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Point sets and sequences with the optimal order of L_p discrepancy

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Abstract

In this thesis we study the problem of finding explicit constructions for low-dimensional finite point sets and infinite sequences in the unit interval with the optimal order of L_p discrepancy for $1 \leq p < \infty$. The L_p discrepancy - defined as the L_p norm of the so-called discrepancy function - is a quantitative measure for the irregularities of distribution of point sets and has strong connections to uniform distribution modulo 1 of sequences, which is a branch of number theory.

Our constructions are based on the Hammersley point set and the van der Corput sequence, which both have a long history in discrepancy theory. While it is well known that the L_∞ norm of their discrepancy function (better known as the star discrepancy) is of best possible order, respectively, the same is not the case for the L_p discrepancy when $1 \leq p < \infty$.

It is the aim of this thesis to consider slightly modified versions of the Hammersley point set and the van der Corput sequence whose L_p discrepancy is of optimal order, respectively. First we try to tackle this problem directly and prove very precise formulas on the L_2 and L_4 discrepancy. In particular, we find an exact formula for the L_4 discrepancy of digitally shifted Hammersley point sets and compute the L_2 discrepancy of symmetrized Hammersley point sets and van der Corput sequences precisely.

To obtain results for all parameters $1 \leq p < \infty$, we employ methods from harmonic analysis, namely Haar functions and Littlewood-Paley theory. These tools enable us to find a large class of point sets and sequences with the optimal order of L_p discrepancy. Furthermore, the approach via Haar functions allows us to study the norm of the discrepancy function in other functions spaces such as Besov spaces of dominating mixed smoothness also, and we will pursue this aim in this thesis.

Kurzfassung

In dieser Dissertation behandeln wir das Problem niedrigdimensionale endliche Punktmengen sowie unendliche Folgen im Einheitsintervall zu konstruieren, deren L_p-Diskrepanz für $1 \leq p < \infty$ die optimale Ordnung aufweist. Die L_p-Diskrepanz - definiert als die L_p-Norm der sogenannten Diskrepanzfunktion - ist ein quantitatives Maß für die Unregelmäßigkeiten einer Punktverteilung und ist stark verknüpft mit der Gleichverteilung modulo 1 von Folgen, welche ein Teilgebiet der Zahlentheorie ist.
Unsere Konstruktionen basieren auf der Hammersley-Punktmenge sowie der van der Corput-Folge, die beide schon sehr lange im Rahmen der Diskrepanztheorie untersucht werden. Während es eine bekannte Tatsache ist, dass die L_∞-Norm der entsprechenden Diskrepanzfunktion (besser bekannt als Sterndiskrepanz) jeweils die optimale Ordnung hat, ist das für die L_p-Norm für $1 \leq p < \infty$ nicht der Fall.
Ziel dieser Dissertation ist es, modifizierte Varianten der Hammersley-Punktmenge und der van der Corput-Folge zu finden, deren L_p-Diskrepanz jeweils von optimaler Ordnung ist. Zuerst versuchen wir dieses Problem direkt anzugehen und beweisen sehr präzise Formeln für die L_2- und die L_4-Diskrepanz. Konkret finden wir eine exakte Formel für die L_4-Diskrepanz von verallgemeinerten Hammersley-Punktmengen und berechnen die L_2-Diskrepanz von symmetrisierten Hammersley-Punktmengen und van der Corput-Folgen sehr genau.
Um für alle Parameter $1 \leq p < \infty$ Resultate zu erhalten, machen wir Gebrauch von Methoden aus der harmonischen Analysis, nämlich Haarfunktionen und Littlewood-Paley-Theorie. Diese Mittel ermöglichen es uns, eine große Klasse von Punktmengen und Folgen mit einer L_p-Diskrepanz von optimaler Ordnung zu erhalten. Zudem erlaubt der Zugang über Haarfunktionen auch das Behandeln der Norm der Diskrepanzfunktion in anderen Funktionenräumen, wie etwa den Besov-Räumen, weshalb wir dieses Ziel ebenfalls in dieser Dissertation verfolgen werden.

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Contents

Preface

Discrepancy theory is a part of number theory and deals with the irregularities of point distributions on certain domains. In this thesis we follow the classical case and consider point sets and sequences in the s-dimensional unit interval $[0,1)^s$. For a given set of N points $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ the discrepancy function is defined as the difference of the actual number of points in a subinterval $I \subseteq [0,1)^s$ (where I is often anchored in the origin) and the expected value under the assumption of hypothetic uniform distribution. By taking some norm of the discrepancy function, e.g. the L_p norm for any $p \in [1,\infty]$, one obtains a quantitative measure for the irregularity of distribution of this point set. We speak of the L_p discrepancy and of the star discrepancy (for $p = \infty$), respectively. For infinite sequences, we consider the discrepancy of its first N elements and observe its behaviour as N increases. It is clear that studying the discrepancy of finite point sets and infinite sequences are two different issues, as for infinite sequences we have to assure that the discrepancy of all initial segments $\{\boldsymbol{x}_0\}$, $\{\boldsymbol{x}_0, \boldsymbol{x}_1\}$, $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2\}$ $\dots$, $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ for $N \geq 2$ is low, whereas N is fixed for finite point sets and only the behaviour of $\{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{N-1}\}$ is relevant. It is often observed that the discrepancy of point sets in dimension $s+1$ is related to the discrepancy of infinite sequences in dimension s. We will find however that this is not necessarily the case for all norms taken of the discrepancy function. Discrepancy theory is not only interesting for itself, but has relations to numerical integration using quasi-Monte Carlo algorithms and to uniform distribution modulo 1.

In this thesis we will investigate the L_p discrepancy of finite point sets in $[0,1)^2$ and infinite sequences in $[0,1)$. Our explicit constructions are modified variants of the well known Hammersley point sets and the van der Corput sequences. These are prominent examples of point sets in $[0,1)^2$ and sequences in $[0,1)$ with the best possible order of star discrepancy in N, which is $\log N$ according to a result of Schmidt (1980). However, it is well known that this rate is not optimal for the L_p discrepancy if $p \in [1,\infty)$, where $\sqrt{\log N}$ is best possible. This is known from famous results by Roth (1954) and Schmidt (1972), where especially Roth's lower bound on the L_2 discrepancy established discrepancy theory as an interesting and well-studied subject. However, the mentioned constructions fail to achieve the optimal L_p discrepancy rate. To this end, we need to apply certain modifications to these point sets and sequences, among which are digital shifts and symmetrization. These modified variants are known to have the optimal order of L_2 discrepancy. It is an important aim of this thesis to extend these results to the L_p discrepancy.

The problems treated in this thesis are twofold:

- We would like to find exact formulas for the L_p discrepancy of point sets achieving the optimal rate of this quantity. Such formulas are rare, since they are difficult to obtain. Results of this kind have previously been found, among others, by Halton, Zaremba, Faure, Kritzer and Pillichshammer, who considered generalized

Hammersley point sets in base 2 or arbitrary bases. All these authors provide exact formulas only for the L_2 discrepancy. For the first time, we will also find an exact formula for $p \neq 2$, namely for the L_4 discrepancy of digit shifted Hammersley point sets. Further, we will study the L_2 discrepancy of symmetrized Hammersley point sets and van der Corput sequences in arbitrary bases thoroughly. The aim of all these results is to compute the leading coefficients of the leading $\sqrt{\log N}$-term. The methods to find such exact formulas are of elementary, number-theoretic nature and require dealing carefully with digit expansions of numbers.

- It will turn out that a precise computation of the L_p discrepancy of modified Hammersley point sets or van der Corput sequences would be extremely difficult and technical for $p \notin \{2, 4\}$. We will therefore exploit other techniques to show the optimal L_p discrepancy rate for certain point sets and sequences, which however will not provide exact constants of the leading term. The necessary tools come from harmonic analysis and involve Haar functions, Littlewood-Paley theory and function spaces of dominating mixed smoothness, which have previously been used for example in works of Triebel, Hinrichs, Markhasin, Dick, Pillichshammer and others. These methods are strong enough to classify those variants of the Hammersley point sets and the van der Corput sequences which achieve the optimal order of L_p discrepancy for all $p \in [1, \infty)$. In particular, this thesis contains the first explicit constructions of infinite sequences in $[0, 1)$ with an L_p discrepancy of order $\sqrt{\log N}$ for all $N \geq 2$ and all $p \in [1, \infty)$, namely symmetrized van der Corput sequences. Further, the approach via Haar functions opens the door to consider the norm of the discrepancy function in spaces of dominating mixed smoothness and further function spaces, and we will do so for our point sets and sequences in interest. The most surprising result which we will obtain is the fact that for positive smoothness parameters infinite sequences in the unit interval $[0, 1)$ can achieve the same optimal discrepancy rate in Besov spaces of dominating mixed smoothness as finite point sets in $[0, 1)$.

The structure of this work is as follows: In the first chapter we give a definition of discrepancy and present several simple examples. We further comment on its relations to uniform distribution modulo 1 and numerical integration and state known facts on general bounds on the discrepancy. In Chapter 2 we introduce several variants of the Hammersley point set and the van der Corput sequence, give a survey on known results and explain the basic tools for our proofs. These tools are exact formulas for the discrepancy function of Hammersley point sets, as well as the basic information on harmonic analysis. Chapter 3 is dedicated to our precise study of discrepancy, as explained in the first point above, whereas Chapter 4 deals with the estimation of L_p discrepancy and beyond via the harmonic analysis approach, as explained in the second point.

Basic notation We briefly introduce the basic notation used throughout this thesis.

We denote by $\mathbb{N}$ the set of positive integers $\{1, 2, 3, \dots\}$ and write $\mathbb{N}_0$ if we also include zero. The set of all integers $\{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ is denoted by $\mathbb{Z}$. By $\mathbb{Q}$ and $\mathbb{R}$ we mean the set of rational and real numbers, respectively.

We will use the basic notation from set theory. For two sets A and B we write their union and intersection as $A \cup B$ and $A \cap B$, respectively, and $A \setminus B$ is the set of all elements, which are contained in A but not in B. We write $x \in A$ if x is an element of the set A. We write $\emptyset$ for the empty set, which does not contain any elements. Let M be a universe and $A \subseteq M$. The function $\mathbf{1}_A : M \to \{0, 1\}$ defined through $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$ is the indicator function of the set A. The cardesian product of two sets is given by

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

For $s \in \mathbb{N}$ we write $A^s := A \times A \times \cdots \times A$. We will mainly consider intervals of the form $[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$ for $a, b \in \mathbb{R}$ with $a \leq b$. For $\mathbf{a} = (a_1, a_2, \dots, a_s) \in \mathbb{R}^s$ and $\mathbf{b} = (b_1, b_2, \dots, b_s) \in \mathbb{R}^s$, where $a_i \leq b_i$ for all $i \in \{1, 2, \dots, s\}$, we set

$$[\mathbf{a}, \mathbf{b}) = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_s, b_s).$$

For a finite set $\mathcal{P}$ we mean by $|\mathcal{P}|$ the number of its elements. For a Lebesgue measurable set A we denote by $|A|$ its Lebesgue measure, e.g. we have $|[a, b)| = b - a$.

The function $\log_b$ is the logarithm in base b. If we omit the lower index b, then we always mean the natural logarithm in base e, where e is Euler's number. We will sometimes consider exponential functions of the form $\mathrm{e}^{2\pi \mathrm{i} \cdot}$, where π is the perimeter of a circle with diameter 1 and where $\mathrm{i} := \sqrt{-1}$.

Let $x \in \mathbb{R}$. By $|x|$ we denote its absolute value. Then $\lfloor x \rfloor$ is the largest integer z such that $z \leq x$, and $\lceil x \rceil$ is the smallest integer z such that $z \geq x$. We speak of $\{x\} = x - \lfloor x \rfloor$ as the fractional part of x. The distance of x to its nearest integer is denoted by $\|x\|$, which can also be defined via $\|x\| := \min(\{x\}, 1 - \{x\})$. For a complexe number $z = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$, its absolute value is given by $|z| := \sqrt{a^2 + b^2}$.

For functions $f, g : \mathbb{N} \to \mathbb{R}^+$, we write $g(N) \lesssim f(N)$ and $g(N) \gtrsim f(N)$, if there exists a $C > 0$ such that $g(N) \leq Cf(N)$ or $g(N) \geq Cf(N)$ for all $N \in \mathbb{N}$, $N \geq 2$, respectively. This constant C is independent of N, but might depend on several other parameters $\alpha_1, \dots, \alpha_i$, which we sometimes emphasize by writing $\lesssim_{\alpha_1, \dots, \alpha_i}$ and $\gtrsim_{\alpha_1, \dots, \alpha_i}$, respectively. Further, we write $f(N) \asymp g(N)$ if the relations $g(N) \lesssim f(N)$ and $g(N) \gtrsim f(N)$ hold simultaneously. We write $f(N) = \mathcal{O}(g(N))$ if there exists a constant $C > 0$ such that $f(N) \leq Cg(N)$ for all $N \in \mathbb{N}$.

Finally, we will make extensive use of the fact that for an integer base $b \geq 2$ every number $n \in \mathbb{N}$ has a unique representation of the form

$$n = n_k b^k + \cdots + n_1 b + n_0,$$

where $n_i \in \{0, 1, \dots, b-1\}$ for all $i \in \{0, \dots, k\}$ and $n_k \neq 0$. We speak of the b-adic (or dyadic, if $b = 2$) expansion of n.

1. Discrepancy of point sets and sequences

1.1. Definition of discrepancy and general facts

1.1.1. Discrepancy and uniform distribution modulo 1

The notion of discrepancy describes the irregularity of point distributions on certain sets. The most studied case deals with points in the multi-dimensional unit interval $[0,1]^s$. The two central questions are:

- For a given integer $N \geq 1$: How can we distribute an N-element set of points $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ as uniformly as possible?
- How can we construct infinite sequences $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ in $[0,1)^s$ such that for every integer $N \geq 1$ the first N elements of $\mathcal{S}$ are well distributed in the unit interval?

To answer these questions, we should first define an adequate measure for the irregularity of point distributions in the unit interval. Intuitively, one would probably consider a set of points $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ to be well distributed in the unit interval, if each measurable subset U of $[0,1]^s$ contains a number of elements in $\mathcal{P}$ which is proportial to the Lebesgue measure of U. However, it turns out that it is too restrictive if we demand this for every arbitrary subset U. In this thesis, we will follow the classical approach and measure the irregularity of point distributions with respect to intervals of the form $[\boldsymbol{0}, \boldsymbol{t})$. Here, for $\boldsymbol{t} = (t_1, t_2, \dots, t_s) \in [0,1]^s$, we mean by $[\boldsymbol{0}, \boldsymbol{t})$ the s-dimensional interval

$$[\boldsymbol{0}, \boldsymbol{t}) = [0, t_1) \times [0, t_2) \times \dots \times [0, t_s).$$

We need the following essential notation.

Definition 1.1. For a given N-element point set $\mathcal{P} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{N-1}\}$ in $[0,1)^s$, the expression

$$\Delta_N(\boldsymbol{t}, \mathcal{P}) := A_N([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}) - N t_1 t_2 \dots t_s$$

is called the discrepancy function of $\mathcal{P}$, where $A_N([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}) := |\mathcal{P} \cap [\boldsymbol{0}, \boldsymbol{t})|$. Note that we can also write $A_N([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}) = \sum_{n=0}^{N-1} \mathbf{1}_{[\boldsymbol{0}, \boldsymbol{t})}(\boldsymbol{x}_n)$, where $\mathbf{1}_I$ denotes the indicator function of the interval I. We will do so in appropriate parts of this thesis. We will speak of $A_N([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P})$ as the counting part of the discrepancy function and of $N t_1 t_2 \dots t_s$ as its volume part.

The discrepancy function describes the difference of the actual number of elements in $\mathcal{P}$ which lie in $[\boldsymbol{0}, \boldsymbol{t})$ and the expected value $N|[\boldsymbol{0}, \boldsymbol{t})| = N t_1 t_2 \dots t_s$. This difference should be as small as possible for all intervals $[\boldsymbol{0}, \boldsymbol{t}) \subset [0,1]^s$. In other words, the supremum $\sup_{\boldsymbol{t} \in [0,1]^s} |\Delta_N(\boldsymbol{t}, \mathcal{P})|$, i.e. the supremum norm of the discrepancy function, should be small. It is therefore reasonable to consider a norm of the discrepancy function. This leads us to the central definition of this thesis.

Definition 1.2. Let $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ be a set of N points in the unit interval $[0,1)^s$. Then the star discrepancy of $\mathcal{P}$ is the supremum norm of the discrepancy function, i.e.

$$L_{\infty,N}(\mathcal{P}) := \sup_{\boldsymbol{t}\in[0,1]^s} |\Delta_N(\boldsymbol{t},\mathcal{P})|. \tag{1.1}$$

For $p \in [1,\infty)$, we define the L_p discrepancy to be the L_p norm of the discrepancy function, which is given by

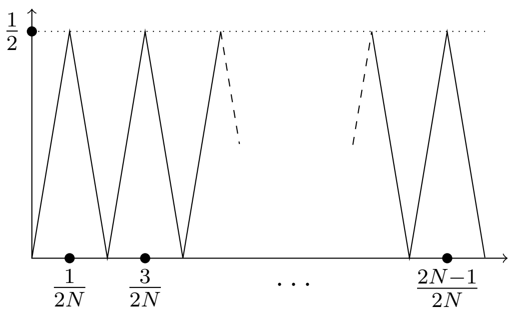
$$L_{p,N}(\mathcal{P}) := \left(\int_{[0,1]^s} |\Delta_N(\boldsymbol{t},\mathcal{P})|^p \,\mathrm{d}\boldsymbol{t} \right)^{\frac{1}{p}}. \tag{1.2}$$

We will often omit the lower index N in the notion of L_p and star discrepancy for concrete point sets and simply write $L_p(\mathcal{P})$ and $L_\infty(\mathcal{P})$, respectively, since the number of elements is fixed and in most cases known from the context. We continue with several discrepancy results on simple point sets.

Example 1.3. We consider the centred regular grid in the one-dimensional unit interval $[0,1)$, which for $N \in \mathbb{N}$ is defined as

$$\Gamma_N^c := \left\{ \frac{2k+1}{2N} \,:\, k \in \{0,1,\ldots,N-1\} \right\}.$$

Our aim is to calculate the L_p discrepancy of this point set. To this end, we consider the absolute value of its discrepancy function. It is easy to convince oneself that $|\Delta_N(t,\Gamma_N^c)|$ is given as shown in the following image:



We see from this image that $\|\Delta_N(\cdot,\Gamma_N^c)\|_\infty = \frac{1}{2}$; thus we have

$$L_\infty(\Gamma_N^c) = \frac{1}{2}.$$

For $1 \leq p < \infty$, we compute

$$\int_0^1 |\Delta_N(t,\Gamma_N^c)|^p \,\mathrm{d}t = 2N \int_0^{\frac{1}{2N}} (Nt)^p \,\mathrm{d}t = \left(\frac{1}{2}\right)^p \frac{1}{p+1}$$

and therefore

$$L_p(\Gamma_N^c) = \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

We have $\lim_{p\to\infty} L_p(\Gamma_N^c) = L_\infty(\Gamma_N^c)$; a relation which must of course be true for any point set $\mathcal{P}$.

Note that there exists an explicit formula for the L_2 discrepancy of point sets in $[0,1)^s$. Let $\mathcal{P}=\{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$ with $\boldsymbol{x}_k=(x_{k,1},\dots,x_{k,s})$ for $k=0,\dots,N-1$. Then we have

$$\begin{aligned}(L_{2,N}(\mathcal{P}))^2=&\int_{[0,1)^s}\left(\sum_{k=0}^{N-1}\mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{x}_k)-N|[\mathbf{0},\boldsymbol{t})|\right)^2\,\mathrm{d}\boldsymbol{t}\\ =&N^2\prod_{i=1}^{s}\int_0^1 t_i^2\,\mathrm{d}t_i-2N\sum_{k=0}^{N-1}\prod_{i=1}^{s}\int_0^1 t_i\mathbf{1}_{[0,t_i)}(x_{k,i})\,\mathrm{d}t_i\\ &+\sum_{k,l=0}^{N-1}\prod_{i=1}^{s}\int_0^1\mathbf{1}_{[0,t_i)}(x_{k,i})\mathbf{1}_{[0,t_i)}(x_{l,i})\,\mathrm{d}t_i\\ =&\frac{N^2}{3^s}-2N\sum_{k=0}^{N-1}\prod_{i=1}^{s}\int_{x_{k,i}}^1 t_i\,\mathrm{d}t_i+\sum_{k,l=0}^{N-1}\prod_{i=1}^{s}\int_{\max\{x_{k,i},x_{l,i}\}}^1 1\,\mathrm{d}t_i\\ =&\frac{N^2}{3^s}-\frac{N}{2^{s-1}}\sum_{k=0}^{N-1}\prod_{i=1}^{s}(1-x_{k,i}^2)+\sum_{k,l=0}^{N-1}\prod_{i=1}^{s}(1-\max\{x_{k,i},x_{l,i}\})\,.\end{aligned}\tag{1.3}$$

This formula is due to Warnock [77].

A central problem in discrepancy theory is to find for a given cardinality N the point set with the lowest discrepancy of all N-element point sets. The following theorem offers a solution to this problem for the L_1, L_2 and the star discrepancy of point sets in the one-dimensional unit interval $[0,1)$.

Theorem 1.4. *Let $N\in\mathbb{N}$. We have*

$$\inf_{|\mathcal{P}|=N}L_{1,N}(\mathcal{P})=\frac{1}{4},\quad \inf_{|\mathcal{P}|=N}L_{2,N}(\mathcal{P})=\frac{1}{\sqrt{12}}\quad\textit{and}\quad\inf_{|\mathcal{P}|=N}L_{\infty,N}(\mathcal{P})=\frac{1}{2},$$

where the infimum is extended over all N-element point sets in $[0,1)$ and where this infimum is attained only for Γ_N^c in all three cases, respectively.

Proof. Throughout the whole proof, let $\mathcal{P}=\{x_0,x_1,\dots,x_{N-1}\}\subset[0,1)$ be an arbitrary N-element point set in the unit interval, where we assume that $x_0\le x_1\le\dots\le x_{N-1}$.

We show the claim on the L_1 discrepancy. We have

$$\begin{aligned}L_{1,N}(\mathcal{P})=&\int_0^{x_0}|-Nt|\,\mathrm{d}t+\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}|k-Nt|\,\mathrm{d}t+\int_{x_{N-1}}^1|N-Nt|\,\mathrm{d}t\\ =&\frac{N}{2}x_0^2+\frac{N}{2}(1-x_{N-1})^2+N\sum_{k=1}^{N-1}\int_{x_{k-1}}^{x_k}\left|\frac{k}{N}-t\right|\,\mathrm{d}t.\end{aligned}\tag{1.4}$$

We show that $\int_{x_{k-1}}^{x_k}\left|\frac{k}{N}-t\right|\,\mathrm{d}t\ge\frac{1}{4}(x_k-x_{k-1})^2$ for all $k\in\{1,\dots,N-1\}$. If $\frac{k}{N}\in(x_{k-1},x_k)$, then with $\epsilon_k=\frac{1}{2}(x_k+x_{k-1})-\frac{k}{N}$ we have

$$\begin{aligned}\int_{x_{k-1}}^{x_k}\left|\frac{k}{N}-t\right|\,\mathrm{d}t=&\int_{x_{k-1}}^{k/N}\left(\frac{k}{N}-t\right)\,\mathrm{d}t+\int_{k/N}^{x_k}\left(t-\frac{k}{N}\right)\,\mathrm{d}t\\ =&\frac{1}{2}\left(\frac{k}{N}-x_{k-1}\right)^2+\frac{1}{2}\left(x_k-\frac{k}{N}\right)^2\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{\Big(\frac{1}{2}(x_k - x_{k-1}) - \epsilon_k\Big)^2 + \Big(\frac{1}{2}(x_k - x_{k-1}) + \epsilon_k\Big)^2\right\} \\
&= \frac{1}{2}\left(\frac{1}{2}(x_k - x_{k-1})^2 + 2\epsilon_k^2\right) \geq \frac{1}{4}(x_k - x_{k-1})^2.
\end{aligned}
$$

In a similar manner one can check that we even have $\int_{x_{k-1}}^{x_k} \left|\frac{k}{N} - t\right| \,\mathrm{d}t \geq \frac{1}{2}(x_k - x_{k-1})^2$ if $\frac{k}{N} \geq x_k$ or $\frac{k}{N} \leq x_{k-1}$. Since $\sum_{k=1}^{N-1}(x_k - x_{k-1}) = x_{N-1} - x_0$, we conclude from (1.4)

$$
\begin{aligned}
L_{1,N}(\mathcal{P}) \geq & \frac{N}{2}\left(\Big(x_0 - \frac{1}{2N}\Big) + \frac{1}{2N}\right)^2 + \frac{N}{2}\left(\Big(1 - x_{N-1} - \frac{1}{2N}\Big) + \frac{1}{2N}\right)^2 \\
& + \frac{N}{4}\sum_{k=1}^{N-1}\left(\Big(x_{k+1} - x_k - \frac{1}{N}\Big) + \frac{1}{N}\right)^2 \\
\geq & \frac{N}{2}\left(\frac{1}{N}\Big(x_0 - \frac{1}{2N}\Big) + \frac{1}{4N^2}\right) + \frac{N}{2}\left(\frac{1}{N}\Big(1 - x_{N-1} - \frac{1}{2N}\Big) + \frac{1}{4N^2}\right) \\
& + \frac{N}{4}\sum_{k=1}^{N-1}\left(\frac{2}{N}\Big(x_k - x_{k-1} - \frac{1}{N}\Big) + \frac{1}{N^2}\right) \\
= & \frac{1}{2}\Big(x_0 - \frac{1}{2N}\Big) + \frac{1}{8N} + \frac{1}{2}\Big(1 - x_{N-1} - \frac{1}{2N}\Big) + \frac{1}{8N} \\
& + \frac{1}{2}\left(x_{N-1} - x_0 - \frac{N-1}{N}\right) + \frac{N-1}{4N} = \frac{1}{4}.
\end{aligned}
$$

Since $\mathcal{P}$ was chosen arbitrarily and since $L_1(\Gamma_N^{\mathrm{c}}) = \frac{1}{4}$, we have shown $\inf_{|\mathcal{P}|=N} L_{1,N}(\mathcal{P}) = \frac{1}{4}$ as claimed. Additionally, it follows from the proof that we have equality if and only if $x_0 = \frac{1}{2N}$, $x_{N-1} = 1 - \frac{1}{2N}$, $x_k - x_{k-1} = \frac{1}{N}$ as well as $\epsilon_k = 0$, i.e. $x_{k-1} + x_k = \frac{2k}{N}$, for all $k \in \{1, \dots, N-1\}$. It is easy to see that all these conditions are satisfied simultaneously if and only if $x_k = \frac{2k+1}{2N}$ for all $k \in \{0, \dots, N-1\}$, i.e. if $\mathcal{P} = \Gamma_N^{\mathrm{c}}$.

In order to verify the result on the L_2 discrepancy, we prove an explicit formula for $L_{2,N}(\mathcal{P})$. From (1.3) for $s = 1$ we already know that

$$
\begin{aligned}
(L_{2,N}(\mathcal{P}))^2 =& \frac{N^2}{3} - N\sum_{k=0}^{N-1}(1 - x_k^2) + \sum_{k,l=0}^{N-1}(1 - \max\{x_k, x_l\}) \\
=& -\frac{2N^2}{3} + N\sum_{k=0}^{N-1} x_k^2 + \sum_{k=0}^{N-1}\sum_{l=0}^{k}(1 - x_k) + \sum_{l=1}^{N-1}\sum_{k=0}^{l-1}(1 - x_l).
\end{aligned}
$$

Applying some elementary algebra, one finds that the L_2 discrepancy of $\mathcal{P}$ is given by the formula

$$
L_{2,N}(\mathcal{P}) = \left(N\sum_{k=0}^{N-1}\left(x_k - \frac{2k+1}{2N}\right)^2 + \frac{1}{12}\right)^{\frac{1}{2}}.
$$

It follows that $L_{2,N}(\mathcal{P}) \geq 1/\sqrt{12}$ with equality if and only if $x_k = (2k+1)/(2N)$ for all $k \in \{0, 1, \dots, N-1\}$, i.e. if $\mathcal{P} = \Gamma_N^{\mathrm{c}}$.

We prove now that Γ_N^{c} is the best distributed N-element point set in $[0, 1)$ with respect to the star discrepancy. We first assume that $x_0 \neq x_1$. We choose $\varepsilon > 0$ such that $\varepsilon < \min\{1/N, x_1 - x_0\}$. Then we have

$$\begin{aligned}2L_{\infty,N}(\mathcal{P}) =&L_{\infty,N}(\mathcal{P}) + L_{\infty,N}(\mathcal{P}) \geq |\Delta_N(x_0+\varepsilon,\mathcal{P})| + |\Delta_N(x_0,\mathcal{P})| \\ \geq& |\Delta_N(x_0+\varepsilon,\mathcal{P}) - \Delta_N(x_0,\mathcal{P})| = |1-N(x_0+\varepsilon)-(-Nx_0)| \\ =& |1-N\varepsilon| = 1-N\varepsilon.\end{aligned}$$

We can choose ε arbitrarily close to zero; hence we have $2L_{\infty,N}(\mathcal{P}) \geq 1$. If $x_0 = x_1$, the above argumentation yields even $2L_{\infty,N}(\mathcal{P}) \geq 2$, and therefore we have $L_{\infty,N}(\mathcal{P}) \geq 1/2$ in any case. Since $\mathcal{P}$ was an arbitrary point set, we have shown that the star discrepancy of any N-element point set in $[0,1)$ is bounded from below by $1/2$. This value is achieved for Γ_N^c as observed in Example 1.3. The result on the star discrepancy follows also from the formula

$$L_{\infty,N}(\mathcal{P}) = N \max_{k=0,1,\dots,N-1} \left| x_k - \frac{2k+1}{2N} \right| + \frac{1}{2},$$

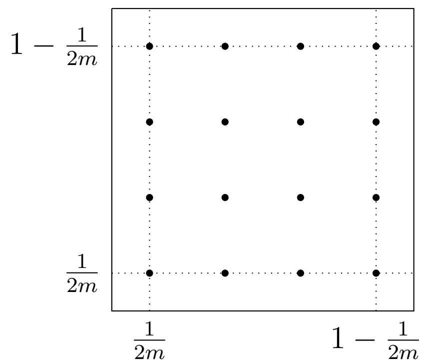
which has been shown by Niederreiter (see [55, Theorem 2.6]). Considering this formula, it is clear that the infimum of the L_∞ discrepancy is attained only for the centred regular grid. □

Theorem 1.4 confirms that in the one-dimensional case the point set one would intuitively consider as the best distributed has indeed minimal L_1, L_2 and star discrepancy for a given number of points. One would hope that a multidimensional version of the centred regular grid achieves the same in higher dimensions.

