinite asymmetrization cost, but density 0.

Theorem 4.3. *Let G be an infinite vertex-transitive cubic graph with two arc orbits C and D, where D consists of a complete matching and G' be the graph obtained from G by contraction of each element of G to a single vertex and by merging multiple edges to single ones. If G' is a tree different from T_2 and K_2, then G has infinite distinguishing cost and $\delta(G) = 0$.*

Proof We first observe that $G' \cong K_2$ if C consists of two-sided infinite paths with more than one edge between neighboring paths. In this case $\rho(G) = 3$. Furthermore, if C consists of quadrangles with two edges between neighboring quadrangles, then $G' \cong T_2$ and $\rho(G) = 3$.

Let G satisfy the assumptions. If C consists of two-sided infinite paths, then G is T_3, $\rho(G) = \infty$ and $\delta(G) = 0$ by [10].

Hence, C consists of cycles of finite length $l \geq 3$ and $G' = T_l$ or $T_{l/2}$, but different from T_2. Then G' has an asymmetrizing 2-coloring of density 0. By coloring one vertex in the preimage of each black vertex of G' black, we obtain the desired coloring of G. □

4.2 Girth 4

In the remainder of the paper the subject of investigation are asynchronously connected graphs with two arc orbits, one of which consists of quadrangles, and the other of disjoint edges, which form a matching. For this class of graphs we do not have complete answers, but will encounter many interesting classes of graphs, in particular a large class of graphs with positive density.

We begin with the case when there are two edges from the set of matching edges between pairs of adjacent quadrangles. Suppose the quadrangles $abcd$ and $uvwz$ are adjacent, and the edges between them are ax and by, where $x, y \in \{u, v, w, z\}$. If x, y are not adjacent, then the other two matching edges originating in $uvwz$ cannot originate from adjacent vertices, but this means that $abcd$ cannot be mapped into $uvwz$, which contradicts the transitivity assumption. Hence xy is an edge. Then the only possible graphs are the cube, the prism, the Moebius ladder, or the infinite ladder. None of these graphs has two arc orbits, where one consists of quadrangles and the other of a matching.

Hence, we can assume that the edges between $abcd$ and $uvwz$ are between opposite vertices of the quadrangles. It is easy to see that the only possible graphs in this case are the ring of at least three quadrangles, see Figure 8, or the chain of quadrangles of Figure 2. As the colorings in the figures indicate, the asymmetrization cost for the ring is the number of quadrangles, and for the chain of quadrangles we have already shown that the asymmetrizing density is $\frac{1}{4}$.

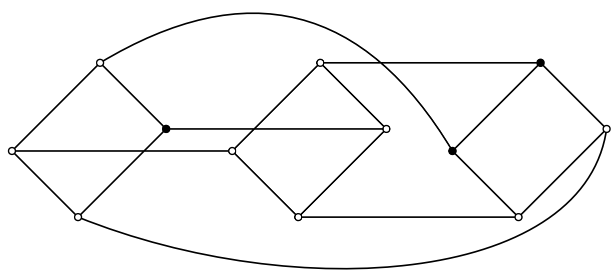

Figure 8: Chain of three quadrangles

Therefore we can assume from now on that there is at most one edge between two quadrangles. For such graphs, and for the chains or rings of quadrangles from above, we define a transformation, which we call *folding*, that reduces them to a smaller graphs. If G folds onto G' we wish to use the information about G' for the construction of asymmetrizing colorings of G.

Given a graph G that is either a chain or ring of quadrangles or a graph with at most one edge between adjacent quadrangles, we partition $V(G)$ into the sets of opposite vertices in the quadrangles, and then form the quotient graph G' of G with respect to this partition. The new vertices are connected by an edge if there is at least one edge between their preimages in G. We call this a folding, because it can be envisaged as an operation on the squares, where we first identify a pair of opposite vertices in each square. This folds the squares edges. Then the paths are folded into single edges. These new edges are disjoint and form a matching.

The edges incident to opposite vertices of the squares remain distinct after folding, but share one endpoint in G', compare Figure 9. It means that they form a subgraph where each vertex has degree 2, that is, a subgraph of cycles of equal lengths. Here we also admit cycles of length 2, that is, double edges, and cycles of infinite length.

Cycles of length 2 occur when we fold a ring or chain of quadrangles, see Figure 10. Double rays appear when G consists of graphs arranged in a tree-like manner.