Example 1.5. Let $m \in \mathbb{N}$. Then the centred regular grid with $N = m^s$ points in the unit interval $[0,1)^s$ is the point set

$$\Gamma_{m,s}^c := \left\{ \left(\frac{2k_1+1}{2m}, \frac{2k_2+1}{2m}, \dots, \frac{2k_s+1}{2m} \right) : k_1,\dots,k_s \in \{0,1,\dots,m-1\} \right\}.$$

This point set is illustrated for $s=2$ and $m=4$ in the following:



As a warm-up for what to come in the main parts of this thesis, we compute the L_2 discrepancy of $\Gamma_{m,s}^c$ exactly. We therefore employ Warnock's formula (1.3) once again to find

$$\begin{aligned}(L_2(\Gamma_{m,s}^c))^2 =& \frac{m^{2s}}{3^s} - 2\left(\frac{m}{2}\right)^s \sum_{k_1,\dots,k_s=0}^{m-1} \prod_{i=1}^{s} \left(1 - \left(\frac{2k_i+1}{2m}\right)^2\right) \\ &+ \sum_{\substack{k_1,\dots,k_s=0 \\ l_1,\dots,l_s=0}}^{m-1} \prod_{i=1}^{s} \left(1 - \max\left\{\frac{2k_i+1}{2m}, \frac{2l_i+1}{2m}\right\}\right).\end{aligned}$$

We interchange products and sums and obtain

$$
\begin{aligned}
(L_2(\Gamma^c_{m,s}))^2 =& \frac{m^{2s}}{3^s} - 2\left(\frac{m}{2}\right)^s \prod_{i=1}^{s} \sum_{k_i=0}^{m-1} \left(1 - \left(\frac{2k_i+1}{2m}\right)^2\right) \\
&+ \prod_{i=1}^{s} \sum_{k_i,l_i=0}^{m-1} \left(1 - \max\left\{\frac{2k_i+1}{2m}, \frac{2l_i+1}{2m}\right\}\right) \\
=& \frac{m^{2s}}{3^s} - 2\left(\frac{m}{2}\right)^s \left(\frac{8m^2+1}{12m}\right)^s + \left(\frac{2m^2+1}{6}\right)^s \\
=& \frac{m^{2s}}{3^s} - \frac{2}{3^s}\left(m^2 + \frac{1}{8}\right)^s + \frac{1}{3^s}\left(m^2 + \frac{1}{2}\right)^s.
\end{aligned}
$$

To see the order of magnitude in N of this expression, we apply the binomial theorem and compute

$$
\begin{aligned}
(L_2(\Gamma^c_{m,s}))^2 =& \frac{m^{2s}}{3^s} - \frac{2}{3^s}\sum_{k=0}^{s}\binom{s}{k}(m^2)^k\left(\frac{1}{8}\right)^{s-k} + \frac{1}{3^s}\sum_{k=0}^{s}\binom{s}{k}(m^2)^k\left(\frac{1}{2}\right)^{s-k} \\
=& \frac{m^{2s}}{3^s} - \frac{2}{3^s}\left(m^{2s} + \frac{s}{8}m^{2s-2} + \mathcal{O}(m^{2s-4})\right) \\
&+ \frac{1}{3^s}\left(m^{2s} + \frac{s}{2}m^{2s-2} + \mathcal{O}(m^{2s-4})\right) = \frac{s}{4\cdot 3^s}m^{2s-2} + \mathcal{O}(m^{2s-4}).
\end{aligned}
$$

We obtain

$$
\lim_{N\to\infty} \frac{L_2(\Gamma^c_{m,s})}{N^{1-\frac{1}{s}}} = \frac{1}{2}\sqrt{\frac{s}{3^s}}.
$$

The L_2 discrepancy is of order $N^{1-\frac{1}{s}}$, which seems to be unsatisfactory large. The star discrepancy of $\Gamma^c_{m,s}$ shows a similar behaviour. It is known that

$$
L_\infty(\Gamma^c_{m,s}) = m^s - \left(m - \frac{1}{2}\right)^s.
$$

The fact that $L_\infty(\Gamma^c_{m,s}) \geq m^s - \left(m - \frac{1}{2}\right)^s$ follows from the fact that all points of $\Gamma^c_{m,s}$ lie in $[0, 1-1/(2m)]^s$. The proof of $L_\infty(\Gamma^c_{m,s}) \leq m^s - \left(m - \frac{1}{2}\right)^s$ can be found in [48, Theorem 2.19]. It follows from this formula that

$$
\begin{aligned}
L_\infty(\Gamma^c_{m,s}) =& m^s - \left(m - \frac{1}{2}\right)^s = m^s - \sum_{k=0}^{s}\binom{s}{k} m^k \left(-\frac{1}{2}\right)^{s-k} \\
=& \frac{s}{2}m^{s-1} - \sum_{k=0}^{s-2}\binom{s}{k} m^k \left(-\frac{1}{2}\right)^{s-k} = \frac{s}{2}m^{s-1} + \mathcal{O}(m^{s-2});
\end{aligned}
$$

i.e. $L_\infty(\Gamma^c_{m,s}) \asymp N^{1-\frac{1}{s}}$. This convergence rate is extremely bad as the dimension gets larger. As we will see in the next Section 1.2, it is already very bad in the two-dimensional case.

Let us now consider infinite sequences. In Theorem 1.4, we completely solved the problem to find the N-element point set in $[0, 1)$ with the minimal L_1, L_2 and star discrepancy. Therefore discrepancy theory for point sets in $[0, 1)$ is almost trivial and not very interesting. This changes completely, if we wish to construct a sequence $\mathcal{S} = \{x_0, x_1, \dots\}$

of points in $[0, 1)$, where the set of the first N elements of $\mathcal{S}$ shall have a small discrepancy for all $N \in \mathbb{N}$. This is not a trivial task at all. The construction of such well distributed sequences has strong connections to the theory of distribution modulo 1. We call a sequence $\mathcal{S} = (\boldsymbol{x}_n)_{n\geq 0}$ in $[0, 1)^s$ uniformly distributed modulo 1, if

$$\lim_{N\to\infty} \frac{A_N([\mathbf{a}, \mathbf{b}), \mathcal{S})}{N} = |[\mathbf{a}, \mathbf{b})| \tag{1.5}$$

for all intervals $[\mathbf{a}, \mathbf{b}) \in \mathbb{R}^s$. In this definition, for $\mathbf{a} = (a_1, \dots, a_s)$ and $\mathbf{b} = (b_1, \dots, b_s)$ with $a_i \leq b_i$ for all $i \in \{1, \dots, s\}$ we set $[\mathbf{a}, \mathbf{b}) = [a_1, b_1) \times \cdots \times [a_s, b_s)$. Further we define

$$A_N([\mathbf{a}, \mathbf{b}), \mathcal{S}) := |\{n \in \mathbb{N}_0 : 0 \leq n \leq N-1 \text{ and } \boldsymbol{x}_n \in [\mathbf{a}, \mathbf{b})\}|.$$

Finally, $|[\mathbf{a}, \mathbf{b})| = \prod_{i=1}^s (b_i - a_i)$ denotes the Lebesgue measure of $[\mathbf{a}, \mathbf{b})$. Roughly speaking, uniform distribution modulo 1 means that for N tending to infinity, the relative number of elements in any interval $[\mathbf{a}, \mathbf{b})$ equals the measure of this interval. Uniform distribution modulo 1 can also be defined for sequences in $\mathbb{R}^s$, but then one has to take the component-wise fractional part of each element of the sequence. We show that it suffices to demand (1.5) only for intervals of the form $[\mathbf{0}, \boldsymbol{t})$ with $\boldsymbol{t} = (t_1, \dots, t_s) \in [0, 1]^s$. This is trivial for one-dimensional intervals, for which we have $[a, b) = [0, b) \setminus [0, a)$. We also consider the two-dimensional case and observe that

$$\begin{aligned}
\lim_{N\to\infty} \frac{A_N([\mathbf{a}, \mathbf{b}), \mathcal{S})}{N} =& \lim_{N\to\infty} \frac{A_N([\mathbf{0}, \mathbf{b}), \mathcal{S})}{N} - \lim_{N\to\infty} \frac{A_N([\mathbf{0}, (a_1, b_2)), \mathcal{S})}{N} \\
&- \lim_{N\to\infty} \frac{A_N([\mathbf{0}, (b_1, a_2)), \mathcal{S})}{N} + \lim_{N\to\infty} \frac{A_N([\mathbf{0}, \mathbf{a}), \mathcal{S})}{N} \\
=& b_1 b_2 - a_1 b_2 - b_1 a_2 + a_1 a_2 = (b_1 - a_1)(b_2 - a_2) = |[\mathbf{a}, \mathbf{b})|.
\end{aligned}$$

The case $s > 2$ can be shown similarly. We observe further that the definition of uniform distribution modulo 1 may also be written in the equivalent form

$$\lim_{N\to\infty} \frac{\Delta_N([\mathbf{0}, \boldsymbol{t}), \mathcal{S})}{N} = 0$$

for all $\boldsymbol{t} \in [0, 1]^s$, where $\Delta_N([\mathbf{0}, \boldsymbol{t}), \mathcal{S})$ is the discrepancy function of the first N elements of $\mathcal{S}$.

Definition 1.6. Let $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ be a sequence of points in $[0, 1)^s$. Then the discrepancy function $\Delta_N(\boldsymbol{t}, \mathcal{S})$ is defined as the discrepancy function of its first N elements. The L_p discrepancy $L_{p,N}(\mathcal{S})$ of $\mathcal{S}$ for $1 \leq p \leq \infty$ and $N \in \mathbb{N}$ is then defined analogously to the case of finite point sets.

A criterion for uniform distribution modulo 1 is then

$$\lim_{N\to\infty} \frac{L_{p,N}(\mathcal{S})}{N} = 0. \tag{1.6}$$

for any $1 \leq p \leq \infty$. For further information on the equivalence of uniform distribution and (1.6) we refer to [48, Theorem 2.15, Corollary 2.23]. Note that we will not omit the lower index N if we deal with the L_p discrepancy of infinite sequences, since in this case the number N is not fixed. The L_p discrepancy quantifies the convergence rate of the limit in the definition of uniform distribution modulo 1. It is therefore a quantitative measure for how well a sequence is distributed in the unit interval. There exist further criteria for uniform distribution modulo 1. As a qualitative measure for distribution modulo 1 we state the famous criterion due to Weyl, who initiated the study of uniform distribution in his celebrated paper [78] from 1916. He showed that a sequence $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\} \subset [0,1)^s$ is uniformly distributed modulo 1 if and only if for all $\mathbf{h} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} \mathbf{h}\cdot \boldsymbol{x}_n} = 0.$$

Here $\mathbf{x} \cdot \mathbf{y}$ denotes the usual inner product of two elements $\mathbf{x}, \mathbf{y} \in \mathbb{R}^s$.

Example 1.7. For $\alpha \in \mathbb{R}$ we consider the sequence $\mathcal{S} = (\{n\alpha\})_{n\geq 0}$ in the unit interval $[0,1)$. We use Weyl's criterion to decide whether this sequence is uniformly distributed modulo 1 or not. Let $h \in \mathbb{Z} \setminus \{0\}$ be arbitrary. Since $x \mapsto \mathrm{e}^{2\pi i h x}$ is a one-periodic function, we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} h\{n\alpha\}} = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} h n\alpha} = \frac{1}{N}\sum_{n=0}^{N-1} \left(\mathrm{e}^{2\pi \mathrm{i} h\alpha}\right)^n.$$

We assume that α is irrational. Then $h\alpha \notin \mathbb{Q}$ and consequently $\mathrm{e}^{2\pi \mathrm{i} h\alpha} \neq 1$. Hence we can apply the formula for geometric sums to obtain

$$\left|\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} h\{n\alpha\}}\right| = \frac{1}{N}\left|\frac{\mathrm{e}^{2\pi \mathrm{i} N h\alpha} - 1}{\mathrm{e}^{2\pi \mathrm{i} h\alpha} - 1}\right| \leq \frac{1}{N}\frac{2}{|\mathrm{e}^{2\pi \mathrm{i} h\alpha} - 1|} \to 0$$

for $N \to \infty$. Since this holds for any $h \in \mathbb{Z}\setminus\{0\}$, the sequence $\mathcal{S}$ is uniformly distributed modulo 1 if $\alpha \notin \mathbb{Q}$. Let now $\alpha \in \mathbb{Q}$, i.e. $\alpha = \frac{p}{q}$ for some $p \in \mathbb{Z}, q \in \mathbb{N}$. Then we can find a non-zero integer, for instance $h^* = q$, for which $h^*\alpha \in \mathbb{Z}$. For h^* we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} h^*\{n\alpha\}} = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} n h^*\alpha} = \frac{1}{N}\sum_{n=0}^{N-1} 1 = 1;$$

hence Weyl's criterion is violated. As a result we have that $\mathcal{S}$ is uniformly distributed if and only if α is irrational. A higher-dimensional version of such sequences can be found in [21, Example 3.6] or [48, Proposition 2.6].

1.1.2. Numerical integration

We consider a function f which is defined on the s-dimensional unit cube $[0,1]^s$. We would like to compute the integral

$$I(f) := \int_{[0,1]^s} f(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t}. \tag{1.7}$$

For most integrands this integral can not be evaluated exactly. Either it is not possible to find an anti-derivative of f or the function values are only partially available. This is why one usually employs numerical methods to find good approximations for $I(f)$. We intend to approximate $I(f)$ by an algorithm of the form

$$Q_N(\mathcal{P}, f) := \frac{1}{N}\sum_{k=0}^{N-1} f(\boldsymbol{x}_k), \tag{1.8}$$

where the set $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ of sample points consists of elements of $[0,1)^s$. The essential question is to find sample points such that $Q_N(\mathcal{P}, f)$ requires only few function evaluations (i.e. N is small), but delivers a reasonably good approximation to the integral of f. For $\mathcal{P} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ we consider the integration error

$$e(f, \mathcal{P}) := I(f) - Q_N(\mathcal{P}, f) \tag{1.9}$$

as a quality measure for $\mathcal{P}$ to approximate $I(f)$.
The concept of Monte Carlo methods is to choose the set of sample points $\mathcal{P}$ independently and identically distributed in $[0,1)^s$. If $f \in L_2([0,1)^s)$, it can be shown that the expected value of the integration error of a Monte Carlo rule satisfies

$$\mathbb{E}[|e(f, \cdot)|] \leq \frac{\sigma[f]}{\sqrt{N}}$$

(see e.g. [48, p. 5]), where $\sigma[f]$ denotes the standard derivation of f. The Monte Carlo method requires the generation of random sample points, which is not an easy task. Furthermore, it is desirable to reduce the somewhat slow convergence rate of $1/\sqrt{N}$.
Let us first consider the one-dimensional case. For $f : [0,1] \to \mathbb{R}$ with continuous first derivative the fundamental theorem of calculus delivers

$$f(x) = f(1) - \int_x^1 f'(t) \, \mathrm{d}t,$$

which we insert into (1.9). Our algorithm $Q_N(\mathcal{P}, f)$ shall use the sample points $\mathcal{P} = \{x_0, \ldots, x_{N-1}\}$ in [0,1). Hence, with the arguments in [48, Section 3.1] we find

$$\begin{aligned} e(f, \mathcal{P}) =& \frac{1}{N} \sum_{n=0}^{N-1} \int_{x_n}^1 f'(t) \, \mathrm{d}t - \int_0^1 \int_x^1 f'(t) \, \mathrm{d}t \, \mathrm{d}x \\ =& \int_0^1 \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{(x_n, 1]}(t) f'(t) \, \mathrm{d}t - \int_0^1 \int_0^t f'(t) \, \mathrm{d}x \, \mathrm{d}t \\ =& \int_0^1 f'(t) \left(\frac{1}{N} \sum_{n=0}^{N-1} \mathbf{1}_{(x_n, 1]}(t) - t \right) \mathrm{d}t. \end{aligned}$$

At this point we note that the expression in the large brackets is exactly the discrepancy function of $\mathcal{P}$ divided by N. We use the triangle inequality and Hölder's inequality, which states that for functions $F, G \in L_1([0,1]^s)$ and parameters $1 \leq p, q \leq \infty$ satisfying $1/p + 1/q = 1$ (where $p = 1$ for $q = \infty$ and $p = \infty$ for $q = 1$) we have

$$\|FG\|_{L_1([0,1]^s)} \leq \|F\|_{L_p([0,1]^s)} \|G\|_{L_q([0,1]^s)}, \tag{1.10}$$

to obtain

$$|e(f, \mathcal{P})| \leq \frac{1}{N} \int_0^1 |f'(t) \Delta_N(\mathcal{P}, t)| \, \mathrm{d}t \leq \frac{1}{N} \|f'\|_{L_q([0,1])} L_{p,N}(\mathcal{P}). \tag{1.11}$$

We remark that (1.11) separates the influence of the integrand f and the sample points $\mathcal{P}$ on the integration error. Additionally, we see that the smaller the L_p discrepancy of $\mathcal{P}$, the smaller the integration error. If we choose $\mathcal{P} = \Gamma_N^c$, then we have

$$|e(f, \Gamma_N^c)| \leq \frac{1}{2N} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \|f'\|_{L_q([0,1))},$$

if f has a continuous first derivative. We achieve a convergence rate of $1/N$, which is much better than using Monte Carlo methods. We see that it is possible to achieve a considerably better convergence rate by using deterministic sample points. We therefore speak of quasi-Monte Carlo algorithms. For completeness, we state a higher-dimensional, much more sophisticated version of inequality (1.11), which is known as the famous Koksma-Hlawka inequality.

Theorem 1.8 (Koksma-Hlawka inequality)**.** *Let $\mathcal{P}$ be an N-element point set in the unit cube $[0,1)^s$. Then for all functions on $[0,1]^s$ with bounded variation $V(f)$ in the sense of Hardy and Krause we have*

$$|e(f,\mathcal{P})| \leq V(f) L_{\infty,N}(\mathcal{P}).$$

A proof and a definition of $V(f)$ can be found in [41, 49]. Note that $V(f)$ is a quantitative measure for the fluctuation of f. We close this section by mentioning that the choice of good sample points does not only depend on their discrepancy, but it is also important to consider properties of the integrands f. For instance, so-called lattice point sets work well for functions in the Korobov space (see [48, Section 4.4]), where a higher smoothness of the integrands leads to a better convergence rate of the integration error. Nonetheless, the results presented in this section demonstrate that the study of the discrepancy of point sets is not only of theoretic interest, but also has applications in numerical integration.

1.2. Bounds on the L_p and star discrepancy

This section is dedicated to the best known upper and lower bounds on the L_p discrepancy of point sets in $[0,1)^s$ for all $1 \leq p \leq \infty$. Clearly, a trivial upper bound is given by N, since $|\Delta_N(\boldsymbol{t},\mathcal{P})| \leq N$ for all $\boldsymbol{t} \in [0,1]^s$ and therefore

$$L_{p,N}(\mathcal{P}) \leq L_{\infty,N}(\mathcal{P}) = \sup_{\boldsymbol{t}\in[0,1]^s} |\Delta_N(\boldsymbol{t},\mathcal{P})| \leq N.$$

for all $1 \leq p \leq \infty$. In this argumentation we considered the monotonicity of the L_p norms, which yields $L_{p,N}(\mathcal{P}) \leq L_{q,N}(\mathcal{P})$ for $1 \leq p < q \leq \infty$. For $s = 1$ we also know a sharp lower bound on the L_1 discrepancy and hence on the L_p discrepancy for all $p \geq 1$ (see Theorem 1.4), which is constant.

The situation for $s \geq 2$ is by far more complex and difficult. In 1954 Roth [63] showed his celebrated lower bound on the L_2 discrepancy, which is probably the best-known result in discrepancy theory. It says that the L_2 discrepancy of any N-element point set $\mathcal{P}$ in the unit cube $[0,1)^s$ satisfies the bound

$$L_{2,N}(\mathcal{P}) \gtrsim_s (\log N)^{\frac{s-1}{2}}. \tag{1.12}$$

Roth's bound demonstrates in particular that in higher dimensions the L_2 discrepancy cannot be bounded in N, as it is the case for $s = 1$. The upper bound holds also for the L_p discrepancy for $p > 2$ due to the monotonicity of L_p norms as explained above. It took more than 20 years until the same lower bound has also been shown for $1 < p < 2$ by Schmidt; i.e. the L_p discrepancy of any N-element point set $\mathcal{P}$ in the unit cube $[0,1)^s$ satisfies the bound

$$L_{p,N}(\mathcal{P}) \gtrsim_{p,s} (\log N)^{\frac{s-1}{2}} \tag{1.13}$$

for all $1 < p \leq \infty$ (see [65]). We will present a proof of Schmidt's result in Section 2.3.4. It turns out that lower bounds for the L_1 discrepancy are much harder to prove. Halász could show that the L_1 discrepancy of any N-element point set $\mathcal{P}$ in the unit square $[0,1)^2$ satisfies

$$L_{1,N}(\mathcal{P}) \gtrsim \sqrt{\log N}. \tag{1.14}$$

This convergence rate matches Roth's and Schmidt's lower bounds on the L_p discrepancy. However, for $s \geq 3$ no better bound than the one of Halász is known.
It is natural to ask whether the given lower bounds are sharp in the order of magnitude in N; i.e. if there exist point sets which match these bounds. The first break-through in this direction came in 1956, when Davenport [16] presented an explicit construction of a point set in the unit square $[0,1)^2$ with an L_2 discrepancy of order $\sqrt{\log N}$.

Example 1.9. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with continued fraction expansion $\alpha = [a_0; a_1, a_2, \dots]$ such that the sequence of partial quotients $(a_k)_{k\geq 1}$ is bounded. We consider the point set

$$\mathcal{L}_N(\alpha) := \left\{ \left(\{n\alpha\}, \frac{n}{N} \right) : n \in \{0, 1, \dots, N-1\} \right\}$$

and a symmetrized version

$$\widetilde{\mathcal{L}}_N(\alpha) := \mathcal{L}_N(\alpha) \cup \{(1-x, y) : (x,y) \in \mathcal{L}_N(\alpha)\}, \tag{1.15}$$

which has $2N$ elements. Then we have

$$L_2(\widetilde{\mathcal{L}}_N(\alpha)) \lesssim_\alpha \sqrt{\log N}.$$

The symmetrization of point sets will also play a major role in this thesis. It turns out that symmetrization is often a useful method to construct point sets with the best possible rate of L_2 and L_p discrepancy. A thorough discussion of Davenport's principle, applied to the Hammersley point set, can be found in [13]. Note however that the symmetrization is not always necessary in the above example. Bilyk [4] could show that the non-symmetrized point set $\mathcal{L}_N(\alpha)$ has the optimal order of L_2 discrepancy, i.e. $L_2(\mathcal{L}_N(\alpha)) \lesssim_\alpha \sqrt{\log N}$, if and only if the bounded partial quotients of $\alpha = [a_0; a_1, a_2, \dots]$ satisfy

$$\left| \sum_{k=0}^{n} (-1)^k a_k \right| \lesssim \sqrt{n}$$

for all $n \in \mathbb{N}$.

There exist point sets in every dimension s with the order $(\log N)^{\frac{s-1}{2}}$ of L_p-discrepancy for all $p \in (1, \infty)$ (see [12] for the first existence result), which shows that the lower bound given in (1.13) is sharp. Further existence results for point sets with optimal order of L_p-discrepancy can be found in [15, 20, 67]. However, all these results for dimension 3 and higher are only existence results obtained by averaging arguments. Chen and Skriganov [14] gave for the first time for every integer $N \geq 2$ and every dimension $s \in \mathbb{N}$, explicit constructions of finite N-element point sets in $[0,1)^s$ whose L_2 discrepancy achieves an order of convergence of $(\log N)^{\frac{s-1}{2}}$. The result in [14] was extended to the L_p-discrepancy for $p \in (1, \infty)$ by Skriganov [66]. Further explicit constructions can be found in [21, 51, 52, 53].

There exist similar lower and upper bounds on the L_p discrepancy of infinite sequences. From the results of Roth and Schmidt Proinov [59] was able to show that for a sequence $\mathcal{S} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots\}$ in $[0,1)^s$ we have

$$L_{p,N}(\mathcal{S}) \gtrsim_{p,s} (\log N)^{\frac{s}{2}} \quad \text{for infinitely many } N \in \mathbb{N}. \tag{1.16}$$

This is true for all $p \in (1,\infty)$. Regarding Halász' result we obtain from Proinov's arguments that the bound holds also for the L_1 discrepancy of sequences in $[0,1)$; i.e. for a sequence $\mathcal{S} = \{x_0, x_1, \dots\}$ in $[0,1)$ we have

$$L_{1,N}(\mathcal{S}) \gtrsim \sqrt{\log N} \quad \text{for infinitely many } N \in \mathbb{N}. \tag{1.17}$$

Again we ask whether these bounds are sharp. Proinov could prove in [60] that symmetrized van der Corput sequences as introduced in Section 2.1 have an L_2 discrepancy of order $\sqrt{\log N}$ for all $N \in \mathbb{N}$. Based on higher order digital sequences, this result on the L_2 discrepancy was extended to arbitrary dimensions several years later by Dick and Pillichshammer (see [22, 23]), whereas the problem for the L_p discrepancy remained still open. In a joint work [45] with Pillichshammer the author could prove that the symmetrized van der Corput sequence achieves even the optimal order of L_p discepancy for every $1 \leq p < \infty$, thereby showing that Proinov's lower bound is sharp in the one-dimensional case. This result is part of this thesis and will be presented in a more general form in Section 4.2. A short time later, Dick, Hinrichs, Markhasin and Pillichshammer [17] could show that higher order digital sequences achieve the optimal order of L_p discrepancy for all $1 < p < \infty$ and in all dimensions.

Apart from sharp bounds on the L_p discrepancy in the order of magnitude in N, one is also interested in the coefficients of the leading term $(\log N)^{\frac{s-1}{2}}$. In dimension two and for $p = 2$, much effort has been put into the investigation of such constants. To be more precise, we are interested in the infimum

$$\inf_{|\mathcal{P}|=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}},$$

which is extended over all N-element point sets in the unit square. Borda [9] could show by considering the lattices $\widetilde{\mathcal{L}}(\alpha)$ from Example 1.9 for $\alpha = (\sqrt{5}+1)/2$ that

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}} \leq 0.176006\dots, \tag{1.18}$$

a constant which was previously discovered in [8] by numerical experiments on symmetriced Fibonacci lattices. The currently best known lower bounds were found in [40] by Hinrichs and Larcher, who showed

$$\inf_{N\geq 2} \inf_{|\mathcal{P}|=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}} \geq 0.051559\dots$$

and

$$\limsup_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}} \geq 0.061073\dots$$

The best implied constants for the L_1 discrepancy of point sets in the unit square are due to Vagharshakyan [73], who improved Halász' proof and showed that

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{1,N}(\mathcal{P})}{\sqrt{\log N}} \quad \geq \quad 0.00854\dots \quad \text{and}$$

$$\limsup_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{1,N}(\mathcal{P})}{\sqrt{\log N}} \geq 0.01137\dots \tag{1.19}$$

We close this section with remarks on the L_1 and L_∞ discrepancy of point sets in $[0,1)^s$ for $s \geq 3$. The sharp order of the L_1 discrepancy is known for point sets in the unit square and sequences in the one-dimensional unit interval by the result of Halász. The problem to find sharp bounds also for $s \geq 3$ is wide open. It is conjectured that the right order should be $(\log N)^{\frac{s-1}{2}}$ and hence matching Roth's and Schmidt's bounds on the L_p discrepancy. It has been shown in [2] that whenever we have $L_{p,N}(\mathcal{P}) \lesssim (\log N)^{\frac{s-1}{2}}$ for an N element point set $\mathcal{P}$ in $[0,1)^s$ and a $p \in (1,\infty)$, then we also have $L_{1,N}(\mathcal{P}) \gtrsim (\log N)^{\frac{s-1}{2}}$. This means that if one tries to find a point set whose L_1 discrepancy is of lower order than $(\log N)^{\frac{s-1}{2}}$, then one has to choose a point which does not have the optimal order of L_p discrepancy.

The best known open problem in discrepancy theory is probably the determination of the exact order of the star discrepancy. This problem is solved for one-dimensional point sets (see Theorem 1.4) and also for two-dimensional point sets. Let $\mathcal{P}$ an N-element point set in the unit square. Then we have

$$L_{\infty,N}(\mathcal{P}) \gtrsim \log N. \tag{1.20}$$

This bound has been shown by Schmidt [64] in 1980. By Bilyk, Lacey and Vagharshakyan [6] we know that for every dimension s there exists a constant $\eta(s) \in \left(0, \frac{1}{2}\right)$ such that for every N-element point set in $[0,1)^s$ we have

$$L_{\infty,N}(\mathcal{P}) \gtrsim (\log N)^{\frac{s-1}{2}+\eta(s)}.$$

This is the best known lower bound in dimension 3 and higher. Note that for $s=3$ the lower bound holds for all $\eta(3) < 0.017357\dots$ as shown in [62]. The exact exponent $\theta(s)$ of $\log N$ is unknown, but it is conjectured to be either $\theta(s) = s-1$ or $\theta(s) = \frac{s}{2}$. The situation for the star discrepancy of infinite sequences in $[0,1)^s$ for $s \geq 2$ is unclear as well. Only for sequences in $[0,1)$ we know that the optimal order of star discrepancy is $\mathcal{O}(\log N)$ for infinitely many N. The lower bound has been shown by Proinov based on Schmidt's corresponding bound for two-dimensional point sets, and the upper bound comes from explicit constructions (e.g. van der Corput sequences or the $n\alpha$-sequences we considered in Example 1.7).

It is in general much harder to find point sets in the unit square with the optimal order $\mathcal{O}(\sqrt{\log N})$ of L_p discrepancy for $p \in [1,\infty)$ than to construct such with the best possible star discrepancy rate $\mathcal{O}(\log N)$. There are several results concerning the L_2 discrepancy like Davenport's theorem or the results on Hammersley point sets in the next chapter, but for arbitrary p the results are much sparser. It is an important aim of this thesis to bring light into this problem and find a large number of point sets in the unit square and sequences in the unit interval which achieve the optimal order of L_p discrepancy.

2. Preliminaries

2.1. Hammersley point sets and van der Corput sequences

2.1.1. Generalized and symmetrized Hammersley point sets

A prominent example of a point set in the unit square with the optimal order of star discrepancy $\mathcal{O}(\log N)$ is the Hammersley point set. For $n \in \mathbb{N}$ the Hammersley point set in base $b \geq 2$, where b is an integer, with $N = b^n$ elements is defined as the point set

$$\mathcal{H}_{b,n} := \left\{ \left(\frac{a_n}{b} + \cdots + \frac{a_1}{b^n}, \frac{a_1}{b} + \cdots + \frac{a_n}{b^n} \right) : a_1, \ldots, a_n \in \{0, 1, \ldots, b-1\} \right\}. \tag{2.1}$$

Note that $n \asymp \log N$; a fact we shall often consider throughout this thesis. However, the Hammersley point set has a major drawback. It does not achieve the optimal order of L_p discrepancy for any $p \in [1, \infty)$. This follows for instance from exact formulas for the L_1 discrepancy of this point set, see (2.3). For this reason, several modifications have been applied to the Hammersley point set in history. Among these are digit shifting, digit scrambling and symmetrization. In the following, we will explain these variants of $\mathcal{H}_{b,n}$.

Let us first consider the dyadic case. For an n-tuple $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{0,1\}^n$, where $n \in \mathbb{N}$, we define the point set

$$\mathcal{H}_{2,n}(\boldsymbol{\sigma}) : \left\{ \left(\frac{a_n \oplus \sigma_n}{2} + \cdots + \frac{a_1 \oplus \sigma_1}{2^n}, \frac{a_1}{2} + \cdots + \frac{a_n}{2^n} \right) : a_1, \ldots, a_n \in \{0, 1\} \right\},$$

where the operation $\oplus$ denotes addition modulo 2. We obtain the classical Hammersley point set $\mathcal{H}_{2,n}$ with 2^n points by choosing $\boldsymbol{\sigma} = (0, 0, \ldots, 0)$. We speak of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ as a digit shifted Hammersley point set.

We would like to generalize the definition of digitally shifted Hammersley point sets to arbitrary bases. To this end, we observe that we can define $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ also in the following way: Instead of a tuple $\boldsymbol{\sigma} \in \{0,1\}^n$ we consider a tuple of permutations $\Sigma \in \{id, \tau_2\}^n$, where id shall denote the identity on $\{0,1\}$ and τ_2 the permutation on $\{0,1\}$ given by $\tau_2(k) = 1 - k$ for $k \in \{0,1\}$. Then we identify a tuple $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{0,1\}^n$ with $\Sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{id, \tau_2\}^n$, where we replace every zero digit in $\boldsymbol{\sigma}$ by id and every one digit by τ_2. It is then obvious that

$$\mathcal{H}_{2,n}(\boldsymbol{\sigma}) = \mathcal{H}_{2,n}^{\Sigma} := \left\{ \left(\frac{\sigma_n(a_n)}{2} + \cdots + \frac{\sigma_1(a_1)}{2^n}, \frac{a_1}{2} + \cdots + \frac{a_n}{2^n} \right) : a_1, \ldots, a_n \in \{0, 1\} \right\}.$$

This definition can be easily transferred to other bases. By $\mathfrak{S}_b$ we mean the set of all permutations on the set $\{0, 1, \ldots, b-1\}$ of b-adic digits. A particular element in $\mathfrak{S}_b$ is the so-called swapping permutation τ_b, which is given by $\tau_b(k) = b - 1 - k$. While in the dyadic case we only had the permutations id and τ_2, there are far more permutations in other bases. We fix a permutation $\sigma \in \mathfrak{S}_b$ and set $\overline{\sigma} = \tau_b \circ \sigma$. For a tuple $\Sigma = (\sigma_1, \ldots, \sigma_n) \in \{\sigma, \overline{\sigma}\}^n$ we define the point set

$$\mathcal{H}_{b,n}^{\Sigma} := \left\{ \left(\frac{\sigma_n(a_n)}{b} + \cdots + \frac{\sigma_1(a_1)}{b^n}, \frac{a_1}{b} + \cdots + \frac{a_n}{b^n} \right) : a_1, \ldots, a_n \in \{0, 1, \ldots, b-1\} \right\}.$$

We call $\mathcal{H}_{b,n}^{\Sigma}$ a generalized Hammersley point set. The choice $\Sigma = (id, \ldots, id)$ leads back to the classical Hammersley point set. Of course it would also be possible to apply more than two different permutations to the digits in the definition of the generalized Hammersley point set, but the case $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ suffices for our purposes and is easier to handle than a more general tuple Σ.
Finally, we introduce the symmetrized Hammersley point set. In base 2 we fix $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{0,1\}^n$ and set $\boldsymbol{\sigma}^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ such that $\sigma_j^* = \sigma_j \oplus 1$ for all $j \in \{1, \ldots, n\}$. The symmetrized Hammersley point set $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$ is then defined as

$$\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}) := \mathcal{H}_{2,n}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}(\boldsymbol{\sigma}^*).$$

This point set has 2^{n+1} elements. For arbitrary bases, we consider a tuple $\Sigma = (\sigma_i)_{i=1}^n \in \{\sigma, \overline{\sigma}\}^n$ and set $\Sigma^* = (\sigma_i^*)_{i=1}^n \in \{\sigma, \overline{\sigma}\}^n$, where $\sigma_i^* = \tau_b \circ \sigma_i$ for all $i \in \{1, \ldots, n\}$. The symmetrized Hammersley point set (associated to Σ) consisting of $2b^n$ elements is then defined as

$$\widetilde{\mathcal{H}}_{b,n}^{\Sigma} = \mathcal{H}_{b,n}^{\Sigma} \cup \mathcal{H}_{b,n}^{\Sigma^*}.$$

We speak of a symmetrized point set, because $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ can also be written as

$$\widetilde{\mathcal{H}}_{b,n}^{\Sigma} = \mathcal{H}_{b,n}^{\Sigma} \cup \left\{ \left(1 - \frac{1}{b^n} - x, y\right) \,:\, (x,y) \in \mathcal{H}_{b,n}^{\Sigma} \right\}. \tag{2.2}$$

Again we have $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}) = \widetilde{\mathcal{H}}_{2,n}^{\Sigma}$. The reader might wonder why we use a special notation for the dyadic case. The reason is that it allows us to use a simpler apparatus to study the L_p discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ and $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$, respectively (see Theorem 2.5 in the next subsection).

We survey several known results on the L_p discrepancy point sets that we have defined above. Let us first have a look at the classical Hammersley point set according to (2.1). We have

$$L_1(\mathcal{H}_{b,n}) = n\frac{b^2-1}{12b} + \frac{1}{2} + \frac{1}{4b^n}, \tag{2.3}$$

$$(L_2(\mathcal{H}_{b,n}))^2 = \\ n^2 \left(\frac{b^2-1}{12b}\right)^2 + n\left(\frac{3b^4+10b^2-13}{720b^2} + \frac{b^2-1}{12b}\left(1 - \frac{1}{2b^n}\right)\right) + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}, \tag{2.4}$$

and for every $p \in \mathbb{N}$ we have

$$(L_p(\mathcal{H}_{b,n}))^p = n^p \left(\frac{b^2-1}{12b}\right)^p + \mathcal{O}(n^{p-1}) \tag{2.5}$$

for $p \in \mathbb{N}$. The result on the L_2 discrepancy for $b = 2$ has been shown in [75], [34] and [58]. The results on the L_1 and the L_p discrepancy for $p \geq 3$ in the dyadic case have been proven in [58]. The generalizations to arbitrary bases can be found in [29]. A new proof of the result on the L_1 discrepancy is provided in Remark 4.9. The formula (2.3) tells us that the classical Hammersley point set does not achieve the optimal order of L_p discrepancy for any $p \in [1,\infty)$. This is why the generalized versions have been introduced. The first authors who studied digit shifted Hammersley point sets in base 2 (although they did not use this name) were Halton and Zaremba in [34]. They considered special shifts $\boldsymbol{\sigma}$, namely

$$\boldsymbol{\sigma} = \begin{cases} (1,0,1,0,\dots,1,0) & \text{if } n \text{ is even,} \\ (1,0,1,0,\dots,1,0,1) & \text{if } n \text{ is odd.} \end{cases}$$

They obtained

$$(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 = \frac{5n}{192} + \frac{3}{8} - \frac{7\varepsilon_n}{64} + \frac{1}{4\cdot 2^n} + \frac{\varepsilon_n}{16\cdot 2^n} - \frac{1}{72\cdot 2^{2n}},$$

where $\varepsilon_n = 0$ for even n and $\varepsilon_n = 1$ for odd n. A result for arbitrary $\boldsymbol{\sigma} \in \{0,1\}^n$ has been shown by Kritzer and Pillichshammer. To state their formula, we need to introduce the parameter $l = l(\boldsymbol{\sigma}) := |\{i \in \{1,\dots,n\} : \sigma_i = 0\}|$, i. e. l is the number of components of $\boldsymbol{\sigma}$ which are equal to zero. Then we have the following result:

Theorem 2.1 (Kritzer and Pillichshammer)**.** *Let $n \in \mathbb{N}$, $\boldsymbol{\sigma} \in \{0,1\}^n$ and l as above. Then we have*

$$(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 = \frac{n^2}{64} - \frac{19n}{192} - \frac{ln}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{n}{16\cdot 2^n} - \frac{l}{8\cdot 2^n} + \frac{1}{4\cdot 2^n} - \frac{1}{72\cdot 4^n}.$$

A remarkable aspect of this result is that the L_2 discrepancy depends only on the number of zero digits in $\boldsymbol{\sigma}$ and not at all on their position. We can write this formula as

$$(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 = \frac{1}{64}(n-2l)^2 + \mathcal{O}(n). \tag{2.6}$$

Now it is easy to see that the point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ achieves the optimal order of L_2 discrepancy if and only if $|n-2l| = \mathcal{O}(\sqrt{n})$. Roughly speaking, the number of zero and one digits in $\boldsymbol{\sigma}$ should be quite balanced to achieve a low L_2 discrepancy. To be more precise, it follows from Theorem 2.1 that the optimal choice for l is $\left\lceil \frac{n-5}{2} + \frac{1}{2^n} \right\rceil$ (see also [43, Corollary 1]), which leads to

$$(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 = \frac{5n}{192} + \mathcal{O}(1). \tag{2.7}$$

The situation of general p is even harder. For even integers p Kritzer and Pillichshammer could at least prove the existence of a shift $\boldsymbol{\sigma}$ such that

$$(L_p(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^p \leq \frac{2S(p,p/2)}{4^p} n^{\frac{p}{2}} + \mathcal{O}(n^{\frac{p}{2}-1}) \tag{2.8}$$

by considering the mean over all possible shifts (see [42, Theorem 1]). The number $S(p,p/2)$ is a Stirling number of second kind. This relation means that for even integers p it is always possible to find a shift $\boldsymbol{\sigma}$ such that $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ achieves the optimal order of L_p discrepancy. Generalizations of these results to the point sets $\mathcal{H}_{b,n}^{\Sigma}$, where $\Sigma \in \{id, \tau_b\}^n$, can be found in [29].
The L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ for the general case $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ has been calculated in [31]. We define $l = l(\Sigma) := |\{i \in \{1,\dots,n\} : \sigma_i = \sigma\}|$ and $\mathcal{A}_b(\tau) = \{\sigma \in \mathfrak{S}_b : \sigma \circ \tau_b = \tau_b \circ \sigma\}$. The numbers Φ_b^{σ} and $\Phi_b^{\sigma,(2)}$, which appear in the following formula, will be explained in Section 2.2 and depend only on b and σ.

Theorem 2.2 (Faure, Pillichshammer, Pirsic, Schmid)**.** *Let* $\sigma \in \mathfrak{S}_b$ *and* $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. *Then we have*

$$(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = \left(\Phi_b^\sigma\right)^2((n-2l)^2 - n) + \mathcal{O}(n).$$

For $\sigma \in \mathcal{A}_b(\tau)$ *we have the exact formula*

$$\begin{aligned}(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 = &\left(\Phi_b^\sigma\right)^2((n-2l)^2 - n) + \Phi_b^\sigma \left(1 - \frac{1}{2b^n}\right)(2l-n) \\ &+ n\Phi_b^{\sigma,(2)} + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}}.\end{aligned}$$

It follows immediately that $L_2(\mathcal{H}_{b,n}^{\Sigma}) = \mathcal{O}(\sqrt{n})$ if and only if either $|n-2l| = \mathcal{O}(\sqrt{n})$ for any σ or if we choose σ such that $\Phi_b^\sigma = 0$. By [31, Lemma 5] we have

$$\Phi_b^\sigma = \frac{1}{b^2}\sum_{a=0}^{b-1}\sigma(a)a - \frac{1}{b}\left(\frac{b-1}{2}\right)^2.$$

Thus, $\Phi_b^\sigma = 0$ is equivalent to

$$\frac{1}{b}\sum_{a=0}^{b-1}\sigma(a)a = \left(\frac{b-1}{2}\right)^2. \tag{2.9}$$

We give examples for permutations σ fulfilling (2.9) that were discovered in [30]. We choose $\sigma = id_r$ for $r \in \{0,1,\dots,b-1\}$, where $id_r(a) := a \oplus_b r$ for $a \in \{0,1,\dots,b-1\}$ ($\oplus_b$ denotes addition modulo b). Then we have

$$\begin{aligned}\sum_{a=0}^{b-1} id_r(a)a &= \sum_{a=0}^{b-1}(a \oplus_b r)a = \sum_{a=0}^{b-r-1}(a+r)a + \sum_{a=b-r}^{b-1}(a+r-b)a \\ &= \sum_{a=0}^{b-1}(a+r)a - b\sum_{a=b-r}^{b-1} a = \frac{b}{6}(1+2b^2+3r^2-3b(1+r)).\end{aligned}$$

Hence, (2.9) is fulfilled if and only if

$$\frac{b^2-1}{12} = \frac{r(b-r)}{2}.$$

The pairs (b,r) for which this equality is satisfied were given in [30, Corollary 1]. One could for instance choose $b=5$ and $r=1$ or $r=4$. In [31], further explicit examples and constructions for permutations which fulfil (2.9) were presented.

Previous results on the symmetrized Hammersley point sets are sparser. However, it is known that $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ achieves the optimal order of L_2 discrepancy for all $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. This has been shown by Proinov [60] in a very general form. He proved the bound

$$(L_{2,N}(\widetilde{\mathcal{H}}_{b,n}^{\Sigma}))^2 \leq \frac{b^2-1}{3\log b}\log N \tag{2.10}$$

for all bases $b \geq 2$ and all tuples $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ and hence gave also an explicit bound for the leading constant.

Note that the kind of symmetrization given in (2.2) is due to technical issues. Davenport's construction from Example 1.9 would rather suggest to investigate the point set

$$\widehat{\mathcal{H}}_{b,n}^{\Sigma} = \mathcal{H}_{b,n}^{\Sigma} \cup \left\{(1-x,y) \,:\, (x,y) \in \mathcal{H}_{b,n}^{\Sigma}\right\}.$$

However, we will show in the following lemma that the L_p discrepancies of $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ and $\widehat{\mathcal{H}}_{b,n}^{\Sigma}$ differ only by a small margin.

Lemma 2.3. *Let $b \geq 2$, $n \in \mathbb{N}$ and $N = 2b^n$. Then for all $p \in [1,\infty]$ we have*

$$|L_p(\widehat{\mathcal{H}}_{b,n}^{\Sigma}) - L_p(\widetilde{\mathcal{H}}_{b,n}^{\Sigma})| \leq 1.$$

Proof. At first we note that

$$A_N([\mathbf{0},\boldsymbol{t})\,, \widehat{\mathcal{H}}_{b,n}^{\Sigma}) \leq A_N([\mathbf{0},\boldsymbol{t})\,, \widetilde{\mathcal{H}}_{b,n}^{\Sigma}) \leq A_N([\mathbf{0},\boldsymbol{t})\,, \widehat{\mathcal{H}}_{b,n}^{\Sigma}) + 1. \tag{2.11}$$

For the proof of this claim we consider an arbitrary interval $[\mathbf{0},\boldsymbol{t}) \subseteq [0,1]^2$. It is evident that the point set $\widehat{\mathcal{H}}_{b,n}^{\Sigma}$ results from $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ if the points in

$$\left\{(1-1/b^n-x,y) : (x,y) \in \mathcal{H}_{b,n}^{\Sigma}\right\}$$

are shifted $1/b^n$ in the positive x-direction and the remaining points (which are the elements of $\mathcal{H}_{b,n}^{\Sigma}$) do not move. Since the x-coordinates of two distinctive elements in $\left\{(1-1/b^n-x,y) : (x,y) \in \mathcal{H}_{b,n}^{\Sigma}\right\}$ differ at least by $1/b^n$, there is at most one element in $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ that might leave the interval $[\mathbf{0},\boldsymbol{t})$ by shifting these points in the described way, whereas we cannot get additional points in this interval. Regarding these observations the above inequalities (2.11) are clear. Therefore we obtain

$$|\Delta_N(\boldsymbol{t},\widetilde{\mathcal{H}}_{b,n}^{\Sigma}) - \Delta_N(\boldsymbol{t},\widehat{\mathcal{H}}_{b,n}^{\Sigma})| \leq |A_N([\mathbf{0},\boldsymbol{t})\,, \widetilde{\mathcal{H}}_{b,n}^{\Sigma}) - A_N([\mathbf{0},\boldsymbol{t})\,, \widehat{\mathcal{H}}_{b,n}^{\Sigma})| \leq 1.$$

From $||x|-|y|| \leq |x-y|$ for all $x,y \in \mathbb{R}$ we get

$$\Big||\Delta_N(\boldsymbol{t},\widetilde{\mathcal{H}}_{b,n}^{\Sigma})| - |\Delta_N(\boldsymbol{t},\widehat{\mathcal{H}}_{b,n}^{\Sigma})|\Big| \leq 1.$$

Hence we have

$$|\Delta_N(\boldsymbol{t},\widetilde{\mathcal{H}}_{b,n}^{\Sigma})| \leq |\Delta_N(\boldsymbol{t},\widehat{\mathcal{H}}_{b,n}^{\Sigma})| + 1 \tag{2.12}$$

and

$$|\Delta_N(\boldsymbol{t},\widehat{\mathcal{H}}_{b,n}^{\Sigma})| \leq |\Delta_N(\boldsymbol{t},\widetilde{\mathcal{H}}_{b,n}^{\Sigma})| + 1. \tag{2.13}$$

Now we take the L_p-norm on both sides of inequality (2.12) and get by applying the triangle inequality

$$\begin{aligned} L_p(\widetilde{\mathcal{H}}_{b,n}^{\Sigma}) &= \left\|\Delta_N(\cdot,\widetilde{\mathcal{H}}_{b,n}^{\Sigma})\right\|_{L_p} \\ &\leq \|\Delta_N(\cdot,\widehat{\mathcal{H}}_{b,n}^{\Sigma})\|_{L_p} + \|1\|_{L_p} \\ &= L_p(\widehat{\mathcal{H}}_{b,n}^{\Sigma}) + 1. \end{aligned}$$

From inequality (2.13) we derive in an analogue way

$$L_p(\widehat{\mathcal{H}}_{b,n}^{\Sigma}) \leq L_p(\widetilde{\mathcal{H}}_{b,n}^{\Sigma}) + 1,$$

which finally yields the desired result concerning the L_p discrepancy of the generalized and symmetrized Hammersley point sets. □

This lemma demonstrates that it is not necessary to treat the point set $\widehat{\mathcal{H}}_{b,n}^{\Sigma}$ separately, since all results on $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ we show in this thesis basically apply also to $\widehat{\mathcal{H}}_{b,n}^{\Sigma}$.
In this thesis we would like to solve the following problems:

- We present two different shifts $\boldsymbol{\sigma}$ for which the digit shifted Hammersley point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ achieves the best possible order of L_4 discrepancy. We do so by proving exact formulas for this quantity. We use a method which is very effective to obtain precise discrepancy results, but is ineffective to classify all shifts $\boldsymbol{\sigma}$ which lead to the best possible order of L_4 or even L_p discrepancy.

- We will find exact formulas for the L_2 discrepancy of the symmetrized Hammersley point sets. We will first consider the dyadic point sets $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$. Our results will show the surprising fact that $L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))$ does not depend on $\boldsymbol{\sigma}$ at all. We will generalize these exact formulas to arbitrary bases and permutations $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ with $\sigma \in \mathcal{A}_b(\tau)$ and thereby determine the exact constants of the leading term $\sqrt{\log N}$, respectvely.

- We use Littlewood-Paley theory to fully classify for any $p \in [1, \infty)$ the tuples $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ such that the L_p discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ has the optimal order.

- By similar means we prove that for any $p \in [1, \infty)$ the symmetrized Hammersley point sets $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$ achieve the optimal order of L_p discrepancy for all possible tuples $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ with $\sigma \in \mathfrak{S}_b$.

2.1.2. Generalized and symmetrized van der Corput sequences

Every nonnegative integer n has a unique b-adic representation of the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0,$$

where the digits $a_0, a_1, \ldots, a_{k-1}, a_k$ are elements of the set $\{0, 1, \ldots, b-1\}$. Recall that $\mathfrak{S}_b$ is the set of all permutations of the set $\{0, 1, \ldots, b-1\}$. Let us here further assume that $\sigma(0) = 0$. Then we define the function $\varphi_b^{\sigma} : \mathbb{N}_0 \to [0, 1)$ by setting

$$\varphi_b^{\sigma}(n) := \frac{\sigma(a_0)}{b} + \frac{\sigma(a_1)}{b^2} + \cdots + \frac{\sigma(a_{k-1})}{b^k} + \frac{\sigma(a_k)}{b^{k+1}}.$$

We speak of φ_b^{σ} as a (generalized) radical inverse function. Instead of φ_b^{id} we simply write φ_b.

Definition 2.4. The (classical) van der Corput sequence in base b is defined as $\mathcal{V}_b := (\varphi_b(n))_{n \geq 0}$. The generalized van der Corput sequence (with respect to σ) is the sequence $\mathcal{V}_b^{\sigma} := (\varphi_b^{\sigma}(n))_{n \geq 0}$. For every sequence $\mathcal{S} = \{x_0, x_1, \ldots\}$ in $[0, 1)$ we understand under its symmetrized version the sequence

$$\widetilde{\mathcal{S}} := \{x_0, 1 - x_0, x_1, 1 - x_1, \ldots\}.$$

For the symmetrized van der Corput sequences we write consequently $\widetilde{\mathcal{V}}_b$ and $\widetilde{\mathcal{V}}_b^{\sigma}$, respectively.

We survey several previous results on the L_p discrepancy of the class of van der Corput sequences which are relevant in our thesis. A more detailed introduction to these sequences is provided in form of the recommendable survey paper [28]. The van der Corput sequences are prominent examples of so-called low-discrepancy sequences, which have star discrepancy of order $\log N$ for all $N \in \mathbb{N}$. Recall that this order is optimal according to the result of Schmidt, see (1.20). This has been shown for $\mathcal{V}_2$ by van der Corput in his paper [74] from 1935, where he introduced these sequences. More precisely, we have for $b = 2$ by Béjian and Faure

$$\limsup_{N\to\infty}\left(L_{\infty,N}(\mathcal{V}_2) - \frac{\log N}{3\log 2}\right) = \frac{4}{9} + \frac{\log 3}{3\log 2} \tag{2.14}$$

(see [3]). However, the classical van der Corput sequence does not achieve the optimal order of L_p discrepancy. Pillichshammer showed in [58] that

$$\limsup_{N\to\infty}\frac{L_{p,N}(\mathcal{V}_2)}{\log N} = \frac{1}{6\log 2} \tag{2.15}$$

for $p \geq 1$ (see also [61] for the case $p = 2$). Since we also have

$$\limsup_{N\to\infty}\frac{L_{2,N}(\mathcal{V}_b^\sigma)}{\log N} = \frac{\beta_{b,\sigma}}{b\log b} \tag{2.16}$$

for a positive constant $\beta_{b,\sigma}$ depending only on the base b and the permutation σ (see [11]), the whole large class of generalized van der Corput sequences fails to have the optimal order of L_2 discrepancy and therefore does not achieve the optimal order of L_p discrepancy for $p > 2$ either. For $\sigma = id$, it is also known from the work of Chaix and Faure [11] that $L_{1,N}(\mathcal{V}_b) \gtrsim_b \log N$ for infinitely many N, and therefore the classical van der Corput sequence has an L_p discrepancy of exact order $\log N$ for infinitely many N and all $p \geq 1$.

However, Proinov [60] could show that the whole class of symmetrized van der Corput sequences $\widetilde{\mathcal{V}}_b^\sigma$ achieves the optimal order of L_2 discrepancy. He proved for all $N \geq 2$ the upper bound

$$(L_{2,N}(\widetilde{\mathcal{V}}_b^\sigma))^2 \leq \frac{b^2-1}{3\log b}\log N$$

for any base b and any $\sigma \in \mathfrak{S}_b$.

We are now interested in the constants which appear as the coefficients of $\sqrt{\log N}$ in the L_2 discrepancy of $\widetilde{\mathcal{V}}_b^\sigma$. In other words, we are interested in the value of

$$l_2(\widetilde{\mathcal{V}}_b^\sigma) := \limsup_{N\to\infty}\frac{L_{2,N}(\widetilde{\mathcal{V}}_b^\sigma)}{\sqrt{\log N}}.$$

It turns out that the exact computation of $l_2(\widetilde{\mathcal{V}}_b^\sigma)$ is very complicated. In previous research, two different approaches to provide good estimates for $l_2(\widetilde{\mathcal{V}}_b^\sigma)$ have been employed.

1. The diaphony of a sequence $\mathcal{S} = \{x_0, x_1, \dots\}$ in $[0,1)$ is given by

$$F_N(\mathcal{S}) := \left(2\sum_{m=1}^{\infty}\frac{1}{m^2}\left|\sum_{n=0}^{N-1}\mathrm{e}^{2\pi\mathrm{i}mx_n}\right|^2\right)^{\frac{1}{2}}$$

for its first N elements. The diaphony provides a further criterion to decide whether a sequence $\mathcal{S}$ is uniformly distributed modulo 1, as $\mathcal{S}$ has this property if and only if

$$\lim_{N\to\infty} \frac{F_N(\mathcal{S})}{N} = 0.$$

The diaphony of $\mathcal{V}_b^\sigma$ is well known by the work of Chaix and Faure, who presented an exact formula for this quantity in their paper [11], from which they could deduce very precise values for

$$f(\mathcal{V}_b^\sigma) := \limsup_{N\to\infty} \frac{F_N(\mathcal{V}_b^\sigma)}{\sqrt{\log N}}.$$

From the relation

$$l_2(\widetilde{\mathcal{V}}_b^\sigma) \leq \frac{1}{\pi} f(\mathcal{V}_b^\sigma) \tag{2.17}$$

one can further derive upper bounds on $l_2(\widetilde{\mathcal{V}}_b^\sigma)$ from the results on the diaphony of $\mathcal{V}_b^\sigma$. The best constants found by this approach are the following:

- Chaix, Faure ([11], 1993): $f(\mathcal{V}_{19}^\sigma) = 1.14706\ldots$,
- Pausinger, Schmid ([56], 2010): $f(\mathcal{V}_{57}^\sigma) = 1.06674\ldots$

for particular permutations σ in $\mathfrak{S}_{19}$ and $\mathfrak{S}_{57}$, respectively. With (2.17) this yields $l_2(\widetilde{\mathcal{V}}_{19}^\sigma) \leq 0.36511...$ and $l_2(\widetilde{\mathcal{V}}_{57}^\sigma) \leq 0.33955\ldots$, respectively.