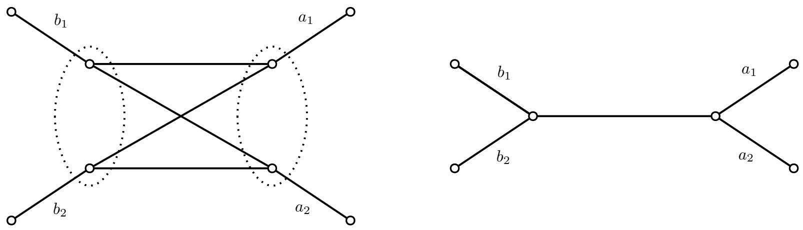

Figure 9: Folding of quadrangles

Figure 10: Chain of single and double edges.

Lemma 4.1. *Let G be a cubic vertex-transitive graph with exactly two arc orbits, one consisting of quadrangles and the other of isolated edges, and G' the corresponding graph after folding. Then G' is vertex-transitive as well and the subgroup of the automorphism group of G' that is induced by* $\mathrm{Aut}(G)$ *contains two arc orbits, one consisting of a matching, and the other one of cycles or double edges.*

Proof G' is formed from G by identifying opposite vertices of each quadrangle and by replacing the four edges between the identified vertices by a single edge. Clearly each automorphism α of G induces an automorphism of G', say $\varphi(\alpha)$, because it preserves pairs of opposite vertices of quadrangles. As $\mathrm{Aut}(G)$ acts transitively on the pairs of opposite vertices of the quadrangles the group $\varphi(\mathrm{Aut}(G)) \subseteq \mathrm{Aut}(G')$ acts transitively on G'.

Clearly $\varphi(\mathrm{Aut}(G))$ acts transitively on the images of the quadrangles and transitively on the images of the matching edges in G, but $\varphi(\mathrm{Aut}(G))$ does not map images of quadrangles into images of matching edges. $\square$

If G consists of quadrangles that are arranged in a tree like manner, then G' has only one arc orbit, despite the fact that $\varphi(\mathrm{Aut}(G))$ has two arc orbits. Clearly G' cannot be folded again.

To find a condition when G' can be folded again, let us consider the case when the images of the matching edges of G form quadrangles. Then no edge of the matching edges of G' can be in a four-cycle, because then there would have to be two edges between two neighboring quadrangles of G, and the origins of the edges would have to be adjacent. But we already excluded this case earlier. Hence, the matching edges of G' form an orbit under $\mathrm{Aut}(G')$ if the images of the matching edges of G consist of quadrangles. In this case we can also fold G'.

Now we show that asymmetrizing colorings of G' induce asymmetrizing colorings of G.

Lemma 4.2. *Let G be a cubic vertex-transitive graph with an arc orbit consisting of quadrangles, and G' be its corresponding graph after folding. Then any asymmetrizing coloring of G' induces an asymmetrizing coloring of G with a bijection between the black vertices in G' and G.*

Proof First we clarify how an asymmetrizing coloring c' of G' induces one of G. Let $v' \in G'$ be a colored vertex and v_1, v_2 be its its preimages in G. Then we choose randomly one of the preimages and color it. Let c be this coloring of G.

Suppose an automorphism α of G preserves c. As α preserves the set of opposite vertices of the quadrangles in G, which are the preimages of the vertices in G', it induces an automorphism $\varphi(\alpha)$ of G'. Moreover, if a preimage v_1, v_2 is mapped into u_1, u_2 by α, then either both pairs contain exactly one colored vertex, or both pairs contain only uncolored vertices. But then $\varphi(\alpha)$ preserves c' and must be the identity mapping, which means that all pairs of vertices of G are stabilized by α and that pairs that have just one colored vertex are fixed pointwise.

This means that we have to consider the possible interchange of the two uncolored opposite vertices u_1, u_2 in a quadrangle. Let u' be the image of $\{u_1, u_2\}$ under folding. Clearly an interchange of u_1,u_2 would induce an interchange of the two edges incident with u', that is, the images of the matching edges incident with u_1, resp. u_2 in G. But this is prohibited as G' as c' is asymmetrizing.

The assertion about the bijection between the black vertices in the colorings of G and G' follows from the construction. $\square$

As a simple application let us have a look at the graph G consisting of quadrangles that are arranged in a tree-like manner. By folding we obtain an infinite cubic tree G'. We know that such trees have asymmetrizing 2-colorings of density zero. Any such coloring induces an asymmetrizing 2-coloring of G, and it is easy to see that the density is still zero.