2. Faure developed a more precise method to find bounds on $l_2(\widetilde{\mathcal{V}}_2)$ by proving an exact formula for the L_2 discrepancy of $\widetilde{\mathcal{V}}_2$. He found

$$(L_{2,N}(\widetilde{\mathcal{V}}_2))^2 = \sum_{j=1}^{n} \left(1 - \|2^j \varphi_2(N)\|\right) \left\|\frac{N}{2^j}\right\|^2 + \frac{N^2}{3\cdot 4^n} \tag{2.18}$$

for $1 \leq N < 2^n$. To find the exact value of $l_2(\widetilde{\mathcal{V}}_2)$, we have to find for every $n \in \mathbb{N}$ the maximum

$$\max_{1\leq N<2^n} \left(\sum_{j=1}^{n} \left(1 - \|2^j \varphi_2(N)\|\right) \left\|\frac{N}{2^j}\right\|^2 + \frac{N^2}{3\cdot 4^n} \right).$$

Unfortunately, this is a difficult task. For reasonably small integers n, the maximum can be searched for with the aid of the computer. It turns out that probably the maximum is attained for $N(n) \in \mathbb{N}_0$ such that

$$\frac{N(n)}{2^n} = 0.00010001\cdots 0001 \quad (n \text{ digits after the comma})$$

in dyadic expansion, assuming that n is a multiple of 4. By inserting these N into the formula for the L_2 discrepancy of $\widetilde{\mathcal{V}}_2$, we find

$$(L_{2,N(n)}(\widetilde{\mathcal{V}}_2))^2 = \frac{421}{6750} n + \mathcal{O}(1).$$

Hence we have

$$l_2(\widetilde{\mathcal{V}}_2) \geq \left(\frac{421}{6750 \log 2}\right)^{\frac{1}{2}} \approx 0.299969\ldots,$$

where it is strongly conjectured that this is the exact value of $l_2(\widetilde{\mathcal{V}}_2)$. To find an upper bound on $l_2(\widetilde{\mathcal{V}}_2)$, Faure proved the following formula:

$$l_2(\widetilde{\mathcal{V}}_2) = \left(\inf_{n\geq 1} \max_{1\leq N<2^n} \left(\frac{1}{n\log 2} \sum_{j=1}^{n} \left(1 - \|2^j \varphi_2(N)\|\right) \left\| \frac{N}{2^j} \right\|^2 \right) \right)^{\frac{1}{2}}.$$

It therefore suffices to compute the above maximum for certain values of n to find a desired upper bound. For $n = 20$ Faure found the value $0.319553\dots$ However, searching the minimum for $n = 24$ (which requires a long runtime) delivers a slightly better bound, namely $0.316373\dots$ Summarizing, we have

$$0.299969\dots \leq l_2(\widetilde{\mathcal{V}}_2) \leq 0.316373\dots$$

We have

$$\inf_{\mathcal{S}\in[0,1)^{\mathbb{N}}} \limsup_{N\to\infty} \frac{L_{2,N}(\mathcal{S})}{\sqrt{\log N}} \leq 0.316373\dots \tag{2.19}$$

from Faure's result. This is currently the best known upper bound for this infimum. It would be desirable to improve upon this constant, and we will pursue this aim in this thesis.

We will treat the following problems concerning the L_p discrepancy of the (symmetrized) van der Corput sequences:

- We will prove an precise formula for the L_2 discrepancy of the symmetrized van der Corput sequences $\widetilde{\mathcal{V}}_b^\sigma$ for any base b and any permutation σ and try to derive some precise statements on $l_2(\widetilde{\mathcal{V}}_b^\sigma)$.
- We will prove that the sequences $\mathcal{V}_b^\sigma$ do not have the optimal order of L_p discrepancy for any $p \in (1,\infty)$, whereas the symmetrized sequences $\widetilde{\mathcal{V}}_b^\sigma$ achieve the optimal order for all $p \in [1,\infty)$ and all b and σ.

2.2. The discrepancy function of generalized Hammersley point sets

The proofs in Section 3 will be based on exact formulas for the discrepancy function of the point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ and $\mathcal{H}_{b,n}^{\Sigma}$.
In the dyadic case, such a formula was found by Larcher and Pillichshammer in [46]. The proof is based on a thorough Walsh analysis of the discrepancy function. Walsh functions have many applications in discrepancy theory. We refer to [21] for their definition and their basic properties. Recall that $\|x\|$ denotes the distance of a real number x to the nearest integer and that $\oplus$ means addition modulo 2. Further, it is reasonable to introduce the notion of n-bit numbers. We call a real number n-bit if and only if it belongs to the set

$$\mathbb{Q}(2^n) := \left\{ \frac{m}{2^n} \,:\, m \in \{0,1,\dots,2^n-1\} \right\}.$$

It is obvious that $\alpha \in \mathbb{Q}(2^n)$ if and only if it has a representation of the form $\alpha = \frac{\alpha_1}{2} + \dots + \frac{\alpha_n}{2^n}$, where $\alpha_1,\dots,\alpha_n \in \{0,1\}$. The following theorem involves $(\sigma_n,\dots,\sigma_1)$ instead of $(\sigma_1,\dots,\sigma_n)$, which is not a problem since our results will only depend on the number of zeroes in $\boldsymbol{\sigma}$.

Theorem 2.5. *For n-bit numbers* $\alpha = \frac{\alpha_1}{2} + \cdots + \frac{\alpha_n}{2^n}$ *and* $\beta = \frac{\beta_1}{2} + \cdots + \frac{\beta_n}{2^n}$ *the discrepancy function* $\Delta(\alpha, \beta)$ *of* $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ *with* $\boldsymbol{\sigma} = (\sigma_n, \ldots, \sigma_1)$ *satisfies*

1. $\Delta(\alpha, \beta) = \sum_{u=0}^{n-1} \|2^u \beta\| (-1)^{\sigma_{u+1}} (\alpha_{n-u} \oplus \alpha_{n+1-j(u)})$, *where we set* $\alpha_{n+1} := 0$ *and where for* $0 \le u \le n-1$ *the numbers* $j(u)$ *are defined as*

$$j(u) = \begin{cases} 0 & \text{if } u = 0, \\ 0 & \text{if } \alpha_{n+1-j} = \beta_j \oplus \sigma_j \text{ for } j = 1, \ldots, u, \\ \max\{j \le u : \alpha_{n+1-j} \neq \beta_j \oplus \sigma_j\} & \text{else.} \end{cases}$$

2. $\Delta(\alpha, 1) = 0$ *for n-bit* α *and* $\Delta(t_1, t_2) = \Delta(t_1(n), t_2(n)) + 2^n (t_1(n) t_2(n) - t_1 t_2)$ *for* $t_1, t_2 \in [0, 1]$, *where* $t_1(n)$ *and* $t_2(n)$ *are the smallest n-bit numbers greater than or equal* t_1 *or* t_2, *respectively. (For* $t_1, t_2 > 1 - 2^{-n}$ *we set* $t_1(n) = 1$ *and* $t_2(n) = 1$, *respectively.)*

In the following, we present a generalization of Theorem 2.5 to arbitrary bases b and permutations $\sigma \in \mathfrak{S}_b$. To this end, we need some notation that was initially introduced by Faure in [26].

Definition 2.6. Let $\sigma \in \mathfrak{S}_b$ and let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \ldots, \sigma(b-1)/b)$. For $h \in \{0, 1, \ldots, b-1\}$ and $x \in [(k-1)/b, k/b)$, where $k \in \{1, \ldots, b\}$, we define

$$\psi_{b,h}^\sigma(x) := \begin{cases} A([0, h/b); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \le h \le \sigma(k-1), \\ (b-h)x - A([h/b, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b. \end{cases}$$

In this definition, for a sequence $X = (x_M)$, $A([x, y); N; X)$ denotes the number of indices M with $1 \le M \le N$ such that $x_M \in [x, y)$. The function $\psi_{b,h}^\sigma$ is extended to the reals by periodicity, i.e. we have $\psi_{b,h}^\sigma(x) = \psi_{b,h}^\sigma(\{x\})$ for all $x \in \mathbb{R}$. We note that $\psi_{b,0}^\sigma = 0$ for any σ and that $\psi_{b,h}^\sigma(0) = 0$ for any σ and any h. We define several other functions which are build of the functions $\psi_{b,h}^\sigma$ and will appear in diverse parts of this thesis. First, we put

$$\psi_b^\sigma := \sum_{h=0}^{b-1} \psi_{b,h}^\sigma \quad \text{and} \quad \psi_b^{\sigma,(2)} := \sum_{h=0}^{b-1} (\psi_{b,h}^\sigma)^2.$$

We also set

$$\widetilde{\psi}_b^\sigma := \sum_{h=0}^{b-1} \psi_{b,h}^\sigma \psi_{b,h}^{\overline{\sigma}}, \quad \widetilde{\psi}_{b,1}^\sigma := \sum_{h=0}^{b-2} \psi_{b,h+1}^\sigma \psi_{b,h}^{\overline{\sigma}} \quad \text{and} \quad \widetilde{\psi}_{b,2}^\sigma := \sum_{h=0}^{b-2} \psi_{b,h}^\sigma \psi_{b,h+1}^{\overline{\sigma}}.$$

Finally, we define $\Phi_b^\sigma := \frac{1}{b} \int_0^1 \psi_b^\sigma(x) \,\mathrm{d}x$ and analogously the numbers $\Phi_b^{\sigma,(2)}$, $\widetilde{\Phi}_b^\sigma$, $\widetilde{\Phi}_{b,1}^\sigma$ and $\widetilde{\Phi}_{b,2}^\sigma$.

The following exact formula for the discrepancy function of $\mathcal{H}_{b,n}^\Sigma$ goes back to the work of Faure [26], where he studied generalized van der Corput sequences, and was first explicitely stated in [29, Lemma 1].

Theorem 2.7. *For integers* $1 \le \lambda, M \le b^n$ *we have*

$$\Delta_N\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}_{b,n}^\Sigma\right) = \sum_{j=1}^n \psi_{b,\varepsilon_j(\lambda,M,\Sigma)}^{\sigma_j}\left(\frac{M}{b^j}\right).$$

The numbers $\varepsilon_j(\lambda, M, \Sigma)$ for $j \in \{1, 2, \ldots, n\}$ are given as follows: For $1 \leq \lambda < b^n$ with b-adic expansion $\lambda = \lambda_1 b^{n-1} + \cdots + \lambda_2 b^{n-2} + \cdots + \lambda_{n-1} b + \lambda_n$, we define

$$\Lambda_{j-1} = \Lambda_{j-1}(\lambda) = \lambda_j b^{n-j} + \cdots + \lambda_n.$$

Then, for $1 \leq M < b^n$ with b-adic expansion $M = M_n b^{n-1} + \cdots + M_1$, we define

$$\nu_j = \nu_j(M, \Sigma) = \sigma_{j+1}(M_{j+1}) b^{n-j-1} + \cdots + \sigma_{n-1}(M_{n-1}) b + \sigma_n(M_n).$$

Now we set $\varepsilon_n = \lambda_n$ and for fixed $1 \leq j \leq n-1$ we set

$$\varepsilon_j = \varepsilon_j(\lambda, M, \Sigma) = \begin{cases} 0 & \text{if } 0 \leq \Lambda_{j-1} \leq \nu_j, \\ h & \text{if } \nu_j + (h-1) b^{n-j} < \Lambda_{j-1} \leq \nu_j + h b^{n-j} \text{ for } 1 \leq h < b, \\ 0 & \text{if } \nu_j + (b-1) b^{n-j} < \Lambda_{j-1} < b^{n-j+1}. \end{cases}$$

For $\lambda = b^n$ or $M = b^n$ we set $\varepsilon_j(\lambda, M, \Sigma) = 0$ for all $1 \leq j \leq n$.

Remark 2.8. Since the components of all points in $\mathcal{H}_{b,n}^{\Sigma}$ are of the form m/b^n for some $m \in \{0, \ldots, b^n - 1\}$, we have in analogy to the dyadic case

$$\Delta_N(t_1, t_2, \mathcal{H}_{b,n}^{\Sigma}) = \Delta_N(t_1(n), t_2(n), \mathcal{H}_{b,n}^{\Sigma}) + b^n (t_1(n) t_2(n) - t_1 t_2)$$

for all $t_1, t_2 \in (0, 1]$, where we set $t_1(n) := \min\{m/b^n \geq t_1 : m \in \{0, \ldots, b^n\}\}$ and analogously $t_2(n)$ for an $t_1, t_2 \in [0, 1)$. This relation has already been remarked in [29, Remark 3].

It is not obvious that Theorem 2.7 is a generalization of Theorem 2.5. We will therefore derive Theorem 2.5 from Theorem 2.7 in the following. We need to investigate

$$\Delta_N\left(\frac{\lambda}{2^n}, \frac{M}{2^n}, \mathcal{H}_{2,n}^{\Sigma}\right) = \sum_{j=1}^{n} \psi_{2,\varepsilon_j(\lambda, M, \Sigma)}^{\sigma_j}\left(\frac{M}{2^j}\right).$$

We write $\frac{\lambda}{2^n} = \alpha = \frac{\alpha_1}{2} + \cdots + \frac{\alpha_n}{2^n}$ and $\frac{M}{2^n} = \beta = \frac{\beta_1}{2} + \cdots + \frac{\beta_n}{2^n}$. From the definition of $\psi_{b,h}^{\sigma}$ it is straightforward to convince oneself that $\psi_{2,0}^{id} = \psi_{2,0}^{\tau_2} = 0$ and $\psi_{2,1}^{id} = \|\cdot\|$ as well as $\psi_{2,1}^{\tau_2} = -\|\cdot\|$. We define the function $\iota : \{id, \tau_2\} \to \{0, 1\}$ by setting $\iota(id) = 0$ and $\iota(\tau_2) = 1$. Then we can write

$$\psi_{2,1}^{\sigma} = (-1)^{\iota(\sigma)} \|\cdot\|$$

for $\sigma \in \{id, \tau_2\}$. Now we have

$$\begin{aligned} \Delta_N\left(\alpha, \beta, \mathcal{H}_{2,n}^{\Sigma}\right) &= \sum_{j=1}^{n} \psi_{2,\varepsilon_j(\alpha,\beta,\Sigma)}^{\sigma_j}\left(\frac{2^n \beta}{2^j}\right) = \sum_{u=0}^{n-1} \psi_{2,\varepsilon_{u+1}(\alpha,\beta,\Sigma)}^{\sigma_{n-u}}\left(\frac{2^n \beta}{2^{n-u}}\right) \\ &= \sum_{\substack{u=0 \\ \varepsilon_{n-u}(\alpha,\beta,\Sigma)=1}}^{n-1} (-1)^{\iota(\sigma_{u+1})} \|2^u \beta\|. \end{aligned}$$

In order to verify Theorem 2.5 it remains to show that $\varepsilon_{n-u}(\alpha, \beta, \Sigma) = 1$ if and only if $\alpha_{n-u} \oplus \alpha_{n+1-j(u)} = 1$. From the definitions in Theorem 2.7 we find

$$\Lambda_{n-u-1}(\alpha) = \alpha_{n-u} 2^u + \cdots + \alpha_n$$

and

$$\nu_{n-u}(\alpha, \beta) = \sigma_u(\beta_u) 2^{u-1} + \cdots + \sigma_1(\beta_1).$$

From the definition of $\varepsilon_j(\lambda, M, \Sigma)$ we see that $\varepsilon_{n-u}(\alpha, \beta, \Sigma) = 1$ if and only if

$$\nu_{n-u} < \Lambda_{n-u-1} \leq 2^u + \nu_{n-u}.$$

We distinguish several cases.

1. Assume that $\alpha_{n-u} = 0$. Then we have $\Lambda_{n-u-1} \leq 2^u + \nu_{n-u}$ for sure. The inequality $\nu_{n-u} < \Lambda_{n-u-1}$ yields the existence of the maximum $j(u) := \max\{j \leq u : \alpha_{n+1-j} \neq \sigma_j(\beta_j)\}$, since otherwise we would have equality. But then $\nu_{n-u} < \Lambda_{n-u-1}$ can only be the case if $\alpha_{n+1-j(u)} = 1$ and consequently $\sigma_{j(u)}(\beta_{j(u)}) = 0$. This yields $\alpha_{n-u} \oplus \alpha_{n+1-j(u)} = 1$. Note that $\sigma_j(\beta_j) = \beta_j \oplus \iota(\sigma_j)$, and hence the definition of $j(u)$ above matches the corresponding definition in Theorem 2.5.
2. Assume now that $\alpha_{n-u} = 1$. Then there are two possible cases:
   - The case that $\Lambda_{n-u-1} = 2^u + \nu_{n-u}$ yields $\alpha_{n+1-j} = \beta_j \oplus \iota(\sigma_j)$ for all $j \in \{1, \dots, u\}$ and hence $j(u) = 0$ (see Theorem 2.5). Then $\alpha_{n+1-j(u)} = \alpha_{n+1} = 0$.
   - The strict inequality $\Lambda_{n-u-1} < 2^u + \nu_{n-u}$ forces the existence of $j(u) := \max\{j \leq u : \alpha_{n+1-j} \neq \sigma_j(\beta_j)\}$ and further the fact that $\alpha_{n+1-j(u)} = 0$ and consequently $\sigma_{j(u)}(\beta_{j(u)}) = 1$.

In all cases we have $\alpha_{n-u} \oplus \alpha_{n+1-j(u)} = 1$, which completes the proof. □

For the symmetrized Hammersley point sets the following simple principle will prove very useful.

Lemma 2.9. *Given two point sets $\mathcal{P}_1$ and $\mathcal{P}_2$ in $[0,1)^s$ with N_1 and N_2 elements, respectively, and their union $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Note that this multiset may contain identical elements, which are all counted separately. Let $N = N_1 + N_2$. Then we have for all $\boldsymbol{t} \in [0,1]^s$*

$$\Delta_N(\boldsymbol{t}, \mathcal{P}) = \Delta_{N_1}(\boldsymbol{t}, \mathcal{P}_1) + \Delta_{N_2}(\boldsymbol{t}, \mathcal{P}_2).$$

Proof. We have

$$\begin{aligned}\Delta_N(\boldsymbol{t}, \mathcal{P}) =& A_{N_1+N_2}([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}) - (N_1 + N_2)|[\boldsymbol{0}, \boldsymbol{t})| \\ =& A_{N_1}([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}_1) - N_1|[\boldsymbol{0}, \boldsymbol{t})| + A_{N_2}([\boldsymbol{0}, \boldsymbol{t}), \mathcal{P}_2) - N_2|[\boldsymbol{0}, \boldsymbol{t})| \\ =& \Delta_{N_1}(\boldsymbol{t}, \mathcal{P}_1) + \Delta_{N_2}(\boldsymbol{t}, \mathcal{P}_2)\end{aligned}$$

for all $\boldsymbol{t} \in [0,1]^s$. □

In particular, it follow from Lemma 2.9 the important facts that

$$\Delta_{2^{n+1}}(\boldsymbol{t}, \widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})) = \Delta_{2^n}(\boldsymbol{t}, \mathcal{H}_{2,n}(\boldsymbol{\sigma})) + \Delta_{2^n}(\boldsymbol{t}, \mathcal{H}_{2,n}(\boldsymbol{\sigma}^*))$$

and

$$\Delta_{2b^n}(\boldsymbol{t}, \widetilde{\mathcal{H}}_{b,n}^{\Sigma}) = \Delta_{b^n}(\boldsymbol{t}, \mathcal{H}_{b,n}^{\Sigma}) + \Delta_{b^n}(\boldsymbol{t}, \mathcal{H}_{b,n}^{\Sigma^*}).$$

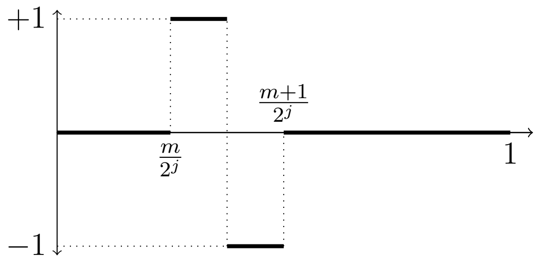
2.3. The Haar function system and several function spaces

2.3.1. The Haar functions

A popular method to estimate the L_2 discrepancy of given point sets or sequences is the following: One takes an orthonormal basis of $L_2([0,1)^s)$, for instance harmonic functions or Walsh functions, tries to find good upper bounds on the Fourier or Walsh coefficients, respectively, and inserts them into Parseval's identity. For our purposes, it is reasonable to work with Haar functions. We give first the dyadic definition of these functions. A dyadic interval of length $2^{-j}, j \in \mathbb{N}_0$, in $[0,1)$ is an interval of the form

$$I = I_{j,m} := \left[\frac{m}{2^j}, \frac{m+1}{2^j}\right) \quad \text{for} \quad m = 0, 1, \ldots, 2^j - 1.$$

We also define $I_{-1,0} = [0,1)$. The left and right half of $I_{j,m}$ are the dyadic intervals $I_{j+1,2m}$ and $I_{j+1,2m+1}$, respectively. The Haar function $h_{j,m}$ is the function on $[0,1)$ which is $+1$ on the left half of $I_{j,m}$, -1 on the right half of $I_{j,m}$ and 0 outside of $I_{j,m}$. The following image shows the Haar function $h_{j,m}$ for $j = 2$ and $m = 1$.



The L_∞-normalized Haar system consists of all Haar functions $h_{j,m}$ with $j \in \mathbb{N}_0$ and $m = 0, 1, \ldots, 2^j - 1$ together with the indicator function $h_{-1,0}$ of $[0,1)$. Normalized in $L_2([0,1))$ we obtain the orthonormal Haar basis of $L_2([0,1))$.
Let $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ and define $\mathbb{D}_j = \{0, 1, \ldots, 2^j - 1\}$ for $j \in \mathbb{N}_0$ and $\mathbb{D}_{-1} = \{0\}$. For $\boldsymbol{j} = (j_1, j_2, \ldots, j_s) \in \mathbb{N}_{-1}^s$ and $\boldsymbol{m} = (m_1, m_2, \ldots, m_s) \in \mathbb{D}_{\boldsymbol{j}} := \mathbb{D}_{j_1} \times \mathbb{D}_{j_2} \times \cdots \times \mathbb{D}_{j_s}$, the Haar function $h_{\boldsymbol{j},\boldsymbol{m}}$ is given as the tensor product

$$h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) = h_{j_1,m_1}(t_1) h_{j_2,m_2}(t_2) \ldots h_{j_s,m_s}(t_s) \quad \text{for } \boldsymbol{t} = (t_1, t_2, \ldots, t_s) \in [0,1)^2. \tag{2.20}$$

We speak of $I_{\boldsymbol{j},\boldsymbol{m}} = I_{j_1,m_1} \times I_{j_2,m_2} \times \cdots \times I_{j_s,m_s}$ as dyadic boxes. We define

$$|\boldsymbol{j}| = \max\{0, j_1\} + \max\{0, j_2\} + \cdots + \max\{0, j_s\}$$

and speak of it as the level of the dyadic box $I_{\boldsymbol{j},\boldsymbol{m}}$.

We extend this definition to arbitrary bases $b \geq 2$ in the following way: For $j \in \mathbb{N}_0$ we define $\mathbb{D}_j := \{0, 1, \ldots, b^j - 1\}$ and $\mathbb{B}_j := \{1, \ldots, b-1\}$. Additionally, we define the sets $\mathbb{D}_{-1} := \{0\}$ and $\mathbb{B}_{-1} := \{1\}$. For $j \in \mathbb{N}_0$ and $m \in \mathbb{D}_j$ we call the interval

$$I_{j,m} := \left[\frac{m}{b^j}, \frac{m+1}{b^j}\right)$$

the m-th b-adic interval on level j. We also define $I_{-1,0} = [0,1)$, which is a b-adic interval on level 0. For $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and any $k \in \{0, 1, \ldots, b-1\}$ we introduce the interval

$$I_{j,m}^k := I_{j+1,bm+k} = \left[\frac{m}{b^j} + \frac{k}{b^{j+1}}, \frac{m}{b^j} + \frac{k+1}{b^{j+1}}\right).$$

It is easy to see that $I_{j,m} = \bigcup_{k=0}^{b-1} I_{j,m}^k$ and $I_{j,m}^{k_1} \cap I_{j,m}^{k_2} = \emptyset$ whenever $k_1 \neq k_2$. We also put $I_{-1,0}^1 = I_{-1,0} = [0,1)$.
For $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and $\ell \in \mathbb{B}_j$ let $h_{j,m,\ell}$ be a function on $[0,1)$ with support in $I_{j,m}$ and the constant value $\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell}$ on $I_{j,m}^k$ for $k \in \{0, 1, \ldots, b-1\}$ and 0 outside of $I_{j,m}$. We call

$h_{j,m,\ell}$ a b-adic Haar function on $[0,1)$. We also put $h_{-1,0,1} = \mathbf{1}_{I_{-1,0}} = \mathbf{1}_{[0,1)}$ on $[0,1)$. It has been shown in [50, Theorem 2.1] that the system

$$\left\{ b^{\frac{\max\{0,j\}}{2}} h_{j,m,\ell} : j \in \mathbb{N}_{-1}, m \in \mathbb{D}_j, \ell \in \mathbb{B}_j \right\}$$

is an orthonormal basis of $L_2([0,1))$ and an unconditional basis of $L_p([0,1))$ for all $p \in (1,\infty)$. We speak of an one-dimensional b-adic Haar basis. The extension to s-dimensional Haar functions is again over tensor products as explained in (2.20).

2.3.2. Littlewood-Paley inequality for Haar functions

In this section we consider the dyadic Haar basis. Let $f \in L_2([0,1)^s)$. Then the Haar coefficients of f are given by the inner product

$$\mu_{\boldsymbol{j},\boldsymbol{m}} := \langle f, h_{\boldsymbol{j},\boldsymbol{m}} \rangle = \int_{[0,1)^s} f(\boldsymbol{t}) h_{\boldsymbol{j},\boldsymbol{m}}(\boldsymbol{t}) \,\mathrm{d}\boldsymbol{t}.$$

Parseval's identity states that

$$\|f\|_{L_2([0,1)^s)} = \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \right)^{\frac{1}{2}}. \tag{2.21}$$

The factor $2^{|\boldsymbol{j}|}$ comes from the L_2 normalization of the Haar functions. We would like to have a similar relation between the L_p norm of a function f and its Haar coefficients. It is provided by the Littlewood-Paley inequality. To this end, we introduce the square function of $f \in L_p([0,1)^s)$ as

$$S(f) := \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right)^{\frac{1}{2}}. \tag{2.22}$$

Proposition 2.10 (Littlewood-Paley inequality)**.** *Let $1 < p < \infty$. For a function $f : [0,1]^s \to \mathbb{R}$ we have*

$$\|f\|_{L_p([0,1)^s)} \asymp_p \|S(f)\|_{L_p([0,1)^s)}.$$

Proofs of these inequalities and further details also yielding the right asymptotic behavior of the involved constants can be found in [10, 68, 76]. For $p = 2$ Proposition 2.10 holds with equality and the Littlewood-Paley inequality is nothing else than Parseval's equality, as we can see as follows. First we write

$$\begin{aligned} \|S(f)\|_{L_2([0,1)^s)} &= \left\| \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right)^{\frac{1}{2}} \right\|_{L_2([0,1)^s)} \\ &= \left(\int_{[0,1)^s} \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}(\boldsymbol{t}) \right) \mathrm{d}\boldsymbol{t} \right)^{\frac{1}{2}}. \end{aligned}$$

Since $\int_{[0,1)^s} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}(\boldsymbol{t}) \,\mathrm{d}\boldsymbol{t} = 2^{-|\boldsymbol{j}|}$, we obtain

$$\|S(f)\|_{L_2([0,1)^s)} = \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \int_{[0,1)^s} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}(\boldsymbol{t}) \,\mathrm{d}\boldsymbol{t} \right)^{\frac{1}{2}} = \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^s, \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{|\boldsymbol{j}|} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^2 \right)^{\frac{1}{2}}.$$

2.3.3. Characterization of Besov spaces with Haar functions

We give a definition of the Besov spaces of dominating mixed smoothness. Let therefore $\mathcal{S}(\mathbb{R}^s)$ denote the Schwartz space and $\mathcal{S}'(\mathbb{R}^s)$ the space of tempered distributions on $\mathbb{R}^s$. For $f \in \mathcal{S}'(\mathbb{R}^s)$ we denote by $\mathcal{F}f$ the Fourier transform of f and by $\mathcal{F}^{-1}f$ its inverse. Let $\phi_0 \in \mathcal{S}(\mathbb{R})$ satisfy $\phi_0(t) = 1$ for $|t| \leq 1$ and $\phi_0(t) = 0$ for $|t| > \frac{3}{2}$. Let

$$\phi_d(t) = \phi_0(2^{-d}t) - \phi_0(2^{-d+1}t),$$

where $t \in \mathbb{R}, d \in \mathbb{N}$, and $\phi_{\boldsymbol{d}}(\boldsymbol{t}) = \phi_{d_1}(t_1)\cdots\phi_{d_s}(t_s)$, where $\boldsymbol{d} = (d_1, \dots, d_s) \in \mathbb{N}_0^s$, $\boldsymbol{t} = (t_1, \dots, t_s) \in \mathbb{R}^s$. We note that $\sum_{\boldsymbol{d}\in\mathbb{N}_0^s} \phi_{\boldsymbol{d}}(\boldsymbol{t}) = 1$ for all $\boldsymbol{t} \in \mathbb{R}^s$, which can be seen as follows. We have $\sum_{\boldsymbol{d}\in\mathbb{N}_0^s} \phi_{\boldsymbol{d}}(\boldsymbol{t}) = \prod_{i=1}^s \sum_{d_i\in\mathbb{N}_0} \phi_{d_i}(t_i)$, and therefore it suffices to show that $\sum_{d\in\mathbb{N}_0} \phi_d(t) = 1$. Define $\tilde{d} = \min\{d \in \mathbb{N}_0 \mid 2^{-d}|t| \leq 1\}$. Then we have

$$\sum_{d\in\mathbb{N}_0} \phi_d(t) = \sum_{d=0}^{\tilde{d}} \phi_d(t) + \sum_{d=\tilde{d}+1}^{\infty} \phi_d(t) = \phi_0(2^{-\tilde{d}}t) + \sum_{d=\tilde{d}+1}^{\infty} \phi_d(t),$$

where we take into account that the sum $\sum_{d=0}^{\tilde{d}} \phi_d(t)$ is a telescoping sum. Further we have $\phi_0(2^{-\tilde{d}}t) = 1$ and, since $|2^{-d}t| \leq 1$ for all $d \geq \tilde{d}$, we have $\phi_d(t) = 0$ for $d > \tilde{d}$. This yields the result.