4.2.1 Finite Graphs with two arc orbits consisting of a set of squares and a matching

We first consider graphs with two arc orbits consisting of a set of squares and a matching that can be reduced to a ring of m single and double edges. As we do not allow triple edges, $m \geq 2$. If G can be folded onto such a ring by n foldings, we denote it by $P(n, m)$. As the processes of folding and defolding yield unique graphs, up to isomorphisms, $P(n, m)$ is uniquely defined. Also, the graphs $P(1, m)$ are the rings of m quadrangles.

$P(1, 2)$ is the cube and not 2-distinguishable, $P(2, 2)$ is the Cartesian product of a C_8 by and edge, it has two arc orbits, one consisting of two cycles of length 8, and is not symmetrically connected. It has 2-distinguishing cost 3. We can still apply defolding and all defolded graphs, that is, all $P(n, 2)$, where $n \geq 2$, will have 2-distinguishing cost at most 3. They are not symmetrically connected.

Hence we are only interested in the case when $m \geq 3$. In the sequel we will show that the $P(n, m)$ graphs with $m \geq 3$ and $1 \leq n \leq m - 1$ are the so-called Split Praeger–Xu graphs, SPX–graphs for short. We will define them now and determine their 2-distinguishing costs.

For the graphs $P(m, m)$, $m > 2$, we will show that they have 2-distinguishing cost 1, and hence three arc orbits.

Definition 2. *Let n and m be positive integers with $m \geq 3$ and $1 \leq n \leq m-1$. The Split Praeger–Xu graph* $\mathrm{SPX}(2, n, m)$ *has vertex-set $\mathbb{Z}_2^n \times \mathbb{Z}_m \times \{+, -\}$ and edge-set*

$$\{\{(i_0, i_1, \ldots, i_{n-1}, x, +), (i_1, i_2, \ldots, i_n, x+1, -)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}$$
$$\cup \{\{(i_0, i_1, \ldots, i_{n-1}, x, +), (i_0, i_1, \ldots, i_{n-1}, x, -)\} \mid i_j \in \mathbb{Z}_2, x \in \mathbb{Z}_m\}.$$

These are cubic, bipartite graphs. For $\mathrm{SPX}(2, 2, m)$, where m is large, compare Figure 11.

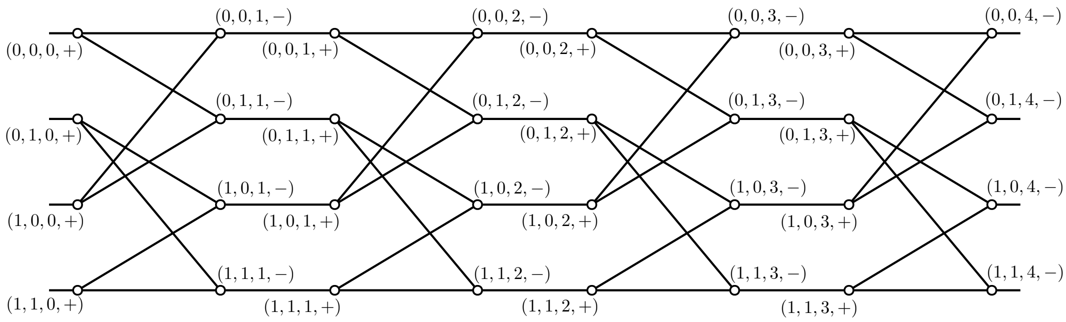

Figure 11: Part of $\mathrm{SPX}(2, 2, m)$ for large m

In [13] it is shown that the wreath product $W = \mathbb{Z}_2^m \rtimes D_m$ acts on the vertex set of $\mathrm{SPX}(2, n, m)$, that is, on $V(\mathrm{SPX}(2, n, m)) = \mathbb{Z}_2^n \times \mathbb{Z}_m \times \{+, -\}$ via the following action: for $g = (g_0, \ldots, g_{m-1}, h) \in W$, with $g_0, \ldots, g_{m-1} \in \mathbb{Z}_2$ and $h \in D_m$, set

$$(i_0, i_1, \dots, i_{n-1}, x, \pm)^g = \begin{cases} (i_0 + g_x, i_1 + g_{x+1}, \dots, i_{n-1} + g_{x+n-1}, x^h, \pm) & \text{if } h \in \mathbb{Z}_m, \\ (i_0 + g_x, i_1 + g_{x+1}, \dots, i_{n-1} + g_{x+n-1}, x^h, \mp) & \text{otherwise.} \end{cases}$$

Here the subscripts are to be understood modulo m and x^h denotes the image of x under h. Clearly the action is vertex transitive and faithful, that is, any two different group elements act differently on $V(\mathrm{SPX}(2, n, m))$.