The functions $\mathcal{F}^{-1}(\phi_{\boldsymbol{d}}\mathcal{F}f)$ are entire analytic functions for any $f \in \mathcal{S}'(\mathbb{R}^s)$. Let $0 < p, q \leq \infty$ and $r \in \mathbb{R}$. The dyadic Besov space $S_{p,q}^r B(\mathbb{R}^s)$ of dominating mixed smoothness consists of all $f \in \mathcal{S}'(\mathbb{R}^s)$ with finite quasi-norm

$$\|f\|_{S_{p,q}^r B(\mathbb{R}^s)} = \left(\sum_{\boldsymbol{d}\in\mathbb{N}_0^s} 2^{r(d_1+\cdots+d_s)q} \left\| \mathcal{F}^{-1}(\phi_{\boldsymbol{d}}\mathcal{F}f) \right\|_{L_p(\mathbb{R}^s)}^q \right)^{\frac{1}{q}},$$

with the usual modification if $q = \infty$. Let $\mathcal{D}([0,1)^s)$ be the set of all complex-valued infinitely differentiable functions on $\mathbb{R}^s$ with compact support in the interior of $[0,1)^s$ and let $\mathcal{D}'([0,1)^s)$ be its dual space of all distributions in $[0,1)^s$. The Besov space $S_{p,q}^r B([0,1)^s)$ of dominating mixed smoothness on the domain $[0,1)^s$ consists of all functions $f \in \mathcal{D}'([0,1)^s)$ with finite quasi norm

$$\|f\|_{S_{p,q}^r B([0,1)^s)} = \inf\left\{ \|g\|_{S_{p,q}^r B(\mathbb{R}^s)} : g \in S_{p,q}^r B(\mathbb{R}^s), g|_{[0,1)^s} = f \right\}.$$

However, this dyadic definition of the Besov space norm is not suitable to estimate the discrepancy of point sets and sequences which are based on the b-adic expansion of integers. To overcome this drawback, b-adic versions of the Besov spaces $S_{p,q}^r B^b(\mathbb{R}^s)$ and $S_{p,q}^r B^b([0,1)^s)$ have been introduced by Markhasin in [50, 52]. We refer to these papers for the definition of the b-adic Besov spaces. It has been shown in [50, Theorem 3.1] that the b-adic Besov space $S_{p,q}^r B^b([0,1)^s)$ is equivalent to the classical dyadic Besov space $S_{p,q}^r B([0,1)^s)$ and that we have the following useful characterization of functions which are contained in this space (see also [69, Theorem 2.41] for the original proof of the dyadic case):

Proposition 2.11. *Let $0 < p, q \leq \infty$ and $\frac{1}{p} - 1 < r < \min\left\{\frac{1}{p}, 1\right\}$. Let $f \in \mathcal{D}'([0,1)^s)$. Then $f \in S_{p,q}^r B^b([0,1)^s)$ if and only if it can be represented as*

$$f = \sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^s} \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}, \boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}} b^{|\boldsymbol{j}|} h_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}$$

for some sequence $(\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}})$ satisfying

$$\left(\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^s} b^{(j_1+\cdots+j_s)\left(r-\frac{1}{p}+1\right)q}\left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty,$$

where the convergence is unconditional in $\mathcal{D}'([0,1)^s)$ and in any $S^{\rho}_{pq}B^b([0,1)^s)$ with $\rho < r$. This representation of f is unique with the b-adic Haar coefficients

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f) := \langle f, h_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}\rangle = \int_{[0,1)^s} f(\boldsymbol{t})h_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(\boldsymbol{t})\,\mathrm{d}\boldsymbol{t} \quad \text{for } \boldsymbol{j}\in\mathbb{N}_{-1}^s,\ \boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}} \text{ and } \boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}. \tag{2.23}$$

The expression on the left-hand-side of the above inequality provides an equivalent quasi-norm on $S^r_{p,q}B^b([0,1)^s)$, i.e.

$$\|f\|_{S^r_{p,q}B^b([0,1)^s)} \asymp \left(\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^s} b^{(j_1+\cdots+j_s)\left(r-\frac{1}{p}+1\right)q}\left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

2.3.4. Triebel-Lizorkin spaces and embedding theorems

It is possible to deduce results on the L_p norm of a function from its Besov norm. The link between these two norms are embedding theorems between Besov spaces and Triebel-Lizorkin spaces with dominating mixed smoothness, where the latter contain the L_p spaces as a special case.
Let $0 < p < \infty$, $0 < q \leq \infty$ and $r \in \mathbb{R}$. The Triebel-Lizorkin space $S^r_{p,q}F(\mathbb{R}^s)$ with dominating mixed smoothness consists of all $f \in \mathcal{S}'(\mathbb{R}^s)$ with finite quasi-norm

$$\|f\|_{S^r_{p,q}F(\mathbb{R}^s)} = \left\|\left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^2} 2^{r(k_1+k_2)q}|\mathcal{F}^{-1}(\Phi_{\boldsymbol{k}}\mathcal{F}f)(\cdot)|^q\right)^{1/q}\right\|_{L_p(\mathbb{R}^2)}$$

with the usual modification if $q = \infty$. The space $S^r_{p,q}F([0,1)^s)$ can be introduced analogously to $S^r_{p,q}B([0,1)^s)$. For $0 < p, q < \infty$ and $r \in \mathbb{R}$ we have the embeddings

$$S^r_{\max\{p,q\},q}B([0,1)^s) \hookrightarrow S^r_{p,q}F([0,1)^s) \hookrightarrow S^r_{\min\{p,q\},q}B([0,1)^s), \tag{2.24}$$

which were proven in [52, Corollary 1.13], based on other embedding theorems from [69, Remark 6.28] and [35, Proposition 2.3.7]. Here, for two sets A and B the notation $A \hookrightarrow B$ means that there exists an injective map $f : A \to B$. For $1 < p < \infty$ the spaces $S^r_pH([0,1)^2) := S^r_{p,2}F([0,1)^2)$ are called Sobolev spaces with dominating mixed smoothness. Further, it is well known that $S^0_pH([0,1)^2) = L_p([0,1)^2)$. We obtain the following Littlewood-Paley type inequalities, which provide an alternative way to Proposition 2.10 to find bounds on the L_p norm of functions and also works for the b-adic Haar function system.

Proposition 2.12. *Let $p \in (1,\infty)$, $f \in L_p([0,1)^s)$ and $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}$ for $\boldsymbol{j}\in\mathbb{N}_{-1}^s$, $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ and $\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}$ its Haar coefficients. Then we have*

$$\|f\|^2_{L_p([0,1)^s)} \lesssim \sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^s} b^{2(j_1+\cdots+j_s)(1-1/\bar{p})}\left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f)|^{\bar{p}}\right)^{2/\bar{p}}$$

and

$$\|f\|^2_{L_p([0,1)^s)} \gtrsim \sum_{\boldsymbol{j}\in\mathbb{N}^s_{-1}} b^{2(j_1+\cdots+j_s)(1-1/p')} \left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f)|^{p'} \right)^{2/p'},$$

where $\bar{p} = \max\{p, 2\}$ *and* $p' = \min\{p, 2\}$, *respectively.*

Proof. With the first embedding in (2.24) and Proposition 2.11 we find

$$\begin{aligned}\|f\|^2_{L_p([0,1)^s)} =& \|f\|^2_{S^0_{p,2}F([0,1)^s)} \lesssim \|f\|^2_{S^0_{\max\{p,2\},2}B([0,1)^s)} \lesssim \|f\|^2_{S^0_{\max\{p,2\},2}B^b([0,1)^s)} \\ &\lesssim \sum_{\boldsymbol{j}\in\mathbb{N}^s_{-1}} b^{2(j_1+\cdots+j_s)(1-1/\bar{p})} \left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f)|^{\bar{p}} \right)^{2/\bar{p}}.\end{aligned}$$

The lower bound can be proven in an analogue way by applying the second embedding in (2.24). □

As an important application of Proposition 2.12 we give a simple proof of Schmidt's famous lower bound on the L_p discrepancy as stated in (1.13).

Proof of Schmidt's lower bound (1.13) Let $p \in (1, \infty)$ and consider an arbitrary N-element point set $\mathcal{P}$ in $[0, 1)^s$. Let $\boldsymbol{j} \in \mathbb{N}_0^s$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$. We compute the Haar coefficients of the volume part $L(\boldsymbol{t}) = Nt_1 \cdots t_s$ of the discrepancy function. We find

$$\begin{aligned}\langle L, h_{\boldsymbol{j},\boldsymbol{m}}\rangle =& N \prod_{i=1}^s \int_0^1 t_i h_{j_i,m_i}(t_i)\,\mathrm{d}t_i = N \prod_{i=1}^s \left(\int_{\frac{m_i}{2^{j_i}}}^{\frac{2m_i+1}{2^{j_i+1}}} t\,\mathrm{d}t - \int_{\frac{2m_i+1}{2^{j_i+1}}}^{\frac{m_i+1}{2^{j_i}}} t\,\mathrm{d}t \right) \\ =& N \prod_{i=1}^s \left(-2^{-2j_i-2}\right) = (-1)^s N 2^{-2|\boldsymbol{j}|-2s};\end{aligned}$$

hence we have $|\langle L, h_{\boldsymbol{j},\boldsymbol{m}}\rangle| = N2^{-2|\boldsymbol{j}|-2s}$. Now we prove $\langle \mathbf{1}_{[\mathbf{0},\cdot)}(\boldsymbol{z}), h_{\boldsymbol{j},\boldsymbol{m}}\rangle = 0$ whenever $\boldsymbol{z} = (z_1, \ldots, z_s) \in [0, 1)^s$ is not contained in the dyadic box $I_{\boldsymbol{j},\boldsymbol{m}}$. Note that

$$\langle \mathbf{1}_{[\mathbf{0},\cdot)}(\boldsymbol{z}), h_{\boldsymbol{j},\boldsymbol{m}}\rangle = \prod_{i=1}^s \int_0^1 \mathbf{1}_{[0,t_i)}(z_i) h_{j_i,m_i}(t_i)\,\mathrm{d}t_i. \tag{2.25}$$

Since $\boldsymbol{z}$ is not contained in $I_{\boldsymbol{j},\boldsymbol{m}}$, there is at least one component z_i of $\boldsymbol{z}$ such that $z_i \notin I_{j_i,m_i}$. If $z_i < \frac{m_i}{2^{j_i}}$, then

$$\int_0^1 \mathbf{1}_{[0,t_i)}(z_i) h_{j_i,m_i}(t_i)\,\mathrm{d}t_i = \int_0^{z_i} 0\,\mathrm{d}t_i = 0.$$

If $z_i \geq \frac{m_i+1}{2^{j_i}}$, then

$$\int_0^1 \mathbf{1}_{[0,t_i)}(z_i) h_{j_i,m_i}(t_i)\,\mathrm{d}t_i = \int_0^{\frac{m_i}{2^{j_i}}} 0\,\mathrm{d}t_i + \int_{\frac{m_i}{2^{j_i}}}^{\frac{2m_i+1}{2^{j_i+1}}} 1\,\mathrm{d}t_i + \int_{\frac{2m_i+1}{2^{j_i+1}}}^{\frac{m_i+1}{2^{j_i}}} (-1)\,\mathrm{d}t_i + \int_{\frac{m_i+1}{2^{j_i}}}^{z_i} 0\,\mathrm{d}t_i = 0.$$

It follows that the product in (2.25) is zero. Choose a level ℓ such that $2^{\ell-1} < 2N \leq 2^\ell$, i.e. $\ell \asymp \log N$, and let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of $\Delta_N(\cdot, \mathcal{P})$. From what we just discussed we observe that $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = N2^{-2|\boldsymbol{j}|-2s}$ whenever $I_{\boldsymbol{j},\boldsymbol{m}}$ does not contain any points of $\mathcal{P}$, because then the counting part of the discrepancy function does not contribute to the Haar coefficient. Now we make use of the second Littlewood-Paley type inequality in Proposition 2.12 and find

$$\begin{aligned}
\|\Delta_N(\cdot,\mathcal{P})\|^2_{L_p([0,1)^s)} &\gtrsim_{p,s} \sum_{\boldsymbol{j}\in\mathbb{N}_0^s} 2^{2|\boldsymbol{j}|(1-1/p')} \left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m}}|^{p'}\right)^{2/p'} \\
&\gtrsim 2^{2\ell(1-1/p')} \sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^s\\ |\boldsymbol{j}|=\ell}} \left(\sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ I_{\boldsymbol{j},\boldsymbol{m}}\cap\mathcal{P}=\emptyset}} (N2^{-2\ell-2s})^{p'}\right)^{2/p'} \\
&= N^2 2^{2\ell(1-1/p')} 2^{-4\ell-4s} \sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^s\\ |\boldsymbol{j}|=\ell}} \left(\sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ I_{\boldsymbol{j},\boldsymbol{m}}\cap\mathcal{P}=\emptyset}} 1\right)^{2/p'},
\end{aligned}$$

where we only sum over those boxes $I_{\boldsymbol{j},\boldsymbol{m}}$, which are of level $|\boldsymbol{j}|=\ell$ and do not contain any points of $\mathcal{P}$. Note that for a fixed $\boldsymbol{j}\in\mathbb{N}_0$ the boxes $I_{\boldsymbol{j},\boldsymbol{m}}$ for $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ are pairwise disjunct. At most N of these 2^ℓ boxes can contain points of $\mathcal{P}$, hence by our choice of ℓ we obtain

$$\sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ I_{\boldsymbol{j},\boldsymbol{m}}\cap\mathcal{P}=\emptyset}} 1 \geq 2^\ell - N \geq 2^{\ell-1}.$$

Basic combinatorics yields that the number of $\boldsymbol{j}\in\mathbb{N}_0^s$ with $|\boldsymbol{j}|=\ell$ is given by $\binom{\ell+s-1}{s-1}\asymp \ell^{s-1}$. Regarding the fact that $N^2 2^{-2\ell}\asymp 1$ we conclude

$$\begin{aligned}
\|\Delta_N(\cdot,\mathcal{P})\|^2_{L_p([0,1)^s)} &\gtrsim_{p,s} N^2 2^{2\ell(1-1/p')} 2^{-4\ell-4s} \ell^{s-1} \left(2^{\ell-1}\right)^{2/p'} \\
&\gtrsim_{p,s} N^2 2^{-2\ell} \ell^{s-1} \asymp_{p,s} (\log N)^{s-1},
\end{aligned}$$

and therefore

$$L_{p,N}(\mathcal{P}) \gtrsim_{p,s} (\log N)^{\frac{s-1}{2}}.$$

Since $\mathcal{P}$ was chosen arbitrarily, we have verified Schmidt's theorem. (Instead of Proposition 2.12 one can also use Proposition 2.10; see e.g. [19].) $\square$

2.3.5. BMO and exponential Orlicz norms

The bounded mean oscillation norm is for an integrable function $f:[0,1]^s\to\mathbb{R}$ defined as

$$\|f\|_{\mathrm{BMO}([0,1)^s)} = \sup_{U\subset[0,1)^s} \left(|U|^{-1}\sum_{\boldsymbol{j}\in\mathbb{N}_0^s} 2^{|\boldsymbol{j}|} \sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ I_{\boldsymbol{j},\boldsymbol{m}}\subset U}} |\langle f, h_{\boldsymbol{j},\boldsymbol{m}}\rangle|^2\right)^{\frac{1}{2}},$$

where the supremum is taken over all measureable subsets of $[0,1)^s$.

We introduce the Orlicz norms. Let therefore (Ω,P) be a probability space and let $\mathbb{E}$ denote the expectation over (Ω,P). Let $\Psi:[0,\infty)\to[0,\infty)$ be a convex function, such that $\Psi(x)=0$ if and only if $x=0$. For a (Ω,P)-measurable real valued function $f:[0,1]^s\to\mathbb{R}$ we define

$$\|f\|_{L^\Psi} := \inf\{K>0 : \mathbb{E}\Psi(|f|/K)\leq 1\},$$

where $\inf \emptyset = \infty$. Let $\alpha > 0$ and let Ψ_α be a convex function which equals $e^{x^\alpha} - 1$ for x sufficiently large, then we denote $\exp(L^\alpha) := L^{\Psi_\alpha}$. We note that for all $1 \leq p < \infty$ we have $L_\infty([0,1)^s) \subset \exp(L^\alpha) \subset L_p([0,1)^s)$, i.e. every bounded function in $L_\infty([0,1)^s)$ is also contained in the exponential Orlicz space $\exp(L^\alpha)$ and every function in $\exp(L^\alpha)$ is also an element of $L_p([0,1)^s)$. The following propositions, which are also mentioned in [7, Proposition 2.2, 2.3], provide tools to estimate the exponential Orlicz norm of a function f.

Proposition 2.13. *For any $\alpha > 0$ and a (Ω, P)-measurable real valued function $f : [0,1]^s \to \mathbb{R}$, the following equivalence holds:*

$$\|f\|_{\exp(L^\alpha)} \simeq \sup_{p>1} p^{-\frac{1}{\alpha}} \|f\|_{L_p([0,1)^s)}.$$

The next proposition follows directly from Proposition 2.10 and Proposition 2.13.

Proposition 2.14. *For a (Ω, P)-measurable real valued function $f : [0,1]^s \to \mathbb{R}$ we have:*

$$\|f\|_{\exp(L^{2/s})} \lesssim \|S(f)\|_{L_\infty([0,1)^s)}.$$

2.4. Discrepancy bounds in several function spaces

For a long time, the only norms of the discrepancy function which have been considered were the L_p norms for $p \in [1, \infty]$. However, in recent years much progress has been made for other norms.

Triebel initiated the study of the discrepancy function in other spaces such as the Besov spaces and Triebel-Lizorkin spaces of dominating mixed smoothness in [69] and [70]. He showed that for all $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$ satisfying $\frac{1}{p} - 1 < r < \frac{1}{p}$ and $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$ and for all $N \geq 2$ the discrepancy function of any N-element point set $\mathcal{P}$ in $[0,1)^s$ satisfies

$$\|\Delta_N(\cdot, \mathcal{P})\|_{S^r_{p,q}B([0,1)^s)} \gtrsim N^r (\log N)^{\frac{s-1}{q}} \tag{2.26}$$

This bound may be proven in a similar manner as Schmidt's lower bound (see Section 2.3.4) by employing Proposition 2.11. Also, for any $N \geq 2$, there exists a point set $\mathcal{P}$ in $[0,1)^s$ with N points such that

$$\|\Delta_N(\cdot, \mathcal{P})\|_{S^r_{p,q}B([0,1)^s)} \lesssim N^r (\log N)^{(s-1)\left(\frac{1}{q}+1-r\right)}.$$

Hinrichs showed in [36] that in two dimensions the gap between the exponents of the lower and the upper bounds can be closed for $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{p}$. For the proof, he considered specific point sets, namely the digit shifted Hammersley point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ for certain shifts $\boldsymbol{\sigma}$, which achieve a $S^r_{p,q}B$-discrepancy of order in accordance to the lower bound (2.26). Markhasin closed the gap in arbitrary dimensions under the same conditions on p, q and r by considering Chen-Skriganov point sets in [51] and higher order digital nets in [53]. Summarizing, for $1 \leq p, q \leq \infty$ and $r \geq 0$ there exist point sets $\mathcal{P}$ in $[0,1)^s$ with N points such that

$$\|\Delta_N(\cdot, \mathcal{P})\|_{S^r_{p,q}B([0,1)^s)} \lesssim N^r (\log N)^{\frac{s-1}{q}},$$

which is best possible. It is interesting that the mentioned point sets do not achieve the optimal order of $S^r_{p,q}B$ discrepancy also for $r < 0$, i.e. for negative smoothness parameters. However, in two dimensions a simple symmetrisation trick of the Hammersley point set can overcome this problem, as we outline in Section 4.1.3.

Until recently, there have not been any concrete results on the Besov norm of the discrepancy function of infinite sequences. The one-dimensional case was first treated in [44]. There we studied the symmetrized van der Corput sequence and showed that for $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{p}$ we have for all $N \geq 2$ that

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_b)\|_{S^r_{p,q}B([0,1))} \lesssim (\log N)^{\frac{1}{q}} \tag{2.27}$$

if $r = 0$ and

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_b)\|_{S^r_{p,q}B([0,1))} \lesssim N^r \tag{2.28}$$

if $0 < r < 1/p$. The surprising aspect of this result is that for positive smoothness $0 < r < 1/p$ sequences in the unit interval $[0,1)$ can achieve the same rate of $S^r_{p,q}B$ discrepancy as point sets in $[0,1)$, whereas the L_p and star discrepancy for sequences in $[0,1)^s$ is related to that of point sets in $[0,1)^{s+1}$. The proofs of (2.27) and (2.28) are part of this thesis and will be given in Section 4.2.2. There exist also higher-dimensional versions of (2.27) and (2.28), which were shown by Dick, Hinrichs, Markhasin and Pillichshammer in [18]. Their result is based on higher order digital sequences $\mathcal{S}$ and states that for $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{p}$ we have for all $N \geq 2$ that

$$\|\Delta_N(\cdot, \mathcal{S})\|_{S^r_{p,q}B([0,1)^s)} \lesssim (\log N)^{\frac{s}{q}} \tag{2.29}$$

if $r = 0$ and

$$\|\Delta_N(\cdot, \mathcal{S})\|_{S^r_{p,q}B([0,1)^s)} \lesssim N^r (\log N)^{\frac{s-1}{q}} \tag{2.30}$$

if $0 < r < 1/p$. We note that the above mentioned curiosity in the case of positive smoothness r appears also in higher dimensions.
For the Triebel-Lizorkin norm of the discrepancy function of higher order digital sequences the authors of [18] obtained the same upper bounds, where the condition on r in (2.30) must be changed to $0 < r < 1/\max\{p, q\}$.

It is convenient to study also the BMO and the exponential Orlicz norm of the discrepancy function, since these norms are in some sense closer to the important L_∞ case than the L_p norms. For all $N \geq 2$ there exist N element point sets in the unit interval $[0,1)^s$ such that

$$\|\Delta_N(\cdot, \mathcal{P})\|_{\mathrm{BMO}([0,1)^s)} \lesssim (\log N)^{\frac{s-1}{2}},$$

namely higher order digital nets. A two-dimensional version of this result based on digit shifted Hammersley point sets can be found in [5]. This upper bound is complemented by a matching lower bound. These results were shown in [7] and demonstrate that the BMO norm of the discrepancy function behaves like the L_p discrepancy. For infinite sequences $\mathcal{S}$, the exact optimal order of magnitude in N of the BMO norm of the discrepancy function is

$$\|\Delta_N(\cdot, \mathcal{S})\|_{\mathrm{BMO}([0,1)^s)} \lesssim (\log N)^{\frac{s}{2}},$$

as shown in [18].

Apart from the BMO norm, also the exponential Orlicz norm of the discrepancy function has been thoroughly studied to gain insight into the behaviour of the star discrepancy. From [7] we know that there exist point sets in $[0,1)^s$ such that

$$\|\Delta_N(\cdot,\mathcal{P})\|_{L^{\frac{2}{s-1}}} \lesssim (\log N)^{\frac{s-1}{2}},$$

which is sharp. Moreover, for every $2/(s-1) \leq \beta < \infty$ we have

$$\|\Delta_N(\cdot,\mathcal{P})\|_{L^\beta} \lesssim (\log N)^{(s-1)-\frac{1}{\beta}},$$

which is known to be sharp only in dimension 2, whereas this is not yet the case for $s \geq 3$. For every infinite sequence $\mathcal{S}$ in $[0,1)^s$, where $s \geq 2$, we have

$$\|\Delta_N(\cdot,\mathcal{S})\|_{L^{\frac{2}{s-1}}} \lesssim (\log N)^{\frac{s}{2}},$$

which is sharp and for every $2/(s-1) \leq \beta < \infty$ we have

$$\|\Delta_N(\cdot,\mathcal{S})\|_{L^\beta} \lesssim (\log N)^{s-\frac{1}{\beta}},$$

This has been shown in [18]. There exists no matching lower bound so far. An one-dimensional version of the last result can be found in Section 4.2.2 (Theorem 4.42).

We would like to close this section by listing all new results in this direction which we will present in this thesis.

- We will investigate which conditions on Σ lead to the optimal order of $S^r_{p,q}B$ and $S^r_{p,q}F$ discrepancy for the generalized Hammersley point sets $\mathcal{H}^\Sigma_{b,n}$ and show that the whole class of symmetrized Hammersley point set $\widetilde{\mathcal{H}}^\Sigma_{b,n}$ does so.
- We will prove that a certain symmetrization of digit shifted Hammersley point sets leads to optimal bounds on its $S^r_{p,q}B$ and $S^r_{p,q}F$ discrepancy even for negative smoothness parameters.
- We will study the $S^r_{p,q}B$ and $S^r_{p,q}F$ discrepancy of the van der Corput sequences $\mathcal{V}^\sigma_b$ and its symmetrized versions $\widetilde{\mathcal{V}}^\sigma_b$ and obtain best possible upper bounds (as announced above).
- We will also show that the BMO and exponential Orlicz norms of the discrepancy function of the sequences $\widetilde{\mathcal{V}}^\sigma_b$ satisfy optimal upper bounds.

3. Precise discrepancy results

3.1. Digit shifted Hammersley point sets in base 2

3.1.1. An exact formula for the L_4 discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$

Statement of the result As already mentioned in Section 2.1, it has been shown in [42] that for every even $p \in \mathbb{N}$ there exists a shift $\boldsymbol{\sigma}$ such that

$$L_p(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) = \mathcal{O}\left(\sqrt{\log N}\right),$$

where $N = 2^n$. The aim of this chapter is to find two shifts $\boldsymbol{\sigma}$ such that the L_4 discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ is of this best possible order and thereby proving exact formulas for this quantity. We consider only even n and study the shifts $\boldsymbol{\sigma}_1 = (\sigma_1, \dots, \sigma_n)$, where $\sigma_j = 1$ for $j \in \{1, \dots, \frac{n}{2}\}$ and $\sigma_j = 0$ otherwise and $\boldsymbol{\sigma}_2 = (\sigma_1', \dots, \sigma_n')$, where $\sigma_j' = 1$ for odd indices j and $\sigma_j' = 0$ for even j. The following theorem shows that the L_4 discrepancy of the point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$ is of the desired low order and that in contrast to the L_2 discrepancy it does not only depend on the number of zero digits in the shift $\boldsymbol{\sigma}$, but also on their position.

Theorem 3.1. *For even $n \in \mathbb{N}$ we have*

$$\begin{aligned}(L_4(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)))^4 =& \frac{25}{12288}n^2 + \left(\frac{1739}{30720} - \frac{13}{144}2^{-n} + \frac{11}{1152}2^{-2n}\right)n \\ &+ \left(\frac{2893}{8640} + \frac{89}{432}2^{-n} - \frac{145}{1728}2^{-2n} - \frac{1}{3600}2^{-4n}\right)\end{aligned}$$

and

$$\begin{aligned}(L_4(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)))^4 =& \frac{25}{12288}n^2 + \left(\frac{5281}{92160} - \frac{1}{12}2^{-n} + \frac{11}{1152}2^{-2n}\right)n \\ &+ \left(\frac{14221}{43200} + \frac{5}{24}2^{-n} - \frac{3481}{43200}2^{-2n} - \frac{1}{3600}2^{-4n}\right).\end{aligned}$$

Remark 3.2. From Theorem 3.1 we get

$$\begin{aligned}(L_4(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)))^4 - (L_4(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)))^4 =& \left(\frac{1}{1440} + \frac{1}{144}2^{-n}\right)n \\ &- \left(\frac{61}{10800} - \frac{1}{432}2^{-n} - \frac{1}{300}2^{-2n}\right).\end{aligned}$$

This difference is 0 for $n = 2$ (which is clear since in this case $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the same tuple, namely $(1, 0)$) and greater than 0 for $n \geq 4$. Hence, with respect to the L_4 discrepancy the shift $\boldsymbol{\sigma}_1$ leads to slightly better results than $\boldsymbol{\sigma}_2$.

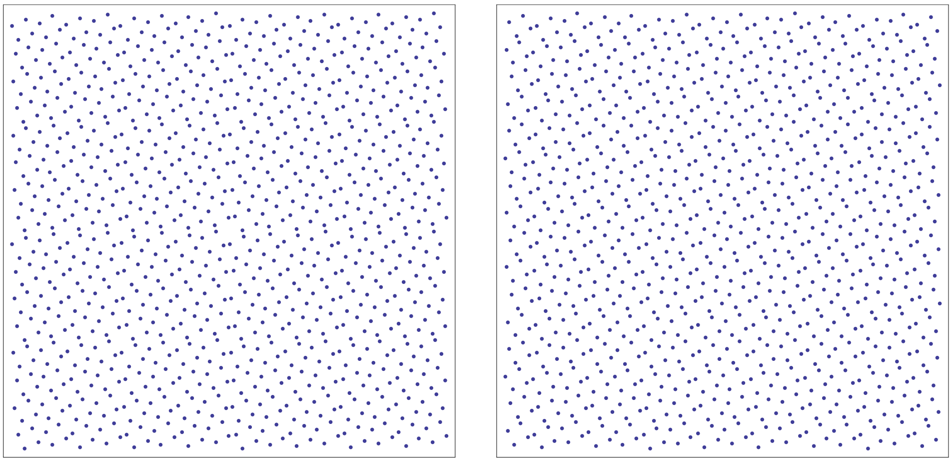


Figure 3.1.: The digit shifted Hammersley point sets $\mathcal{H}_{2,10}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,10}(\boldsymbol{\sigma}_2)$. While the L_2 discrepancy of these point sets has the same value of $0.797283\ldots$, their L_4 discrepancies differ slightly. We have $L_4(\mathcal{H}_{2,10}(\boldsymbol{\sigma}_1)) = 1.024971\ldots$ and $L_4(\mathcal{H}_{2,10}(\boldsymbol{\sigma}_2)) = 1.025288\ldots$.

Auxiliary results The proof of Theorem 3.1 relies strongly on Theorem 2.5 and uses techniques developed and employed in the papers [42, 43, 46, 58]. To show Theorem 3.1, we need various auxiliary results. Lemma 3.3 states that the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$, for which we will simply write $\Delta(\alpha,\beta)$ throughout this section, fulfils an interesting relation, if the shift $\boldsymbol{\sigma}$ has a certain property. This lemma will simplify many calculations later on in the proof of Theorem 3.1.

Lemma 3.3. *If $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ fulfils the property*

$$\mathbf{P} : \sigma_{n+1-j} = \sigma_j \oplus 1 \text{ for all } j \in \{1, \ldots, n\},$$

then the discrepancy function Δ of the shifted Hammersley point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ satisfies

$$\Delta(\alpha,\beta) = -\Delta(1-\beta,\alpha)$$

for all n-bit α and β.

Proof. For a point $(c,d) \in [0,1)^2$ with n-bit components c and d, we define the functions $T(c,d) = T^1(c,d) = (1 - \frac{1}{2^n} - d, c)$ and $T^{k+1}(c,d) = T(T^k(c,d))$ for $k \in \mathbb{N}$. The shift $\boldsymbol{\sigma}$ shall fulfil property $\mathbf{P}$. Then we have

1. $T^4 = id$, thus $T^3 = T^{-1}$,
2. $(c,d) \in \mathcal{H}_{2,n}(\boldsymbol{\sigma})$ if and only if $T(c,d) \in \mathcal{H}_{2,n}(\boldsymbol{\sigma})$.

The first assertion is straightforward, since

$$\begin{aligned} T^4(c,d) =& T^3\left(1 - \frac{1}{2^n} - d, c\right) = T^2\left(1 - \frac{1}{2^n} - c, 1 - \frac{1}{2^n} - d\right) \\ =& T\left(d, 1 - \frac{1}{2^n} - c\right) = (c,d). \end{aligned}$$

We turn to the second assertion. From the definition of $H_{2,n}(\boldsymbol{\sigma})$ we derive that the point $(c,d) \in [0,1)^2$ is an element of this point set if and only if the digits of $c = \frac{c_1}{2} + \cdots + \frac{c_n}{2^n}$ and $d = \frac{d_1}{2} + \cdots + \frac{d_n}{2^n}$ satisfy the relations

$$d_j = c_{n+1-j} \oplus \sigma_{n+1-j} \text{ for all } j \in \{1, \ldots, n\}.$$

That leads to

$$c_j = d_{n+1-j} \oplus \sigma_j \text{ for all } j \in \{1, \dots, n\}.$$

Now we need the condition that $\boldsymbol{\sigma}$ fulfils property **P**, as it yields

$$c_j = (d_{n+1-j} \oplus 1) \oplus (\sigma_j \oplus 1) = (1 - d_{n+1-j}) \oplus \sigma_{n+1-j} \text{ for all } j \in \{1, \dots, n\}.$$

From that we conclude that the point

$$\left(\frac{1-d_1}{2} + \dots + \frac{1-d_n}{2^n}, c \right) = \left(1 - \frac{1}{2^n} - d, c \right) = T(c,d)$$

is also an element of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$. From *1.* we obtain the other implication in *2.*
The clue of the rest of the proof is to show

$$A_N([0,\alpha) \times [0,\beta)) + A_N([0,1-\beta) \times [0,\alpha)) = N\alpha \tag{3.1}$$

for all n-bit α and β, where A_N refers to the point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$. Let therefore (c,d) be an element of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ in $[0,\alpha) \times [0,\beta)$. We consider the point $T(c,d) = \left(1 - \frac{1}{2^n} - d, c\right)$ which is also in $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$. From $0 \le d < \beta$ and the fact that β is n-bit, we get

$$1 - \frac{1}{2^n} - \beta < 1 - \frac{1}{2^n} - d \le 1 - \frac{1}{2^n}, \text{ thus } 1 - \beta \le 1 - \frac{1}{2^n} - d < 1.$$

We obtain $T(c,d) \in [1-\beta, 1) \times [0,\alpha)$. On the other hand, for every point (e,f) of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ in $[1-\beta,1) \times [0,\alpha)$ we have a point of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ in $[0,\alpha) \times [0,\beta)$, namely $T^3(e,f)$. We have found $A_N([0,\alpha) \times [0,\beta)) = A_N([1-\beta,1) \times [0,\alpha))$. It is simple to see that $A_N([0,1) \times [0,\alpha)) = N\alpha$. We conclude

$$\begin{aligned} A_N([0,1-\beta) \times [0,\alpha)) =& A_N([0,1) \times [0,\alpha)) - A_N([1-\beta,1) \times [0,\alpha)) \\ =& N\alpha - A_N([0,\alpha) \times [0,\beta)) \end{aligned}$$

which results in (3.1). Now we can finish the proof since

$$\begin{aligned} &\Delta(\alpha,\beta) + \Delta(1-\beta,\alpha) \\ =& A_N([0,\alpha) \times [0,\beta)) + A_N([0,1-\beta) \times [0,\alpha)) - N\alpha\beta - N(1-\beta)\alpha \\ =& N\alpha - N\alpha\beta - N\alpha + N\alpha\beta = 0 \end{aligned}$$

as claimed. □

We are able to derive several useful consequences from Lemma 3.3, which are stated in the following corollary. Recall the definition of $\mathbb{Q}(2^n)$ from the lines before Theorem 2.5.

Corollary 3.4. Let $\Delta(\alpha,\beta)$ be the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ for n-bit α, β, where $\boldsymbol{\sigma}$ fulfils property **P**. Then we have

$$\sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \alpha^r \beta^s \Delta(\alpha,\beta)^\ell = (-1)^\ell \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} (1-\alpha)^s \beta^r \Delta(\alpha,\beta)^\ell \tag{3.2}$$

for all $r, s \in \mathbb{N}_0$ and $\ell \in \mathbb{N}$. Especially, for $s = 0$ we have

$$\sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \alpha^r \Delta(\alpha,\beta)^\ell = (-1)^\ell \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \beta^r \Delta(\alpha,\beta)^\ell \tag{3.3}$$

and for $r = s = 0$ and odd ℓ

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell = 0. \tag{3.4}$$

We also have

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell = \frac{1}{2}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell \tag{3.5}$$

for all $\ell \in \mathbb{N}$, especially

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell = \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\Delta(\alpha,\beta)^\ell = 0 \tag{3.6}$$

for odd $\ell \in \mathbb{N}$. Further we have

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell = \frac{1}{4}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell \tag{3.7}$$

for even $\ell \in \mathbb{N}$ and

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta^2\Delta(\alpha,\beta)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha^2\beta\Delta(\alpha,\beta)^\ell = 2\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell \tag{3.8}$$

for odd $\ell \in \mathbb{N}$.

Proof. We use Lemma 3.3 and obtain

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha^r\beta^s\Delta(\alpha,\beta)^\ell =&(-1)^\ell \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha^r\beta^s\Delta(1-\beta,\alpha)^\ell \\ =&(-1)^\ell \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha^r(1-\beta)^s\Delta(\beta,\alpha)^\ell \\ =&(-1)^\ell \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} (1-\alpha)^s\beta^r\Delta(\alpha,\beta)^\ell\end{aligned}$$

which yields (3.2), (3.3) and (3.4). To verify (3.5) and (3.6), we write in the case that ℓ is even

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell =& \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} (\beta-1)\Delta(1-\beta,\alpha)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell \\ =&-\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\beta,\alpha)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell \\ =&-\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\Delta(\alpha,\beta)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell \\ =&-\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell\end{aligned}$$

and we obtain (3.5) for even exponents ℓ. In the case of odd exponents ℓ, we have

$$\begin{aligned}0 =& \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell = \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} (1-\beta)\Delta(\alpha,\beta)^\ell \\ =& \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell + \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,1-\beta)^\ell \\ =& \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell - \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\beta,\alpha)^\ell\end{aligned}$$

$$= \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell - \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\Delta(\alpha,\beta)^\ell = 2\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell,$$

which results in (3.6) together with (3.3). We turn to the last two claims in this corollary. If ℓ is even, we have

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell &= \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(1-\beta,\alpha)^\ell = \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha(1-\beta)\Delta(\beta,\alpha)^\ell \\ &= \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta(1-\alpha)\Delta(\alpha,\beta)^\ell \\ &= \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell - \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell,\end{aligned}$$

which yields

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell = \frac{1}{2}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell = \frac{1}{4}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^\ell$$

together with (3.5). If ℓ is odd, we can write

$$\begin{aligned}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta^2\Delta(\alpha,\beta)^\ell &= -\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta^2\Delta(1-\beta,\alpha)^\ell = -\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} (1-\alpha)^2\beta\Delta(\alpha,\beta)^\ell \\ &= -\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta\Delta(\alpha,\beta)^\ell + 2\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta)^\ell \\ &\quad - \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha^2\beta\Delta(\alpha,\beta)^\ell,\end{aligned}$$

which yields (3.8) by regarding (3.6). □

In the following, we study sums which involve the expressions which appear in Theorem 2.5 and will be fundamental for our proof.

Lemma 3.5. *With the definitions as in Theorem 2.5 we have*

$$\sum_{\alpha\in\mathbb{Q}(2^n)} \alpha(\alpha_{n-u}\oplus\alpha_{n+1-j(u)}) = 2^{n-2}+2^{u-2}-\frac{1}{4}-\frac{1}{2}\sum_{j=1}^{u}(\beta_j\oplus\sigma_j\oplus 1)2^{j-1}.$$

Let now $1\le k\le n-1$ *and* $u_1,\dots,u_k\in\{0,\dots,n-1\}$ *with* $u_i\ne u_j$ *for* $1\le i\ne j\le n-1$. *Then we have*

$$\sum_{\alpha\in\mathbb{Q}(2^n)}\prod_{i=1}^{k}(\alpha_{n-u_i}\oplus\alpha_{n+1-j(u_i)}) = 2^{n-k}.$$

Proof: The first formula is [43, Lemma 1] and the second one is [42, Lemma 2].

Lemma 3.6. *Choose an* n*-bit number* β. *Let* $1\le u\le n-1$ *be an integer. Let* $1\le k\le n-1$ *be an integer and* $u_1<u_2<\cdots<u_k\in\{0,1,\dots,n-1\}$. *Then we have*

$$\sum_{\beta\in\mathbb{Q}(2^n)}\prod_{j=1}^{k}\|2^{u_i}\beta\| = \frac{2^n}{2^{2k}},\ \sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\| = \frac{2^n}{8},\ \sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2 = \frac{2^{2n}+2^{2u+1}}{3\cdot 2^{n+2}},$$

$$\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^4 = \frac{1}{16}\left(\frac{1}{5}2^n+\frac{1}{3}2^{-n+2u+2}-\frac{1}{15}2^{-3n+4u+3}\right),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^3\|2^{u_2}\beta\| = \frac{1}{128}(2^n - 2^{n+2u_1-2u_2} - 2^{-n+2u_1+2}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|\|2^{u_2}\beta\|^3 = \frac{1}{128}(2^n + 2^{-n+2u_2+2}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|^2 = \frac{1}{1440}(5\cdot 2^{n+1} + 5\cdot 2^{-n+2u_2+2} - 7\cdot 2^{n+2u_1-2u_2} - $$
$$- 5\cdot 2^{-n+2u_1+2} - 3\cdot 2^{-3n+2u_1+2u_2+4}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|\|2^{u_3}\beta\| = \frac{1}{384}(2^{n+2u_1-2u_3} + 2^{-n+2u_1+2} + 2^{n+1} - 2^{n+2u_1-2u_2}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|\|2^{u_2}\beta\|^2\|2^{u_3}\beta\| = \frac{1}{384}(2^{n+1} - 2^{n+2u_2-2u_3} - 2^{-n+2u_2+2}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|\|2^{u_2}\beta\|\|2^{u_3}\beta\|^2 = \frac{1}{192}(2^n + 2^{-n+2u_3+1}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| = \frac{1}{384}(2^{n+3} + 2^{n-2u_2} - 2^{n-2u_1} + 2^{-n+2}),$$

$$\sum_{\beta\in\mathbb{Q}(2^n)} \beta^2\|2^{u}\beta\|^2 = \frac{1}{45}(5\cdot 2^{2u-n-1} - 3\cdot 2^{2u-3n-1} - 7\cdot 2^{-2u+n-5} + 5\cdot 2^{n-2} - 5\cdot 2^{-n-3}).$$

Proof. The first formula is [58, Lemma 3 a)], the second one has already been shown in the proof of [58, Theorem 2] and the third one is [58, Lemma 3 b)]. To verify the results for the other sums, we show a formula for $\Sigma := \sum_{\beta\in\mathbb{Q}(2^n)} \beta^q\|2^{u_1}\beta\|^r\|2^{u_2}\beta\|^s\|2^{u_3}\beta\|^t$, where $u_1 < u_2 < u_3 < m$ and $q, r, s, t \in \mathbb{N}_0$. This formula is

$$\begin{aligned}\Sigma = \sum_{k=0}^{2^{u_1}-1} \Bigg\{ \sum_{l=0}^{2^{u_2-u_1-1}-1} \Bigg[\sum_{m=0}^{2^{u_3-u_2-1}-1} \Bigg(\sum_{b=0}^{2^{n-u_3-1}-1} A^qB^rC^sD^t + \sum_{b=2^{n-u_3-1}}^{2^{n-u_3}-1} A^qB^rC^s(1-D)^t \Bigg) \\ + \sum_{m=2^{u_3-u_2-1}}^{2^{u_3-u_2}-1} \Bigg(\sum_{b=0}^{2^{n-u_3-1}-1} A^qB^r(1-C)^sD^t + \sum_{b=2^{n-u_3-1}}^{2^{n-u_3}-1} A^qB^r(1-C)^s(1-D)^t \Bigg) \Bigg] \\ + \sum_{l=2^{u_2-u_1-1}}^{2^{u_2-u_1}-1} \Bigg[\sum_{m=0}^{2^{u_3-u_2-1}-1} \Bigg(\sum_{b=0}^{2^{n-u_3-1}-1} A^q(1-B)^rC^sD^t \\ + \sum_{b=2^{n-u_3-1}}^{2^{n-u_3}-1} A^q(1-B)^rC^s(1-D)^t \Bigg) \\ + \sum_{m=2^{u_3-u_2-1}}^{2^{u_3-u_2}-1} \Bigg(\sum_{b=0}^{2^{n-u_3-1}-1} A^q(1-B)^r(1-C)^sD^t \\ + \sum_{b=2^{n-u_3-1}}^{2^{n-u_3}-1} A^q(1-B)^r(1-C)^s(1-D)^t \Bigg) \Bigg] \Bigg\},\end{aligned}$$

where we use the abbreviations $A := \frac{b}{2^n} + \frac{k}{2^{u_1}} + \frac{l}{2^{u_2}} + \frac{m}{2^{u_3}}$, $B := 2^{u_1-n}b + 2^{u_1-u_2}l + 2^{u_1-u_3}m$, $C := 2^{u_2-n}b + 2^{u_2-u_3}m$ and $D := 2^{u_3-n}b$. We can now prove all the formulas in this lemma by calculating this term for suitable choices of the numbers q, r, s and t with the aid of a computer algebra system. To show the idea how the formula above can be found, we derive a similar formula for $\sum_{\beta\in\mathbb{Q}(2^n)} \beta^q\|2^u\beta\|^r$, $u < n$ instead. To show the above formula this method can be adapted easily. The proof however is quite lengthy.

We can write $\sum_{\beta\in\mathbb{Q}(2^n)}\beta^q\|2^u\beta\|^r=\sum_{b=0}^{2^n-1}\left(\frac{b}{2^n}\right)^q\|2^{u-n}b\|^r$ and thus

$$\sum_{\beta\in\mathbb{Q}(2^n)}\beta^q\|2^u\beta\|^r=\sum_{k=0}^{2^u-1}\Bigg\{\sum_{b=(2k)2^{n-u-1}}^{(2k+1)2^{n-u-1}-1}\left(\frac{b}{2^n}\right)^q\|2^{u-n}b\|^r$$
$$+\sum_{b=(2k+1)2^{n-u-1}}^{(2k+2)2^{n-u-1}-1}\left(\frac{b}{2^n}\right)^q\|2^{u-n}b\|^r\Bigg\}.$$

If $(2k)2^{n-u-1}\le b<(2k+1)2^{n-u-1}$, then $k\le 2^{u-n}b<k+\frac{1}{2}$, from which we derive $\lfloor 2^{u-n}b\rfloor=k$ and $\{2^{u-n}b\}=2^{u-n}b-k<\frac{1}{2}$. But that implies $\|2^{u-n}b\|=2^{u-n}b-k$ in this case. If $(2k+1)2^{n-u-1}\le b<(2k+2)2^{n-u-1}$, we get $k+\frac{1}{2}\le 2^{u-n}b<k+1$, which delivers $\lfloor 2^{u-n}b\rfloor=k$ again, but since $\frac{1}{2}\le\{2^{u-n}b\}<1$, we have $\|2^{u-n}b\|=1-2^{u-n}b+k$ in this case. Therefore we have

$$\sum_{\beta\in\mathbb{Q}(2^n)}\beta^q\|2^u\beta\|^r=\sum_{k=0}^{2^u-1}\left\{\sum_{b=(2k)2^{n-u-1}}^{(2k+1)2^{n-u-1}-1}\left(\frac{b}{2^n}\right)^q(2^{u-n}b-k)^r+\right.$$
$$\left.+\sum_{b=(2k+1)2^{n-u-1}}^{(2k+2)2^{n-u-1}-1}\left(\frac{b}{2^n}\right)^q(1-2^{u-n}b+k)^r\right\}.$$

An index shift yields

$$\sum_{\beta\in\mathbb{Q}(2^n)}\beta^q\|2^u\beta\|^r=\sum_{k=0}^{2^u-1}\left\{\sum_{b=0}^{2^{n-u-1}-1}\left(\frac{b}{2^n}+\frac{k}{2^u}\right)^q(2^{u-n}b)^r+\right.$$
$$\left.+\sum_{b=2^{n-u-1}}^{2^{n-u}-1}\left(\frac{b}{2^n}+\frac{k}{2^u}\right)^q(1-2^{u-n}b)^r\right\}.$$

We note the similar structure as the above formula for Σ. □

In the following lemmas, we restrict our calculations to the special shifts $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. It is evident that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ both have property **P** as explained in Lemma 3.3. Recall that the number $l=l(\boldsymbol{\sigma})$ is the number of zero digits in $\boldsymbol{\sigma}$. Note that $l(\boldsymbol{\sigma}_1)=l(\boldsymbol{\sigma}_2)=\frac{n}{2}$.

Lemma 3.7. *Let $\Delta(\alpha,\beta)$ be the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ or of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$. Then we have*

$$\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\alpha\beta\Delta(\alpha,\beta)=-\frac{n}{2^{n+7}}.$$

Proof. We use Theorem 2.5 to write

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\alpha\beta\Delta(\alpha,\beta)=\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\alpha\beta\sum_{u=0}^{n-1}\|2^u\beta\|(-1)^{\sigma_{u+1}}(\alpha_{n-u}\oplus\alpha_{n+1-j(u)})$$
$$=\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\|\sum_{\alpha\in\mathbb{Q}(2^n)}\alpha(\alpha_{n-u}\oplus\alpha_{n+1-j(u)})$$
$$-\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\|\sum_{\alpha\in\mathbb{Q}(2^n)}\alpha(\alpha_{n-u}\oplus\alpha_{n+1-j(u)}).$$

By Lemma 3.5 and the second formula in Lemma 3.6 we find

$$\begin{aligned}
&\sum_{\beta\in\mathbb{Q}(2^n)} \beta\|2^u\beta\| \sum_{\alpha\in\mathbb{Q}(2^n)} \alpha(\alpha_{n-u}\oplus\alpha_{n+1-j(u)})\\
&\quad= \sum_{\beta\in\mathbb{Q}(2^n)} \beta\|2^u\beta\| \left(2^{n-2}+2^{u-2}-\frac{1}{4}-\frac{1}{2}\sum_{j=1}^{u}(\beta_j\oplus\sigma_j\oplus 1)2^{j-1}\right)\\
&\quad= \frac{2^n}{8}\left(2^{n-2}+2^{u-2}-\frac{1}{4}\right)-\frac{1}{2}\sum_{j=1}^{u}2^{j-1}\sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\|(\beta_j\oplus\sigma_j\oplus 1). \qquad (3.9)
\end{aligned}$$

In order to compute the last sum we use the short-hand $B_{u+i} := \frac{\beta_{u+i}}{2^i}+\cdots+\frac{\beta_n}{2^{n-u}}$ for $\beta = \frac{\beta_1}{2}+\cdots+\frac{\beta_n}{2^n}$. Then we obviously have $\|2^u\beta\| = \|B_{u+1}\|$. Since $B_{u+2} < \frac{1}{2}$, we also have $\|B_{u+2}\| = B_{u+2}$ and $\left\|\frac{1}{2}+B_{u+2}\right\| = \frac{1}{2}-B_{u+2}$. That yields

$$\sum_{\beta_{u+1},\dots,\beta_n=0}^{1}\|B_{u+1}\| = \sum_{\beta_{u+2},\dots,\beta_n=0}^{1}\left(\|B_{u+2}\|+\left\|\frac{1}{2}+B_{u+2}\right\|\right) = \sum_{\beta_{u+2},\dots,\beta_n=0}^{1}\frac{1}{2} = 2^{n-u-2}$$

and

$$\begin{aligned}
\sum_{\beta_{u+1},\dots,\beta_n=0}^{1} B_{u+1}\|B_{u+1}\| &= \sum_{\beta_{u+2},\dots,\beta_n=0}^{1}\left(B_{u+2}\|B_{u+2}\|+\left(\frac{1}{2}+B_{u+2}\right)\left\|\frac{1}{2}+B_{u+2}\right\|\right)\\
&= \sum_{\beta_{u+2},\dots,\beta_n=0}^{1}\frac{1}{4} = 2^{n-u-3}.
\end{aligned}$$

From that we obtain

$$\begin{aligned}
&\sum_{j=1}^{u}2^{j-1}\sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\|(\beta_j\oplus\sigma_j\oplus 1)\\
&\quad=\sum_{j=1}^{u}2^{j-1}\sum_{\substack{\beta_1,\dots,\beta_u=0\\ \beta_j=\sigma_j}}^{1}\sum_{\beta_{u+1},\dots,\beta_n=0}^{1}\left(\frac{\beta_1}{2}+\cdots+\frac{\beta_u}{2^u}+\frac{1}{2^u}B_{u+1}\right)\|B_{u+1}\|\\
&\quad=\sum_{j=1}^{u}2^{j-1}\left\{\sum_{\substack{\beta_1,\dots,\beta_u=0\\ \beta_j=\sigma_j}}^{1}\left(\frac{\beta_1}{2}+\cdots+\frac{\sigma_j}{2^j}+\cdots+\frac{\beta_u}{2^u}\right)\sum_{\beta_{u+1},\dots,\beta_n=0}^{1}\|B_{u+1}\|\right.\\
&\qquad\left.+\sum_{\substack{\beta_1,\dots,\beta_u=0\\ \beta_j=\sigma_j}}^{1}\left(\frac{1}{2^u}\sum_{\beta_{u+1},\dots,\beta_n=0}^{1}B_{u+1}\|B_{u+1}\|\right)\right\}\\
&\quad=\sum_{j=1}^{u}2^{j-1}\left\{\left(\frac{2^{u-1}}{2^j}\sigma_j+\frac{1}{4}(2^u-2^{u-j}-1)\right)2^{n-u-2}+2^{n-u-4}\right\}\\
&\quad=\sum_{j=1}^{u}2^{j-1}\left\{2^{n-j-3}\sigma_j+2^{n-4}-2^{n-j-4}\right\} = 2^{n-4}\sum_{j=1}^{u}\sigma_j+2^{n-5}(2^{u+1}-u-2).
\end{aligned}$$

We put this result into (3.9), which leads to

$$\sum_{\beta\in\mathbb{Q}(2^n)}\beta\|2^u\beta\|\sum_{\alpha\in\mathbb{Q}(2^n)}\alpha(\alpha_{n-u}\oplus\alpha_{n+1-j(u)}) = 2^{2n-5}+2^{n-5}\left(\frac{u}{2}-\sum_{j=1}^{u}\sigma_j\right)$$

and finally we see that

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta) = 2^{2n-5}(2l-n)+2^{n-5}\left(\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1}\left(\frac{u}{2}-\sum_{j=1}^{u}\sigma_j\right)-\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1}\left(\frac{u}{2}-\sum_{j=1}^{u}\sigma_j\right)\right).$$

This formula is valid for any shift $\boldsymbol{\sigma}$. In particular, for $\boldsymbol{\sigma}_1$ we find

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta) = 2^{n-5}\left(\sum_{u=0}^{n/2-1}\frac{u}{2}-\sum_{u=n/2}^{n-1}\left(\frac{u}{2}-\left(u-\frac{n}{2}\right)\right)\right) = -\frac{n}{128}2^n$$

and for $\boldsymbol{\sigma}_2$ we have

$$\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta) = 2^{n-5}\left(\sum_{u=0}^{n/2-1}\left(\frac{2u+1}{2}-(u+1)\right)-\sum_{u=1}^{n/2}\left(\frac{2u}{2}-u\right)\right) = -\frac{n}{128}2^n.$$

This yields the claimed result. □

Lemma 3.8. *Let $\Delta(\alpha,\beta)$ be the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ or the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$. Then we have*

$$\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\Delta(\alpha,\beta)^2 = \frac{1}{576}(15n+16-2^{-2n+4}).$$

Proof. We put $l=\frac{n}{2}$ in [42, Lemma 6] and obtain the result. □

Lemma 3.9. *Let $\Delta_1(\alpha,\beta)$ and $\Delta_2(\alpha,\beta)$ be the discrepancy functions of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$, respectively. Then we have*

$$\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\beta^2\Delta_1(\alpha,\beta)^2 = \left(\frac{5}{576}-\frac{1}{288}2^{-n}-\frac{11}{1152}2^{-2n}\right)n + \frac{77}{8640}-\frac{25}{1728}2^{-2n}+\frac{1}{180}2^{-4n}$$

and

$$\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\beta^2\Delta_2(\alpha,\beta)^2 = \left(\frac{5}{576}-\frac{11}{1152}2^{-2n}\right)n + \frac{61}{8640}-\frac{109}{8640}2^{-2n}+\frac{1}{180}2^{-4n}.$$

Proof. Let first $\Delta(\alpha,\beta)$ be the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ for any shift $\boldsymbol{\sigma}$. Considering Theorem 2.5, we can write

$$\begin{aligned}\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\beta^2\Delta(\alpha,\beta)^2 &= \frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\beta^2\Bigg(\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1}\|2^u\beta\|(\alpha_{n-u}\oplus\alpha_{n+1-j(u)}) \\ &\qquad -\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1}\|2^u\beta\|(\alpha_{n-u}\oplus\alpha_{n+1-j(u)})\Bigg)^2 \\ &=:\frac{1}{2^{2n}}(A-2B+C),\end{aligned}$$

where

$$A = \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta^2 \Big(\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \|2^u\beta\|(\alpha_{n-u}\oplus\alpha_{n+1-j(u)})\Big)^2$$
$$=2^n \sum_{\beta\in\mathbb{Q}(2^n)} \Big\{ \frac{1}{2} \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \beta^2\|2^u\beta\|^2 + \frac{1}{4} \sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=0,\sigma_{u_2+1}=0\\ u_1\neq u_2}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| \Big\},$$

$$B = \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Big\{ \beta^2 \Big(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\|(\alpha_{n-u_1}\oplus\alpha_{n+1-j(u_1)})$$
$$\times \sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1} \|2^{u_2}\beta\|(\alpha_{n-u_2}\oplus\alpha_{n+1-j(u_2)})\Big)\Big\}$$
$$=2^n \sum_{\beta\in\mathbb{Q}(2^n)} \Big\{ \frac{1}{4} \sum_{\substack{u_1=0,u_2=0\\ \sigma_{u_1+1}=0,\sigma_{u_2+1}=1}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| \Big\}$$

and

$$C = 2^n \sum_{\beta\in\mathbb{Q}(2^n)} \Big\{ \frac{1}{2} \sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1} \beta^2\|2^u\beta\|^2 + \frac{1}{4} \sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=1\\ u_1\neq u_2}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| \Big\}.$$

Note that we used Lemma 3.5 to simplify these expressions. We put the results together and obtain

$$\frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \beta^2\Delta(\alpha,\beta)^2 = 2^{-n} \sum_{\beta\in\mathbb{Q}(2^n)} \Big\{ \frac{1}{2}\sum_{u=0}^{n-1} \beta^2\|2^u\beta\|^2 + \frac{1}{4} \sum_{\substack{u_1,u_2=0\\ u_1\neq u_2}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\|$$
$$- \sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=0}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| \Big\}. \tag{3.10}$$

The first two sums in the last expression can now be computed with aid of Lemma 3.6. Therefore we define the functions

$$E(x) := \frac{1}{45}\left(5\cdot 2^{2x-n-1} - 3\cdot 2^{2x-3n-1} - 7\cdot 2^{-2x+n-5} + 5\cdot 2^{n-2} - 5\cdot 2^{-n-3}\right) \text{ and}$$
$$F(x,y) := \frac{1}{384}\left(2^{n+3} + 2^{n-2y} - 2^{n-2x} + 2^{-n+2}\right)$$

and have

$$\sum_{\beta\in\mathbb{Q}(2^n)} \sum_{u=0}^{n-1} \beta^2\|2^u\beta\|^2 = \sum_{u=0}^{n-1} E(u) \text{ and}$$
$$\sum_{\beta\in\mathbb{Q}(2^n)} \sum_{\substack{u_1,u_2=0\\ u_1\neq u_2}}^{n-1} \beta^2\|2^{u_1}\beta\|\|2^{u_2}\beta\| = \sum_{\substack{u_1,u_2=0\\ u_1<u_2}}^{n-1} F(u_1,u_2) + \sum_{\substack{u_1,u_2=0\\ u_1>u_2}}^{n-1} F(u_2,u_1).$$