In fact, by [13, Lemma 2.8] W is the full group of automorphisms of $\mathrm{SPX}(2, n, m)$ if $m \geq 5$ and $1 \leq n \leq m - 1$.

If we choose g such that $g_0, g_1, \dots, g_{n-2}, g_n, \dots, g_{m-1}$ and h are equal to 0, then

$$(i_0, i_1, \dots, i_{n-1}, x, \pm)^g = (i_0, i_1, \dots, i_{n-1}, x, \pm)$$

for $n \leq x \leq m$. Calling the subgraph of $\mathrm{SPX}(2, n, m)$ that is spanned by the vertices with the same x the *x-th column,* this means that g fixes all vertices in the last $m - n$ columns. Because $n \leq m - 1$, at least one column is fixed pointwise. In other words, at least one vertex is moved in columns 0 to $n - 1$, and all vertices in the other columns are fixed pointwise.

In [13] it is observed that the subgraphs spanned by the vertices whose $(n + 1)$-th coordinates are x and $x + 1$ consist of disjoint quadrangles. We call these subgraphs columns of quadrangles, not to be confused with the columns of matching edges. If we fold an $\mathrm{SPX}(2, n, m)$ graph, where $n \geq 2$, then we obtain an $\mathrm{SPX}(2, n - 1, m)$ graph. Clearly $\mathrm{SPX}(2, 1, m)$ is a ring of quadrangles and, folding $\mathrm{SPX}(2, 1, m)$, we reach a ring consisting of m double and m single edges.

Theorem 4.4. *Let G be an* $\mathrm{SPX}(2, n, m)$ *graph where $m \geq 5$ and $1 \leq n \leq m-1$. Then G admits an asymmetrizing 2-coloring with $\rho(G) = \lceil \frac{m}{n} \rceil$, unless $\lceil \frac{m}{n} \rceil = 2$. Then $\rho(G) = 3$.*

Proof By [13, Lemma 2.8] the group W defined above is $\mathrm{Aut}(\mathrm{SPX}(2, n, m))$. Its action on the columns is that of D_m. We choose vertex $v_0^- = (0, \dots, 0, x = 0, -)$ in the half column $(0, -)$ and its images under the action of $(0, n), (0, 2n), \dots, (0, (\lfloor \frac{m}{n} \rfloor - 1)n) \in \mathbb{Z}_2^m \rtimes \mathbb{Z}_m < \mathbb{Z}_2^m \rtimes D_m$. We color these vertices black, together with the vertex v_0^+ in column $(0, +)$ that is adjacent to v_0^- in column $(0, -)$. Notice that v_0^- is fixed when v_0^+ is fixed, and that setting v_0^+ black prevents shifting and switching of the columns by an element of order 2 in D_m if $\rho(G) = \lceil \frac{m}{n} \rceil \neq 2$.

The color preserving automorphisms stabilize the columns (and half columns). Suppose g is an automorphism moving a vertex in half column $(x, \pm)$.

It is of the form $g = (g_0, \dots, g_{m-1}, 0)$ and its action is

$$(i_0, i_1, \dots, i_{n-1}, x, \pm)^g = (i_0 + g_x, i_1 + g_{x+1}, \dots, i_{n-1} + g_{x+n-1}, x, \pm).$$

Hence there is some $0 \leq k \leq n - 1$ such that $i_k \neq i_k + g_{x+k}$, which means that $g_{x+k} = 1$. Consider the largest multiple tn of n such that $tn \leq x + k$ and observe that $tn \leq x + k < (t + 1)n$. Now, the vertex $v_{tn}^- = (0, \dots, 0, tn, -)$ is colored black. We see that $(v_{tn}^-)^g$ has a

1 at entry $x+k$ and thus $(v_{tn}^-)^g \neq v_{tn}^-$, but v_{tn}^- is the only black vertex in the half column $(tn,-)$, meaning that g is not color preserving.