The third sum in (3.10) needs to be calculated individually for the shifts $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. For $\boldsymbol{\sigma}_1$ we compute

$$\sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=0}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \beta^2 \|2^{u_1}\beta\| \|2^{u_2}\beta\| = \sum_{u_1=0}^{n/2-1} \sum_{u_2=n/2}^{n-1} F(u_1,u_2)$$

and for $\boldsymbol{\sigma}_2$ we have to evaluate the sum

$$\sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=0}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \beta^2 \|2^{u_1}\beta\| \|2^{u_2}\beta\|$$
$$= \sum_{u_1=0}^{n/2-1} \sum_{u_2=u_1}^{n/2-1} F(2u_1, 2u_2+1) + \sum_{u_1=1}^{n/2-1} \sum_{u_2=0}^{u_1-1} F(2u_2+1, 2u_1).$$

By calculating and combining all these expressions we obtain the claimed result. □

Lemma 3.10. *Let $\Delta_1(\alpha,\beta)$ and $\Delta_2(\alpha,\beta)$ be the discrepancy functions of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$, respectively. Then we have*

$$\frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta_1(\alpha,\beta)^4 = \frac{25}{12288}n^2 + \left(\frac{217}{92160} + \frac{1}{144}2^{-n} + \frac{11}{1152}2^{-2n}\right) n$$
$$+ \left(-\frac{31}{43200} - \frac{1}{432}2^{-n} + \frac{11}{1728}2^{-2n} - \frac{1}{300}2^{-4n}\right)$$

and

$$\frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta_2(\alpha,\beta)^4 = \frac{25}{12288}n^2 + \left(\frac{281}{92160} + \frac{11}{1152}2^{-2n}\right) n$$
$$+ \left(\frac{1}{960} + \frac{11}{4800}2^{-2n} - \frac{1}{300}2^{-4n}\right).$$

Proof. Let first $\Delta(\alpha,\beta)$ be the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ for any shift $\boldsymbol{\sigma}$. To avoid too large expressions we use the short-hand $A_n(u) = \alpha_{n-u} \oplus \alpha_{n+1-j(u)}$ throughout this proof. We can write

$$\frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^4 = \frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left\{ \sum_{u=0}^{n-1} \|2^u\beta\| (-1)^{\sigma_{u+1}} A_n(u) \right\}^4$$
$$= \frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left\{ \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \|2^u\beta\| A_n(u) - \sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1} \|2^u\beta\| A_n(u) \right\}^4$$
$$=: \frac{1}{2^{2n}}(A - 4B + 6C - 4D + E).$$

where

$$A := \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left(\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \|2^u\beta\| A_n(u) \right)^4,$$

$$B := \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\| A_n(u_1) \right)^3 \left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1} \|2^{u_2}\beta\| A_n(u_2) \right),$$

$$C := \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\| A_n(u_1) \right)^2 \left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1} \|2^{u_2}\beta\| A_n(u_2) \right)^2,$$

$$D := \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\| A_n(u_1) \right) \left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1} \|2^{u_2}\beta\| A_n(u_2) \right)^3 \quad \text{and}$$

$$E := \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \left(\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1} \|2^{u}\beta\| A_n(u) \right)^4.$$

In the following, the abbreviation (p.d.) shall always indicate that the indices of the sum are pairwise distinct. We analyze the expression A and apply Lemma 3.5. We obtain

$$\begin{aligned} A =& 2^n \sum_{\beta\in\mathbb{Q}(2^n)} \Bigg\{ \frac{1}{2} \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \|2^u\beta\|^4 + \sum_{\substack{u_1,u_2=0\,(p.d.)\\ \sigma_{u_1+1},\sigma_{u_2+1}=0}}^{n-1} \|2^{u_1}\beta\|^3 \|2^{u_2}\beta\| + \\ &+ \frac{3}{4} \sum_{\substack{u_1,u_2=0\,(p.d.)\\ \sigma_{u_1+1},\sigma_{u_2+1}=0}}^{n-1} \|2^{u_1}\beta\|^2 \|2^{u_2}\beta\|^2 + \\ &+ \frac{3}{4} \sum_{\substack{u_1,u_2,u_3=0\,(p.d.)\\ \sigma_{u_1+1},\sigma_{u_2+1},\sigma_{u_3+1}=0}}^{n-1} \|2^{u_1}\beta\|^2 \|2^{u_2}\beta\| \|2^{u_3}\beta\| + \\ &+ \frac{1}{16} \sum_{\substack{u_1,u_2,u_3,u_4=0\,(p.d.)\\ \sigma_{u_1+1},\sigma_{u_2+1},\sigma_{u_3+1},\sigma_{u_4+1}=0}}^{n-1} \|2^{u_1}\beta\| \|2^{u_2}\beta\| \|2^{u_3}\beta\| \|2^{u_4}\beta\|. \Bigg\} \\ =&: A_1 + A_2 + A_3 + A_4 + A_5 \end{aligned}$$

Lemma 3.6 delivers

$$\begin{aligned} A_1 =& \frac{2^n}{32} \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} \left(\frac{1}{5} 2^n + \frac{1}{3} 2^{-n+2u+2} - \frac{1}{15} 2^{-3n+4u+3} \right) \\ =& \frac{l}{160} 2^{2n} + \frac{1}{24} \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} 2^{2u} - \frac{1}{60} 2^{-2n} \sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1} 2^{4u} \end{aligned}$$

and

$$A_5 = \frac{2^n}{16} \sum_{\substack{u_1,u_2,u_3,u_4=0\,(p.d.)\\ \sigma_{u_1+1},\sigma_{u_2+1},\sigma_{u_3+1},\sigma_{u_4+1}=0}}^{n-1} \frac{2^n}{2^8} = \frac{2^{2n}}{2^{12}} l(l-1)(l-2)(l-3).$$

We get the analogue results for E, with the exception that $\sigma_{u_i+1} = 1$ for $i \in \{1,2,3,4\}$ and especially

$$E_1 = \frac{n-l}{160} 2^{2n} + \frac{1}{24} \sum_{\substack{u=0 \\ \sigma_{u+1}=1}}^{n-1} 2^{2u} - \frac{1}{60} 2^{-2n} \sum_{\substack{u=0 \\ \sigma_{u+1}=1}}^{n-1} 2^{4u}$$

and

$$E_5 = \frac{2^{2n}}{2^{12}} (n-l)(n-l-1)(n-l-2)(n-l-3).$$

We turn to the expression B and employ Lemma 3.5 to find

$$\begin{aligned}
B = \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \Bigg\{ \Bigg(& \sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\|^3 A_n(u_1) + 3 \sum_{\substack{u_1,u_2=0\,(p.d.) \\ \sigma_{u_1+1},\sigma_{u_2+1}=0}}^{n-1} \|2^{u_1}\beta\|^2 \|2^{u_2}\beta\| A_n(u_1) A_n(u_2) + \\
& + \sum_{\substack{u_1,u_2,u_3=0\,(p.d.) \\ \sigma_{u_1+1},\sigma_{u_2+1},\sigma_{u_3+1}=0}}^{n-1} \prod_{i=1}^{3} \|2^{u_i}\beta\| A_n(u_i) \Bigg) \sum_{\substack{u_4=0 \\ \sigma_{u_4+1}=1}}^{n-1} \|2^{u_4}\beta\| A_n(u_4) \Bigg\} \\
= 2^n \sum_{\beta \in \mathbb{Q}(2^n)} \Bigg\{ & \frac{1}{4} \sum_{\substack{u_1,u_4=0 \\ \sigma_{u_1+1}=0,\sigma_{u_4+1}=1}}^{n-1} \|2^{u_1}\beta\|^3 \|2^{u_4}\beta\| \\
& + \frac{3}{8} \sum_{\substack{u_1,u_2,u_4=0\,(p.d.) \\ \sigma_{u_1+1},\sigma_{u_2+1}=0,\sigma_{u_4+1}=1}}^{n-1} \|2^{u_1}\beta\|^2 \|2^{u_2}\beta\| \|2^{u_4}\beta\| \\
& + \frac{1}{16} \sum_{\substack{u_1,u_2,u_3,u_4=0\,(p.d.) \\ \sigma_{u_1+1},\sigma_{u_2+1},\sigma_{u_3+1}=0,\sigma_{u_4+1}=1}}^{n-1} \|2^{u_1}\beta\| \|2^{u_2}\beta\| \|2^{u_3}\beta\| \|2^{u_4}\beta\| \Bigg\} =: B_1 + B_2 + B_3,
\end{aligned}$$

where

$$B_3 = \frac{2^{2n}}{2^{12}} l(l-1)(l-2)(n-l).$$

We obtain the same expressions for D, where we have to change the conditions for the u_i to $\sigma_{u_1+1} = 0, \sigma_{u_2+1} = 1, \sigma_{u_3+1} = 1, \sigma_{u_4+1} = 1$. Especially, we have

$$D_3 = \frac{2^{2n}}{2^{12}} l(n-l)(n-l-1)(n-l-2).$$

It remains to examine C. In a similar manner as before we obtain

$$\begin{aligned}
C = \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \Bigg\{ \Bigg(& \sum_{\substack{u_1=0 \\ \sigma_{u_1+1}=0}}^{n-1} \|2^{u_1}\beta\|^2 A_n(u_1) + \sum_{\substack{u_2,u_3=0\,(p.d.) \\ \sigma_{u_2+1},\sigma_{u_3+1}=0}}^{n-1} \prod_{i=2,3} \|2^{u_i}\beta\| A_n(u_i) \Bigg) \times \\
& \times \Bigg(\sum_{\substack{u_4=0 \\ \sigma_{u_4+1}=1}}^{n-1} \|2^{u_4}\beta\|^2 A_n(u_4) + \sum_{\substack{u_5,u_6=0\,(p.d.) \\ \sigma_{u_5+1},\sigma_{u_6+1}=1}}^{n-1} \prod_{i=5,6} \|2^{u_i}\beta\| A_n(u_i)) \Bigg) \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
=&2^n \sum_{\beta\in\mathbb{Q}(2^n)} \left\{ \frac{1}{4} \sum_{\substack{u_1,u_4=0\\ \sigma_{u_1+1}=0,\sigma_{u_4+1}=1}}^{n-1} \|2^{u_1}\beta\|^2\|2^{u_4}\beta\|^2 + \right.\\
&+\frac{1}{8} \sum_{\substack{u_2,u_3,u_4=0\,(p.d.)\\ \sigma_{u_2+1},\sigma_{u_3+1}=0,\sigma_{u_4+1}=1}}^{n-1} \|2^{u_2}\beta\|\|2^{u_3}\beta\|\|2^{u_4}\beta\|^2+\\
&+\frac{1}{8} \sum_{\substack{u_1,u_5,u_6=0\,(p.d.)\\ \sigma_{u_1+1}=0,\sigma_{u_5+1},\sigma_{u_6+1}=1}}^{n-1} \|2^{u_1}\beta\|^2\|2^{u_5}\beta\|\|2^{u_6}\beta\|+\\
&\left.+\frac{1}{16} \sum_{\substack{u_2,u_3,u_5,u_6=0\,(p.d.)\\ \sigma_{u_2+1},\sigma_{u_3+1}=0,\sigma_{u_5+1},\sigma_{u_6+1}=1}}^{n-1} \|2^{u_2}\beta\|\|2^{u_3}\beta\|\|2^{u_5}\beta\|\|2^{u_6}\beta\| \right\}\\
=:&C_1+C_2+C_3+C_4.
\end{aligned}
$$

We find $C_4 = \frac{2^{2n}}{2^{12}} l(l-1)(n-l)(n-l-1)$. Now we put all the results together. Obviously we have

$$
\begin{aligned}
&A_1+E_1 = \frac{n}{160}2^{2n} + \frac{1}{24}\sum_{u=0}^{n-1}2^{2u} + \frac{2^{-2n}}{160}\sum_{u=0}^{n-1}2^{4u} = \frac{n}{160}2^{2n} + \frac{97}{7200}2^{2n} - \frac{1}{72} + \frac{1}{2400}2^{-2n},\\
&A_5-4B_3+6C_4-4D_3+E_5 = \frac{2^{2n}}{2^{12}}\{l(l-1)(l-2)(l-3) - 4l(l-1)(l-2)(n-l)\\
&+6l(l-1)(n-l)(n-l-1) - 4l(n-l)(n-l-1)(n-l-2)\\
&+(n-l)(n-l-1)(n-l-2)(n-l-3)\} = \frac{2^{2n}}{2^{12}}3n(n-2) \ \text{ for } \ l=\frac{n}{2},\\
&A_3+6C_1+E_3 = \frac{3}{4}2^n \sum_{\substack{u_1,u_2=0\\ u_1\neq u_2}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|^2,\\
&A_2-4B_1-4D_1+E_2 = 2^n \sum_{\substack{u_1,u_2=0\\ u_1\neq u_2}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^3\|2^{u_2}\beta\|\\
&-2^{n+1}\left(\sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=0,\sigma_{u_2+1}=1}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^3\|2^{u_2}\beta\| + \sum_{\substack{u_1,u_2=0\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=0}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^3\|2^{u_2}\beta\| \right),\\
&A_4-4B_2+6C_2+6C_3-4D_2+E_4 = \frac{3}{4}2^n \sum_{u_1,u_2,u_3=0}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|\|2^{u_3}\beta\|\\
&-3\cdot 2^n \left(\sum_{\substack{u_1,u_2,u_3=0\,(p.d.)\\ \sigma_{u_1+1}=0,\sigma_{u_2+1}=0,\sigma_{u_3+1}=1}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|\|2^{u_3}\beta\| \right.\\
&\left. + \sum_{\substack{u_1,u_2,u_3=0\,(p.d.)\\ \sigma_{u_1+1}=1,\sigma_{u_2+1}=0,\sigma_{u_3+1}=1}}^{n-1} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^{u_1}\beta\|^2\|2^{u_2}\beta\|\|2^{u_3}\beta\| \right).
\end{aligned}
$$

We compute the sums in the expression above. Therefore we introduce several functions:

$$\begin{aligned}
f(x,y) :=& \frac{1}{1440}(5\cdot 2^{n+1} + 5\cdot 2^{-n+2y+2} - 7\cdot 2^{n+2x-2y} \\
&\qquad - 5\cdot 2^{-n+2x+2} - 3\cdot 2^{-3n+2x+2y+4}),\\
g_1(x,y) :=& \frac{1}{128}(2^n - 2^{n+2x-2y} - 2^{-n+2x+2}),\\
g_2(x) :=& \frac{1}{128}(2^n + 2^{-n+2x+2}),\\
h_1(x,y,z) :=& \frac{1}{384}(2^{n+2x-2z} + 2^{-n+2x+2} + 2^{n+1} - 2^{n+2x-2y}),\\
h_2(x,y) :=& \frac{1}{384}(2^{n+1} - 2^{n+2x-2y} - 2^{-n+2x+2}) \text{ and}\\
h_3(x) :=& \frac{1}{192}(2^n + 2^{-n+2x+1}).
\end{aligned}$$

Then Lemma 3.6 yields

$$\sum_{\substack{u_1,u_2=0\\u_1\neq u_2}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^{u_1}\beta\|^2\|2^{u_2}\beta\|^2 = \sum_{u_2=1}^{n-1}\sum_{u_1=0}^{u_2-1} f(u_1,u_2) + \sum_{u_2=0}^{n-2}\sum_{u_1=u_2+1}^{n-1} f(u_2,u_1).$$

We also get

$$\sum_{\substack{u_1,u_2=0\\u_1\neq u_2}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^{u_1}\beta\|^3\|2^{u_2}\beta\| = \sum_{u_2=1}^{n-1}\sum_{u_1=0}^{u_2-1} g_1(u_1,u_2) + \sum_{u_2=0}^{n-2}\sum_{u_1=u_2+1}^{n-1} g_2(u_1)$$

and

$$\begin{aligned}
&\sum_{u_1,u_2,u_3=0\ (p.d.)}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^{u_1}\beta\|^2\|2^{u_2}\beta\|\|2^{u_3}\beta\|\\
&\quad= \sum_{\substack{u_1,u_2,u_3=0\\u_1<u_2<u_3}}^{n-1} h_1(u_1,u_2,u_3) + \sum_{\substack{u_1,u_2,u_3=0\\u_1<u_3<u_2}}^{n-1} h_1(u_1,u_3,u_2)+\\
&\quad+ \sum_{\substack{u_1,u_2,u_3=0\\u_2<u_1<u_3}}^{n-1} h_2(u_1,u_3) + \sum_{\substack{u_1,u_2,u_3=0\\u_3<u_1<u_2}}^{n-1} h_2(u_1,u_2)+\\
&\quad+ \sum_{\substack{u_1,u_2,u_3=0\\u_2<u_3<u_1}}^{n-1} h_1(u_1) + \sum_{\substack{u_1,u_2,u_3=0\\u_3<u_2<u_1}}^{n-1} h_1(u_1).
\end{aligned}$$

All these sums can be computed straightforward. The remaining sums have to be evaluated individually for the special shifts $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ in a similar way. □

Proof of Theorem 3.1

Proof. We split the integral in the definition of the L_4 discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ into four parts and write

$$\int_0^1\int_0^1 \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2$$

$$
\begin{aligned}
&= \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_0^{1-2^{-n}} \int_{1-2^{-n}}^{1} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\quad + \int_{1-2^{-n}}^{1} \int_0^{1-2^{-n}} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_{1-2^{-n}}^{1} \int_{1-2^{-n}}^{1} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

Now we make use of the second part of Lemma 2.5. Then we can calculate I_2, I_3 and I_4 easily. We start with I_2 and obtain

$$
\begin{aligned}
I_2 &= \int_0^{1-2^{-n}} \int_{1-2^{-n}}^{1} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_0^{1-2^{-n}} \int_{1-2^{-n}}^{1} (\Delta(t_1(n),1) + 2^n(t_1(n) - t_1t_2))^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_0^{1-2^{-n}} \int_{1-2^{-n}}^{1} (2^n(t_1(n) - t_1t_2))^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= 2^{4n} \sum_{a=1}^{2^n-1} \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{1-2^{-n}}^{1} \left(\frac{a}{2^n} - t_1t_2\right)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \frac{1}{1800 \cdot 2^{6n}} \left(1507 \cdot 2^{5n} - 4440 \cdot 2^{4n} + 5060 \cdot 2^{3n} - 2775 \cdot 2^{2n} + 720 \cdot 2^n - 72\right).
\end{aligned}
$$

The integral I_3 can be computed in the same way and has exactly the same value as I_2. We evaluate I_4 and find

$$
\begin{aligned}
I_4 &= \int_{1-2^{-n}}^{1} \int_{1-2^{-n}}^{1} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_{1-2^{-n}}^{1} \int_{1-2^{-n}}^{1} (\Delta(1,1) + 2^n(1 - t_1t_2))^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= 2^{4n} \int_{1-2^{-n}}^{1} \int_{1-2^{-n}}^{1} (1 - t_1t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \frac{1}{300 \cdot 2^{6n}} \left(620 \cdot 2^{4n} - 840 \cdot 2^{3n} + 465 \cdot 2^{2n} - 120 \cdot 2^n + 12\right).
\end{aligned}
$$

We turn to I_1. Using the relation $\Delta(\alpha,\beta) = \Delta(t_1(n), t_2(n)) + 2^n(t_1(n)t_2(n) - t_1t_2)$ again yields

$$
\begin{aligned}
I_1 &= \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1,t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} (\Delta(t_1(n), t_2(n)) + 2^n(t_1(n)t_2(n) - t_1t_2))^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1(n), t_2(n))^4 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\quad + 2^{n+2} \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1(n), t_2(n))^3 \cdot (t_1(n)t_2(n) - t_1t_2) \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\quad + 3 \cdot 2^{2n+1} \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1(n), t_2(n))^2 \cdot (t_1(n)t_2(n) - t_1t_2)^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\quad + 2^{3n+2} \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} \Delta(t_1(n), t_2(n)) \cdot (t_1(n)t_2(n) - t_1t_2)^3 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&\quad + 2^{4n} \int_0^{1-2^{-n}} \int_0^{1-2^{-n}} (t_1(n)t_2(n) - t_1t_2)^4 \,\mathrm{d}t_1\,\mathrm{d}t_2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{a,b=1}^{2^n-1} \Delta\left(\frac{a}{2^n}, \frac{b}{2^n}\right)^4 \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&\quad + 2^{n+2} \sum_{a,b=1}^{2^n-1} \Delta\left(\frac{a}{2^n}, \frac{b}{2^n}\right)^3 \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \left(\frac{ab}{2^{2n}} - t_1 t_2\right) \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&\quad + 3 \cdot 2^{2n+1} \sum_{a,b=1}^{2^n-1} \Delta\left(\frac{a}{2^n}, \frac{b}{2^n}\right)^2 \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \left(\frac{ab}{2^{2n}} - t_1 t_2\right)^2 \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&\quad + 2^{3n+2} \sum_{a,b=1}^{2^n-1} \Delta\left(\frac{a}{2^n}, \frac{b}{2^n}\right) \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \left(\frac{ab}{2^{2n}} - t_1 t_2\right)^3 \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&\quad + 2^{4n} \sum_{a,b=1}^{2^n-1} \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \left(\frac{ab}{2^{2n}} - t_1 t_2\right)^4 \mathrm{d}t_1 \, \mathrm{d}t_2 \\
&=: \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5.
\end{aligned}
$$

The integrals can be computed easily. We get (by writing $\Delta := \Delta\left(\frac{a}{2^n}, \frac{b}{2^n}\right)$)

$$
\begin{aligned}
\Sigma_1 &= \frac{1}{2^{2n}} \sum_{a,b=1}^{2^n-1} \Delta^4, \\
\Sigma_2 &= \frac{1}{2^{3n}} \left\{ 2 \sum_{a,b=1}^{2^n-1} a\Delta^3 + 2 \sum_{a,b=1}^{2^n-1} b\Delta^3 - \sum_{a,b=1}^{2^n-1} \Delta^3 \right\}, \\
\Sigma_3 &= \frac{1}{3} 2^{-4n} \left\{ 6 \sum_{a,b=1}^{2^n-1} a^2\Delta^2 + 6 \sum_{a,b=1}^{2^n-1} b^2\Delta^2 - 6 \sum_{a,b=1}^{2^n-1} a\Delta^2 \right. \\
&\quad \left. - 6 \sum_{a,b=1}^{2^n-1} b\Delta^2 + 9 \sum_{a,b=1}^{2^n-1} ab\Delta^2 + 2 \sum_{a,b=1}^{2^n-1} \Delta^2 \right\} \quad \text{and} \\
\Sigma_4 &= \frac{1}{12} 2^{-5n} \left\{ 12 \sum_{a,b=1}^{2^n-1} a^3\Delta - 18 \sum_{a,b=1}^{2^n-1} a^2\Delta + 24 \sum_{a,b=1}^{2^n-1} a^2 b\Delta + 12 \sum_{a,b=1}^{2^n-1} a\Delta - 32 \sum_{a,b=1}^{2^n-1} ab\Delta \right. \\
&\quad \left. + 24 \sum_{a,b=1}^{2^n-1} ab^2\Delta + 12 \sum_{a,b=1}^{2^n-1} b^3\Delta - 18 \sum_{a,b=1}^{2^n-1} b^2\Delta + 12 \sum_{a,b=1}^{2^n-1} b\Delta - 3 \sum_{a,b=1}^{2^n-1} \Delta \right\}.
\end{aligned}
$$

The value of Σ_5 can be found by a straightforward calculation and is

$$
\Sigma_5 = \frac{(2^n-1)^2}{10800 \cdot 2^{6n}} \left(3014 \cdot 2^{4n} - 9206 \cdot 2^{3n} + 9209 \cdot 2^{2n} - 3456 \cdot 2^n + 432\right).
$$

Via the correspondences $a = 2^n\alpha$ and $b = 2^n\beta$, Corollary 3.4 now comes to full effect. The relations (3.4) and (3.6) deliver $\Sigma_2 = 0$ immediately. In order to simplify Σ_3 we apply (3.3), (3.5) and (3.7) and obtain

$$
\Sigma_3 = 2^{-2n+2} \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \beta^2 \Delta(\alpha,\beta)^2 + \left(\frac{3}{4} - 2^{-n+1} + \frac{1}{3} 2^{-2n+1}\right) 2^{-2n} \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \Delta(\alpha,\beta)^2.
$$

Using (3.3), (3.4), (3.6), (3.8) and Lemma 3.7 allows us to transform the large expression for Σ_4 to

$$
\Sigma_4 = \frac{4}{3} 2^{-2n} (3 \cdot 2^n - 2) \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \alpha\beta\Delta(\alpha,\beta) = -\frac{n}{96} 2^{-n} (3 - 2^{-n+1}).
$$

Putting all results together, Lemma 3.8, Lemma 3.9 and Lemma 3.10 lead to the formulas in Theorem 3.1. □

Conclusions and open problems In Section 1.2 we answered the question on the best known upper bounds on

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{2,N}(\mathcal{P})}{\sqrt{\log N}}$$

and stated several results concerning this expression. Although there are several precise results on the L_2 discrepancy, there are hardly any similar results on the L_p discrepancy for other parameters p. From Theorem 3.1 however we can derive a new upper bound on

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{4,N}(\mathcal{P})}{\sqrt{\log N}}.$$

Previously, the best upper bound was due to (2.8), which yields

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{4,N}(\mathcal{P})}{\sqrt{\log N}} \leq \frac{1}{4\sqrt{\log 2}}(2S(p,p/2))^{\frac{1}{p}} = 0.580844\ldots$$

for $p=4$. This bound has been obtained in [42] by computing the mean of the L_p discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ over all possible digital shifts. However, from our exact discrepancy results on the point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$ we derive the improved constant

$$\liminf_{N\to\infty} \inf_{|\mathcal{P}|=N} \frac{L_{4,N}(\mathcal{P})}{\sqrt{\log N}} \leq \frac{1}{\sqrt{\log 2}}\left(\frac{25}{12288}\right)^{\frac{1}{4}} = 0.255095\ldots,$$

which is the best upper bound on the L_4 discrepancy of point sets in the unit square known so far. It can probably be further improved by considering generalized Hammersley point sets for arbitrary bases b and permutations σ, but the computations would be very technical.

It is natural to ask for corresponding results on the L_p discrepancy at least for even integers p. However, the combinatorial complexity of the involved calculations explodes fast as p increases. Already the L_6 discrepancy is hard to handle. We propose an open question:

Open Problem 3.11. For any even $p \in \mathbb{N}$: Determine the exact value of the constant

$$c(p) := \limsup_{N\to\infty} \frac{L_{p,N}(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))}{\sqrt{\log N}}$$

for a suitable shift $\boldsymbol{\sigma}$, e.g. $\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}_2$. Note that the existence of $c(p)$ is confirmed for these shifts by Theorem 4.2 in Section 4.1.1. Prove or disprove that in order to find $c(p)$ it suffices to consider only the sum

$$\frac{1}{2^{2n}} \sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta(\alpha,\beta)^p,$$

as it was the case for the L_2 and L_4 discrepancy, and provide precise results on this sum. We conjecture that maybe $c(p)=\frac{5^{p/2}}{3\cdot 8^p}$, which would be in accordance to the results for the L_2 and the L_4 discrepancy.

3.1.2. Bounds on the L_1 discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$

An exact computation of the L_1 discrepancy of the digit shifted Hammersley point set is very difficult, since one has to determine for which intervals $[\mathbf{0},\boldsymbol{t})$ the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ is positive and for which it is negative. This fact makes things far more complicated as it was the case for the L_2 or the L_4 discrepancy, where it was not necessary to take care of the sign of the local discrepancy. As a trivial upper bound on the L_1 discrepancy we can take the result for the L_2 discrepancy. The result of Vagharshakyan (1.19) gives us an unsatisfactory general lower bound. Surprisingly, it is easily possible to find an improved lower bound on the L_1 discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ and $\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$ by applying Hölder's inequality (1.10). We set $F=|f|^{2/3}$ and $G=|f|^{4/3}$ for a function $f\in L_1([0,1)^2)$ with $\|f\|_{L_1([0,1)^2)}\neq 0$ and choose $p=3/2$ and $q=3$. Then we have

$$\begin{aligned}\int_{[0,1)^2}|f(\boldsymbol{t})|^2\,\mathrm{d}\boldsymbol{t}&=\int_{[0,1)^2}|F(\boldsymbol{t})G(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}\leq\left(\int_{[0,1)^2}|F(\boldsymbol{t})|^{\frac{3}{2}}\,\mathrm{d}\boldsymbol{t}\right)^{\frac{2}{3}}\left(\int_{[0,1)^2}|G(\boldsymbol{t})|^{3}\,\mathrm{d}\boldsymbol{t}\right)^{\frac{1}{3}}\\&=\left(\int_{[0,1)^2}|f(\boldsymbol{t})|\,\mathrm{d}\boldsymbol{t}\right)^{\frac{2}{3}}\left(\int_{[0,1)^2}|f(\boldsymbol{t})|^4\,\mathrm{d}\boldsymbol{t}\right)^{\frac{1}{3}}.\end{aligned}$$

We have shown $\|f\|^2_{L_2([0,1)^2)}\leq\|f\|^{\frac{2}{3}}_{L_1([0,1)^2)}\|f\|^{\frac{4}{3}}_{L_4([0,1)^2)}$ and hence

$$\|f\|_{L_1([0,1)^2)}\geq\frac{\|f\|^3_{L_2([0,1)^2)}}{\|f\|^2_{L_4([0,1)^2)}}.$$

This relation between the L_1, L_2 and L_4 norms has proven to be useful in the context of uniform distribution and discrepancy theory before, e.g. in [1] or [2]. We set $f=\Delta_N(\cdot,\mathcal{P})$ for a point set $\mathcal{P}$, which yields the following interesting relation between its L_1, L_2 and L_4 discrepancy:

$$L_{1,N}(\mathcal{P})\geq\frac{(L_{2,N}(\mathcal{P}))^3}{(L_{4,N}(\mathcal{P}))^2}.$$

We combine (2.7) and Theorem 3.1 to obtain

$$L_{1,N}(\mathcal{P})\geq\frac{(5/192)^{\frac{3}{2}}}{(25/12288)^{\frac{1}{2}}}\sqrt{n}+o\,(1)=(0.111902\dots)\sqrt{\log N}+o\,(1)$$

for $\mathcal{P}=\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1)$ or $\mathcal{P}=\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2)$, where $o(1)$ denotes an expression of order 1 or of lower order. Altogether, we have

$$0.111902\dots\leq\liminf_{N\to\infty}\frac{L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))}{\sqrt{\log N}}\leq\limsup_{N\to\infty}\frac{L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))}{\sqrt{\log N}}\leq 0.1938\dots$$

for $\boldsymbol{\sigma}=\boldsymbol{\sigma}_1$ or $\boldsymbol{\sigma}=\boldsymbol{\sigma}_2$, where the upper bound stems from the L_2 discrepancy.