Hence our coloring breaks $\mathrm{Aut}(\mathrm{SPX}(2,n,m))$. □

Proposition 4.5. *The 2-distinguishing cost of the* $\mathrm{SPX}(2,n,m)$ *graphs that are not covered by Theorem 4.4 is 3.*

Proof We have to treat the cases $m=3$ and 4.

$\mathrm{SPX}(2,1,3)$ has distinguishing cost 3, as depicted in Figure 8. By Lemma 4.2 this implies that $\rho(\mathrm{SPX}(2,2,3)) \leq 3$. As we know that it cannot be 2, we infer that it is $\rho(\mathrm{SPX}(2,2,3)=3$.

$\mathrm{SPX}(2,1,4)$ has distinguishing cost 4, as is easily seen by extending the argument for $\mathrm{SPX}(2,1,3)$. $\rho(\mathrm{SPX}(2,2,4)=3$, which can be checked directly. Then $\rho(\mathrm{SPX}(2,3,4))=3$ by the same arguments as before.

Proposition 4.6. *The distinguishing cost of the* $P(n,m)$ *graphs for* $n \geq m \geq 3$ *is 1.*

Proof $P(m,m)$, $m \geq 3$, is obtained from $\mathrm{SPX}(2,m-1,m)$ by defolding. It has the same structure as $\mathrm{SPX}(2,m,m)$ if we relax the condition that $n \leq m-1$ in the definition of SPX graphs. Also W acts on $\mathrm{SPX}(2,m,m)$.

Let us consider the case $m=3$ first. Suppose we color $(1,1,1,0,+)$ in $G=\mathrm{SPX}(2,3,3)$ black. Since G has two arc orbits there must be an automorphism that maps $(1,1,1,1,-)$ into $(1,1,0,1,-)$, and $(1,1,1,1,+)$ into $(1,1,0,1,+)$. Thus the set of neighbors of the vertex $(1,1,1,1,+)$ is mapped into into the set of neighbors $(1,1,0,1,+)$, that is the set of vertices $\{(1,1,1,2,-),(1,1,0,2,-)\}$ is mapped into $\{(1,0,1,2,-),(1,0,0,2,-)\}$, and hence mapping the set $\{(1,1,1,2,+),(1,1,0,2,+)\}$ into $\{(1,0,1,2,+,(1,0,0,2,+)\}$. But this is not possible, because $(1,1,1,2,+)=(1,1,1,0,+)$, and $(1,1,1,0,+)$ is fixed.

But then $P(3,3)$ has three arc orbits, 2-distinguishing cost 1 and thus all $P(n,3)$ graphs for $n>3$ by Lemma 4.2.

Similarly we show that $\gamma(P(m,m))=1$ for $m>3$, and hence this also holds for all $P(n,m)$ with $n \geq m > 3$. □

We can now characterize SPX graphs.

Theorem 4.7. *The Split Praeger–Xu graphs* $\mathrm{SPX}(2,n,m)$*, where* $m \geq 3$ *and* $1 \leq n \leq m-1$*, are exactly those* $P(n,m)$ *graphs with two arc-orbits that are asynchronously connected.*

Proof We first observe that $P(1,2)$ is the cube, which has only one arc-orbit, and that $P(n,2)$ is synchronously connected for $n \geq 2$, as we have shown. Furthermore, by Proposition 4.6, the distinguishing cost of the $P(n,m)$ graphs for $n \geq m \geq 3$ is 1. Hence they have three arc orbits. □

4.2.2 Infinite Split Praeger–Xu graphs

We now extend the definition of Split Praeger–Xu graphs to infinite graphs, show that they have uncountable automorphism groups, and determine their 2-distinguishing density.

If we replace $\mathbb{Z}_m$ in Definition 2 by $\mathbb{Z}$, the we obtain infinite graphs, for which we introduce the notation $\mathrm{SPX}(2,n)$. Their vertex sets are $\mathbb{Z}_2^n \times \mathbb{Z} \times \{+,-\}$.

Theorem 4.8. *Each* $\mathrm{SPX}(2,n)$ *graph admits an asymmetrizing 2-coloring of density* $\delta(G) = \frac{1}{n2^{n+1}}$.

Proof Let W be the group $\mathbb{Z}_2^\infty \rtimes D_\infty$ with the following action on $V(\mathrm{SPX}(2,n))$: for $g = (\dots, g_{-1}, g_0, g_1, \dots, h) \in W$, with $\dots, g_{-1}, g_0, g_1, \dots \in \mathbb{Z}_2$ and $h \in D_\infty$, let

$$(v_0, v_1, \dots, v_{n-1}, x, \pm)^g = \begin{cases} (v_0 + g_x, v_1 + g_{x+1}, \dots, v_{n-1} + g_{x+n-1}, x^h, \pm) & \text{if } h \in \mathbb{Z}_\infty, \\ (v_0 + g_x, v_1 + g_{x+1}, \dots, v_{n-1} + g_{x+n-1}, x^h, \mp) & \text{otherwise.} \end{cases}$$

As in the finite case one sees that the action is faithful, vertex transitive, and that the set of columns is stabilized. By the same arguments as before one also sees that there are group elements that move at least one vertex in columns 0 to $n-1$, but fix all other vertices. Hence there are automorphisms that move at least one vertex in columns kn to $(k+1)n-1$ and fix all other vertices. Let A be the set of these automorphism. A has infinitely many elements, the product of the elements in any subset of A is well defined, and different subsets yield different products. Hence the number of automorphism of $\mathrm{SPX}(2,n)$ is uncountable. By a result of [3] this means that $\mathrm{SPX}(2,n)$ has no finite asymmetrizing set.

Although W stabilizes the set of columns, we have not shown this for $\mathrm{Aut}(\mathrm{SPX}(2,n))$ yet. To see this, we first observe that folding $\mathrm{SPX}(2,n)$ results in $\mathrm{SPX}(2,n-1)$, and that the folding preserves columns. Furthermore, any automorphism of $\mathrm{SPX}(2,n)$ induces an automorphism of $\mathrm{SPX}(2,n-1)$ by Lemma 4.1. Hence, if columns are not preserved in $\mathrm{SPX}(2,n)$, then they are not preserved in $\mathrm{SPX}(2,n-1)$, and consequently not in $\mathrm{SPX}(2,1)$, which is not the case. Therefore the set of columns is preserved.

We now choose an integer k that is a multiple of n larger than 5 and consider the columns $(ik,\pm)$, $i \in \mathbb{N}$. There are 2^n edges in each column, that is, a finite number, and there are only $2^n!$ ways to order them. By the vertex transitivity of $\mathrm{SPX}(2,n)$ there are infinitely many automorphism φ_i that map column $(0,\pm)$ into column $(ik,\pm)$, hence at least two of them preserve the order of the edges in the columns, say φ_r and φ_s, where $r < s$. But then $\varphi_s\varphi_r^{-1}$ maps $(0,\pm)$ into $(s-r,\pm)$. We now identify the half column $(s-r,+)$ with the half column $(0,-)$ to obtain a graph H isomorphic to $\mathrm{SPX}(2,n,(s-r)n)$.

Clearly the action of the vertex stabilizer of v_0 on the first n columns of $\mathrm{SPX}(2,n)$ is the same as that of the vertex stabilizer of v_0 in H on its first n columns. For this case we have shown, that fixing v_0 (and the order of the columns) fixes each element of the first n columns.

Similarly, by vertex transitivity, fixing a vertex in the half column $(jn,+)$ fixes all vertices in the n columns $(jn,+),\dots,((j+1)n-1,+)$. Coloring v_0 black and one vertex each in the half columns $(jn, +)$, where $n \neq 0$, prevents translation of the columns and inverting their order, so all the black vertices remain fixed, and by the above remark also all vertices in their columns and the following $n-1$ columns. This fixes all vertices of $\mathrm{SPX}(2, n)$.

Because the number of vertices in n columns is $n2^{n+1}$, the density of this coloring is $\frac{1}{n2^{n+1}}$. Let us just note that it is enough to show this for one root vertex v, because $|B(v, n+k)| = |B(v, n)| + k2^n$ for $k > 2m$, and therefore we can apply Lemma 2.2 to see that the density is well defined. □

If one folds an $\mathrm{SPX}(2, n)$ graph, where $n > 1$, one obtains the graph $\mathrm{SPX}(2, n-1)$, and if one folds $\mathrm{SPX}(2, 1)$ the result is a chain of single and double edges. The $\mathrm{SPX}(2, n)$ are thus exactly the graphs with two arc orbits consisting of a set of quadrangles and a matching that can be reduced to a chain of single and double edges by a finite number of foldings.

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