Open Problem 3.12. Improve the upper bound on $L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_1))$ or $L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma}_2))$ or find even the exact value of

$$\limsup_{N\to\infty}\frac{L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))}{\sqrt{\log N}}.$$

This result would probably lead to an improved upper bound on the L_1 discrepancy of point sets in the unit square, since the currently best known upper bound is the same as for the L_2 discrepancy according to (1.18).

3.2. Symmetrized Hammersley point sets

3.2.1. An exact formula for the L_2 discrepancy of $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$

Statement of the result In this subsection we consider the symmetrized (digit shifted) Hammersley point sets in base 2. We show an exact formula for the L_2 discrepancy of $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$, which gives not only a concrete constant for the leading term, but also demonstrates that $L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))$ solely depends on the number of elements $N = 2^{n+1}$ of $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$ and not on the shift $\boldsymbol{\sigma}$ whatsoever.

Theorem 3.13. *Let $n \in \mathbb{N}$ and $\boldsymbol{\sigma} \in \{0,1\}^n$. Then we have*

$$(L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})))^2 = \frac{n}{24} + \frac{11}{8} + \frac{1}{2^n} - \frac{1}{9 \cdot 2^{2n+1}},$$

which can be displayed in terms of the number of elements $N = 2^{n+1}$ as

$$L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})) = \left(\frac{\log N}{24 \log 2} + \frac{4}{3} + \frac{2}{N} - \frac{2}{9N^2} \right)^{\frac{1}{2}}.$$

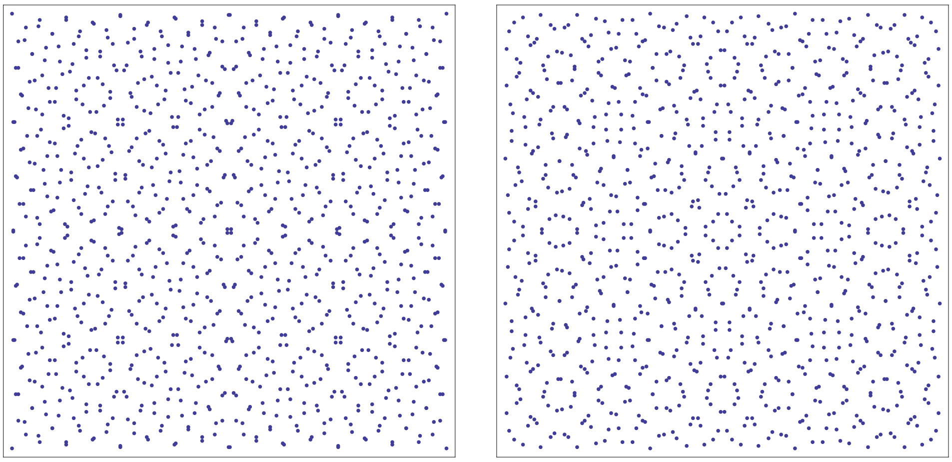


Figure 3.2.: The symmetrized Hammersley point sets $\widetilde{\mathcal{H}}_{2,9}(\boldsymbol{\sigma}_I)$ and $\widetilde{\mathcal{H}}_{2,9}(\boldsymbol{\sigma}_{II})$, where $\boldsymbol{\sigma}_I = (0,0,0,0,0,0,0,0,0)$ and $\boldsymbol{\sigma}_{II} = (0,1,0,1,0,1,0,1,0)$. The L_2 discrepancies of these two point sets have the same value of $1.323613\ldots$

The proof of Theorem 3.13 relies again strongly on the methods in the papers [42, 43, 46, 58]. The fact that we can write the symmetrized Hammersley point set as a union of two shifted Hammersley point sets allows us to employ the same techniques for $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$ by employing Lemma 2.9.

Remark 3.14. Theorem 3.13 shows that the effect of the digital shift $\boldsymbol{\sigma}$ on the L_2 discrepancy of $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$ cancels out completely. We can therefore simply symmetrize the classical Hammersley point set itself. This is a remarkably easy construction of a point set with very low L_2 discrepancy. However, the coefficient of the leading term $\sqrt{\log N}$ of $L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))$ is $\sqrt{1/(24 \log 2)} \approx 0.2451\ldots$, which is slightly higher than for the shifted Hammersley point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ under the condition that the number of ones and zeros in $\boldsymbol{\sigma}$ is more or less balanced. In this case $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ achieves an L_2 discrepancy of optimal order of magnitude in N (see Theorem 2.1). The coefficient of the leading term of $L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ is then $\sqrt{5/(192 \log 2)} \approx 0.1938\ldots$ (see (2.7)).

Auxiliary results In the following, we collect several auxiliary results which will be required in order to prove Theorem 3.13. Like in Section 3.1.1, we must again study sums involving the terms $\|2^u\beta\|$ as well as $\alpha_{n-u}\oplus\alpha_{n+1-j(u)}$, which stem from Theorem 2.5. We write $\Delta_1(\alpha,\beta)$ for the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ and $\Delta_2(\alpha,\beta)$ for the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}^*)$.
Throughout the next lemma, we always write $j_1(u)$ if the function $j(u)$ appearing in the first part of Theorem 2.5 refers to $\Delta_1(\alpha,\beta)$ and $j_2(u)$ if it refers to $\Delta_2(\alpha,\beta)$.

Lemma 3.15. *Let $\alpha = \frac{\alpha_1}{2}+\cdots+\frac{\alpha_n}{2^n}$ and $\beta = \frac{\beta_1}{2}+\cdots+\frac{\beta_n}{2^n}$ be n-bit.*

1. *For $u_1,u_2\in\{0,\dots,n-1\}$ with $u_1\neq u_2$ we have*

$$\sum_{\alpha\in\mathbb{Q}(2^n)}(\alpha_{n-u_1}\oplus\alpha_{n+1-j_1(u_1)})(\alpha_{n-u_2}\oplus\alpha_{n+1-j_2(u_2)})=2^{n-2}.$$

2. *For $u\in\{0,\dots,n-1\}$ we have*

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Q}(2^n)}(\alpha_{n-u}\oplus\alpha_{n+1-j_1(u)})(\alpha_{n-u}\oplus\alpha_{n+1-j_2(u)})\\&=\begin{cases}2^{n-u-1} & \text{if } u\in\{0,1\},\\ 2^{n-u-1}\left(1+\sum_{j=1}^{u-1}2^j((\gamma_j\oplus 1)\gamma_u+\gamma_j(\gamma_u\oplus 1))\right) & \text{if } u\in\{2,\dots,n-1\}.\end{cases}\end{aligned}$$

In the last expression, we define $\gamma_j := \beta_j\oplus\sigma_j$ for all $j\in\{1,\dots,n-1\}$.

Proof. We show the first assertion. W.l.o.g. we may assume that $u_1<u_2$. Since $j_1(u_1)$ does only depend on the digits $\alpha_{n-u_1+1},\dots,\alpha_n$ and $j_2(u_2)$ only on the digits $\alpha_{n-u_2+1},\dots,\alpha_n$, we have

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Q}(2^n)}(\alpha_{n-u_1}\oplus\alpha_{n+1-j_1(u_1)})(\alpha_{n-u_2}\oplus\alpha_{n+1-j_2(u_2)})\\&=\sum_{\alpha_1,\dots,\alpha_{n-u_2-1}=0}^{1}\ \sum_{\alpha_{n-u_2+1},\dots,\alpha_n=0}^{1}\alpha_{n-u_1}\oplus\alpha_{n+1-j_1(u_1)}\\&=\sum_{\alpha_1,\dots,\alpha_{n-u_2-1}=0}^{1}\ \sum_{\alpha_{n-u_2+1},\dots,\alpha_{n-u_1-1}=0}^{1}\ \sum_{\alpha_{n-u_1+1},\dots,\alpha_n=0}^{1}1=2^{n-2}.\end{aligned}$$

We show the second claim. For $u=0$ we have $j_1(u)=0$ and $j_2(u)=0$ by definition and hence

$$\sum_{\alpha\in\mathbb{Q}(2^n)}(\alpha_n\oplus\alpha_{n+1})(\alpha_n\oplus\alpha_{n+1})=\sum_{\alpha_1,\dots,\alpha_n=0}^{1}\alpha_n=\sum_{\alpha_1,\dots,\alpha_{n-1}=0}^{1}1=2^{n-1}=2^{n-u-1}.$$

If $u=1$, we use the fact that $j_1(1)$ and $j_2(1)$ only depend on α_n and write

$$\begin{aligned}&\sum_{\alpha\in\mathbb{Q}(2^n)}(\alpha_{n-1}\oplus\alpha_{n+1-j_1(1)})(\alpha_{n-1}\oplus\alpha_{n+1-j_2(1)})\\&=\sum_{\alpha_n=0}^{1}\left(\sum_{\alpha_1,\dots,\alpha_{n-1}=0}^{1}(\alpha_{n-1}\oplus\alpha_{n+1-j_1(1)})(\alpha_{n-1}\oplus\alpha_{n+1-j_2(1)})\right)\\&=2^{n-2}\sum_{\alpha_n=0}^{1}\left(\alpha_{n+1-j_1(1)}\alpha_{n+1-j_2(1)}+(\alpha_{n+1-j_1(1)}\oplus 1)(\alpha_{n+1-j_2(1)}\oplus 1)\right).\end{aligned}$$

We have to distinguish between the cases $\alpha_n = \gamma_1$ and $\alpha_n = \gamma_1 \oplus 1$. In the first case we obviously have $j_1(1) = 0$ and $j_2(1) = 1$ whereas in the second case we have $j_1(1) = 1$ and $j_2(1) = 0$. We conclude

$$\begin{aligned}
&2^{n-2} \sum_{\alpha_n=0}^{1} \Big(\alpha_{n+1-j_1(1)}\alpha_{n+1-j_2(1)} + (\alpha_{n+1-j_1(1)} \oplus 1)(\alpha_{n+1-j_2(1)} \oplus 1)\Big) \\
&\quad= 2^{n-2} \sum_{\alpha_n=\gamma_1} (\alpha_n \oplus 1) + 2^{n-2} \sum_{\alpha_n=\gamma_1\oplus 1} (\alpha_n \oplus 1) \\
&\quad= 2^{n-2}(\gamma_1 \oplus 1) + 2^{n-2}\gamma_1 = 2^{n-2} = 2^{n-u-1}.
\end{aligned}$$

We turn to the case $u \in \{2, \dots, n-1\}$. Since $j_1(u)$ and $j_2(u)$ only depend on the digits $\alpha_{n+1-u}, \dots, \alpha_n$ but not on $\alpha_1, \dots, \alpha_{n-u}$, we observe that

$$\begin{aligned}
&\sum_{\alpha \in \mathbb{Q}(2^n)} (\alpha_{n-u} \oplus \alpha_{n+1-j_1(u)})(\alpha_{n-u} \oplus \alpha_{n+1-j_2(u)}) \\
&= \sum_{\alpha_{n+1-u},\dots,\alpha_n=0}^{1} \left(\sum_{\alpha_1,\dots,\alpha_{n-u}=0}^{1} (\alpha_{n-u} \oplus \alpha_{n+1-j_1(u)})(\alpha_{n-u} \oplus \alpha_{n+1-j_2(u)}) \right) \\
&= 2^{n-u-1} \sum_{\alpha_{n+1-u},\dots,\alpha_n=0}^{1} \Big(\alpha_{n+1-j_1(u)}\alpha_{n+1-j_2(u)} + (\alpha_{n+1-j_1(u)} \oplus 1)(\alpha_{n+1-j_2(u)} \oplus 1)\Big) \\
&= 2^{n-u-1} \sum_{j_1=0}^{u-1} \sum_{\substack{\alpha_{n+1-u},\dots,\alpha_n=0 \\ j_1(u)=j_1}}^{1} \Big(\alpha_{n+1-j_1}\alpha_{n+1-j_2(u)} + (\alpha_{n+1-j_1} \oplus 1)(\alpha_{n+1-j_2(u)} \oplus 1)\Big) \\
&\quad + 2^{n-u-1} \sum_{j_2=0}^{u-1} \sum_{\substack{\alpha_{n+1-u},\dots,\alpha_n=0 \\ j_2(u)=j_2}}^{1} \Big(\alpha_{n+1-j_1(u)}\alpha_{n+1-j_2} + (\alpha_{n+1-j_1(u)} \oplus 1)(\alpha_{n+1-j_2} \oplus 1)\Big) \\
&=: T_1 + T_2.
\end{aligned}$$

One might wonder why the sums over j_1 and j_2 end in $u-1$ instead of u and why they do not coincide. The reason is that $j_1(u) \in \{0, \dots, u-1\}$ implies $j_2(u) = u$ and $j_2(u) \in \{0, \dots, u-1\}$ implies $j_1(u) = u$. This can be seen as follows: $j_1(u) \in \{0, \dots, u-1\}$ implies $a_{n+1-u} = \gamma_u$, because otherwise we would have $j_1(u) = u$. But from the fact that $a_{n+1-u} = \gamma_u \neq \gamma_u \oplus 1$, we immediately derive $j_2(u) = u$. The other way round can be explained analogously. This means that the case $j_2(u) = u$ is actually contained in the sum over j_1 and reversely. We find

$$\begin{aligned}
T_1 &= 2^{n-u-1} \sum_{\substack{\alpha_{n+1-u}=\gamma_u \\ \vdots \\ \alpha_{n-1}=\gamma_2 \\ \alpha_n=\gamma_1}} \Big(\alpha_{n+1}\alpha_{n+1-j_2(u)} + (\alpha_{n+1} \oplus 1)(\alpha_{n+1-j_2(u)} \oplus 1)\Big) \\
&\quad + 2^{n-u-1} \sum_{j_1=1}^{u-1} \sum_{\alpha_{n+2-j_1},\dots,\alpha_n=0}^{1} \\
&\qquad \sum_{\substack{\alpha_{n+1-u}=\gamma_u \\ \vdots \\ \alpha_{n-j_1}=\gamma_{j_1+1} \\ \alpha_{n+1-j_1}=\gamma_{j_1}\oplus 1}} \Big(\alpha_{n+1-j_1}\alpha_{n+1-j_2(u)} + (\alpha_{n+1-j_1} \oplus 1)(\alpha_{n+1-j_2(u)} \oplus 1)\Big) \\
&= 2^{n-u-1} (\gamma_u \oplus 1) + 2^{n-u-1} \sum_{j_1=1}^{u-1} 2^{j_1-1} \left((\gamma_{j_1} \oplus 1)\gamma_u + \gamma_{j_1}(\gamma_u \oplus 1)\right).
\end{aligned}$$

Similarly we argue that

$$T_2 = 2^{n-u-1}\gamma_u + 2^{n-u-1}\sum_{j_2=1}^{u-1} 2^{j_2-1}\left((\gamma_{j_2}\oplus 1)\gamma_u + \gamma_{j_2}(\gamma_u \oplus 1)\right).$$

Adding T_1 and T_2 completes the proof of the second item of this lemma. □

The next lemma involves again the parameter $l = l(\boldsymbol{\sigma}) := |\{i \in \{1,\dots,n\} : \sigma_i = 0\}|$.

Lemma 3.16. *We have*

$$\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta_1(\alpha,\beta)\Delta_2(\alpha,\beta) = -\frac{n^2}{64} - \frac{l^2}{16} + \frac{ln}{16} - \frac{n}{192} - \frac{5}{144} - \frac{1}{9\cdot 2^{2n+2}}.$$

Proof. In this proof we write for the sake of simplicity $A(\alpha,\beta,\boldsymbol{\sigma},u) := \alpha_{n-u}\oplus\alpha_{n+1-j(u)}$, where we emphasize the dependence of $j(u)$ on α, β and $\boldsymbol{\sigma}$. With the first part of Theorem 2.5 we get

$$\begin{aligned}
&\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)} \Delta_1(\alpha,\beta)\Delta_2(\alpha,\beta)\\
=&\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{u_1=0}^{n-1}\|2^{u_1}\beta\|(-1)^{\sigma_{u_1+1}}A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\\
&\qquad\times\left(\sum_{u_2=0}^{n-1}\|2^{u_2}\beta\|(-1)^{\sigma^*_{u_2+1}}A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
=&-\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{u_1=0}^{n-1}\|2^{u_1}\beta\|(-1)^{\sigma_{u_1+1}}A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\\
&\qquad\times\left(\sum_{u_2=0}^{n-1}\|2^{u_2}\beta\|(-1)^{\sigma_{u_2+1}}A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
=&-\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1}\|2^{u_1}\beta\|A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=0}}^{n-1}\|2^{u_2}\beta\|A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
&+\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1}\|2^{u_1}\beta\|A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1}\|2^{u_2}\beta\|A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
&+\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=1}}^{n-1}\|2^{u_1}\beta\|A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=0}}^{n-1}\|2^{u_2}\beta\|A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
&-\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\left(\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=1}}^{n-1}\|2^{u_1}\beta\|A(\alpha,\beta,\boldsymbol{\sigma},u_1)\right)\left(\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1}\|2^{u_2}\beta\|A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)\right)\\
=:&-R_1+R_2+R_3-R_4.
\end{aligned}$$

With the first part of Lemma 3.15 and the first formula in Lemma 3.6, respectively, we obtain

$$R_2 = \frac{1}{2^{2n}}\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1}\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^{u_1}\beta\|\|2^{u_2}\beta\|\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u_1)A(\alpha,\beta,\boldsymbol{\sigma}^*,u_2)$$

$$=\frac{1}{2^{2n}}\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1}\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=1}}^{n-1}\frac{2^n}{2^4}2^{n-2}=\frac{1}{64}l(n-l).$$

In the same way we show $R_3=\frac{1}{64}l(n-l)$. To calculate R_1 and R_4, we need to distinguish between the cases where $u_1=u_2$ and where $u_1\neq u_2$. This leads to

$$\begin{aligned}R_1=&\frac{1}{2^{2n}}\underbrace{\sum_{\substack{u_1=0\\ \sigma_{u_1+1}=0}}^{n-1}\sum_{\substack{u_2=0\\ \sigma_{u_2+1}=0}}^{n-1}}_{u_1\neq u_2}\frac{2^n}{2^4}2^{n-2}\\ &+\frac{1}{2^{2n}}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u)A(\alpha,\beta,\boldsymbol{\sigma}^*,u)\\ =&\frac{1}{64}l(l-1)+\frac{1}{2^{2n}}\sum_{\substack{u=0\\ \sigma_{u+1}=0}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u)A(\alpha,\beta,\boldsymbol{\sigma}^*,u).\end{aligned}$$

Similarly, we obtain

$$R_4=\frac{1}{64}(n-l)(n-l-1)+\frac{1}{2^{2n}}\sum_{\substack{u=0\\ \sigma_{u+1}=1}}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u)A(\alpha,\beta,\boldsymbol{\sigma}^*,u).$$

Our results for R_1, R_2, R_3 and R_4 yield

$$\begin{aligned}&\frac{1}{2^{2n}}\sum_{\alpha,\beta\in\mathbb{Q}(2^n)}\Delta_1(\alpha,\beta)\Delta_2(\alpha,\beta)\\ &=-\frac{1}{64}(n^2+4l^2-4ln-n)\\ &\quad-\frac{1}{2^{2n}}\sum_{u=0}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u)A(\alpha,\beta,\boldsymbol{\sigma}^*,u).\end{aligned}$$

Hence, our final task is to compute the last expression in the above line. We employ the second part of Lemma 3.15 and the third formula in Lemma 3.6, respectively, to obtain

$$\begin{aligned}&\frac{1}{2^{2n}}\sum_{u=0}^{n-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{\alpha\in\mathbb{Q}(2^n)}A(\alpha,\beta,\boldsymbol{\sigma},u)A(\alpha,\beta,\boldsymbol{\sigma}^*,u)\\ =&\frac{1}{2^{2n}}\sum_{u=0}^{n-1}2^{n-u-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\\ &+\frac{1}{2^{2n}}\sum_{u=2}^{n-1}2^{n-u-1}\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2\sum_{j=1}^{u-1}2^j((\gamma_j\oplus 1)\gamma_u+\gamma_j(\gamma_u\oplus 1))\\ =&\frac{1}{2^{n+1}}\sum_{u=0}^{n-1}2^{-u}\frac{2^{2n}+2^{2u+1}}{3\cdot 2^{n+2}}\\ &+\frac{1}{2^{n+1}}\sum_{u=2}^{n-1}2^{-u}\sum_{j=1}^{u-1}2^j\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2(\beta_j\oplus\sigma_j\oplus 1)(\beta_u\oplus\sigma_u)\\ &+\frac{1}{2^{n+1}}\sum_{u=2}^{n-1}2^{-u}\sum_{j=1}^{u-1}2^j\sum_{\beta\in\mathbb{Q}(2^n)}\|2^u\beta\|^2(\beta_j\oplus\sigma_j)(\beta_u\oplus\sigma_u\oplus 1)=:\Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Finding the value of Σ_1 is a matter of straightforward calculation. We have

$$\Sigma_1 = \frac{1}{12}\left(1 - \frac{1}{2^{2n}}\right).$$

For Σ_2 we find

$$\Sigma_2 = \frac{1}{2^{n+1}} \sum_{u=2}^{n-1} 2^{-u} \sum_{j=1}^{u-1} 2^j \sum_{\substack{\beta_1,\dots,\beta_{j-1},\beta_{j+1},\dots,\beta_{u-1}=0 \\ \beta_j=\sigma_j \\ \beta_u=\sigma_u\oplus 1}}^{1} \sum_{\beta_{u+1},\dots,\beta_n=0}^{1} \|2^u\beta\|^2.$$

We remark at this point that $\|2^u\beta\|^2$ depends only on $\beta_{u+1},\dots,\beta_n$. Hence,

$$\sum_{\beta_{u+1},\dots,\beta_n=0}^{1} \|2^u\beta\|^2 = 2^{-u} \sum_{\beta_1,\dots,\beta_n=0}^{1} \|2^u\beta\|^2 = 2^{-u} \sum_{\beta\in\mathbb{Q}(2^n)} \|2^u\beta\|^2 = 2^{-u}\frac{2^{2n}+2^{2u+1}}{3\cdot 2^{n+2}}.$$

We arrive at

$$\begin{aligned}
\Sigma_2 =& \frac{1}{2^{n+1}} \sum_{u=2}^{n-1} 2^{-u} \sum_{j=1}^{u-1} 2^j 2^{u-2} 2^{-u} \frac{2^{2n}+2^{2u+1}}{3\cdot 2^{n+2}} \\
=& \frac{1}{2^{n+1}} \sum_{u=2}^{n-1} 2^{-u}(2^u-2)2^{u-2}2^{-u} \frac{2^{2n}+2^{2u+1}}{3\cdot 2^{n+2}} \\
=& \frac{n}{96} - \frac{7}{288} + \frac{1}{9\cdot 2^{2n+1}}.
\end{aligned}$$

It is clear that $\Sigma_3 = \Sigma_2$. Thus, after adding all the results the proof of the lemma is finally complete. □

Proof of Theorem 3.13 We apply Lemma 2.9 to write

$$\begin{aligned}
(L_2(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})))^2 =& \int_0^1\int_0^1 (\Delta(t_1,t_2,\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
=& \int_0^1\int_0^1 (\Delta_1(t_1,t_2))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_0^1\int_0^1 (\Delta_2(t_1,t_2))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\
& + 2\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
=& (L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 + (L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma}^*)))^2 \\
& + 2\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2. \qquad (3.11)
\end{aligned}$$

We know the values of $(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2$ and $(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma}^*)))^2$ already from Theorem 2.1 (where in the latter case we have to insert $n-l$ instead of l in this formula). This yields

$$(L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma})))^2 + (L_2(\mathcal{H}_{2,n}(\boldsymbol{\sigma}^*)))^2 = \frac{n^2}{32} + \frac{l^2}{8} - \frac{ln}{8} + \frac{5n}{96} + \frac{3}{4} + \frac{1}{2^{n+1}} - \frac{1}{9\cdot 2^{2n+2}}.$$

We split the integrals in (3.11) in four parts:

$$\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 = \int_0^{1-2^{-n}}\int_0^{1-2^{-n}} \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+\int_0^{1-2^{-n}}\int_{1-2^{-n}}^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+\int_{1-2^{-n}}^1\int_0^{1-2^{-n}} \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+\int_{1-2^{-n}}^1\int_{1-2^{-n}}^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$=: I_1+I_2+I_3+I_4.$$

We can calculate I_2, I_3 and I_4 with aid of the second part of Lemma 2.5. Since this proceeds analogously as in the proof of [42, Theorem 1], we only give the results here. We have

$$I_2 = I_3 = \frac{25}{36\cdot 2^n} - \frac{5}{9\cdot 4^n} - \frac{25}{36\cdot 4^n} + \frac{2}{3\cdot 8^n} - \frac{1}{9\cdot 16^n}$$

and

$$I_4 = \frac{7}{6\cdot 4^n} + \frac{1}{9\cdot 16^n} - \frac{2}{3\cdot 8^n}.$$

It remains to evaluate I_1. We use the second part of Lemma 2.5 to obtain

$$I_1 = \int_0^{1-2^{-n}}\int_0^{1-2^{-n}} (\Delta_1(t_1(n),t_2(n)) + 2^n(t_1(n)t_2(n)-t_1t_2))$$
$$\times (\Delta_2(t_1(n),t_2(n)) + 2^n(t_1(n)t_2(n)-t_1t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$= \int_0^{1-2^{-n}}\int_0^{1-2^{-n}} \Delta_1(t_1(n),t_2(n))\Delta_2(t_1(n),t_2(n))\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+2^n\int_0^{1-2^{-n}}\int_0^{1-2^{-n}} \Delta_1(t_1(n),t_2(n))(t_1(n)t_2(n)-t_1t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+2^n\int_0^{1-2^{-n}}\int_0^{1-2^{-n}} \Delta_2(t_1(n),t_2(n))(t_1(n)t_2(n)-t_1t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$
$$+2^{2n}\int_0^{1-2^{-n}}\int_0^{1-2^{-n}} (t_1(n)t_2(n)-t_1t_2)^2\,\mathrm{d}t_1\,\mathrm{d}t_2 = S_1+S_2+S_3+S_4.$$

The value of S_4 can be calculated in a straightforward way and is

$$S_4 = \frac{1}{72\cdot 16^n}(2^n-1)^2(25\cdot 4^n - 32\cdot 2^n + 8).$$

The expression S_2 was computed in the proof of [43, Theorem 1] and is given by

$$S_2 = 2^{n-1}\frac{2^{n+1}-1}{4^n}\left(\frac{l(\boldsymbol{\sigma})}{8} - \frac{n}{16}\right).$$

Analogously, we have

$$S_3 = 2^{n-1}\frac{2^{n+1}-1}{4^n}\left(\frac{l(\boldsymbol{\sigma}^*)}{8} - \frac{n}{16}\right),$$

where $l(\boldsymbol{\sigma}^*)$ is the number of components in $\boldsymbol{\sigma}^*$ which are equal to zero. Since we obviously have $l(\boldsymbol{\sigma}^*) = n - l(\boldsymbol{\sigma})$, we find $S_2 + S_3 = 0$. So far we have

$$I_1 = S_1 + \frac{1}{72\cdot 16^n}(2^n-1)^2(25\cdot 4^n - 32\cdot 2^n + 8).$$

But since

$$S_1 = \sum_{a,b=1}^{2^n-1} \int_{\frac{a-1}{2^n}}^{\frac{a}{2^n}} \int_{\frac{b-1}{2^n}}^{\frac{b}{2^n}} \Delta_1\left(\frac{a}{2^n}, \frac{b}{2^n}\right) \Delta_2\left(\frac{a}{2^n}, \frac{b}{2^n}\right) \mathrm{d}t_1 \, \mathrm{d}t_2$$
$$= \frac{1}{2^{2n}} \sum_{\alpha,\beta \in \mathbb{Q}(2^n)} \Delta_1(\alpha,\beta)\Delta_2(\alpha,\beta),$$

we also know the value of S_1 from Lemma 3.16. Putting all results together, we obtain the claimed formula in Theorem 3.13. □

3.2.2. An exact formula for the L_2 discrepancy of $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$

Statement of the result The following theorem generalizes Theorem 3.13 to arbitrary bases $b \geq 2$ and permutations $\sigma \in \mathcal{A}_b(\tau)$. Recall the notation from the lines before Theorem 2.2 and from Definition 2.6. In particular, $\mathcal{A}_b(\tau)$ is the set of all permutations in $\mathfrak{S}_b$ which commutate with τ_b.

Theorem 3.17. *Let $\sigma \in \mathcal{A}_b(\tau)$, $n \in \mathbb{N}$ and $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. Then we have*

$$\left(L_2(\widetilde{\mathcal{H}}_{b,n}^{\Sigma})\right)^2 = nc_b^{\sigma} + \frac{11}{8} + \frac{1}{b^n} + \frac{1 - 9 \cdot (-1)^b}{144b^{2n}},$$

where $c_b^{\sigma} := 2\Phi_b^{\sigma,(2)} + \widetilde{\Phi}_b^{\sigma} + \frac{1}{2}\widetilde{\Phi}_{b,1}^{\sigma} + \frac{1}{2}\widetilde{\Phi}_{b,2}^{\sigma}$.

Remark 3.18. Theorem 3.17 demonstrates that $L_2(\widetilde{\mathcal{H}}_{b,n}^{\Sigma})$ does not depend on the positions of σ and $\overline{\sigma}$ in the tuple $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ at all, but only on the base b, on the permutation $\sigma \in \mathcal{A}_b(\tau)$ we choose and on the number of elements $N = 2b^n$. Hence, for a fixed $\sigma \in \mathcal{A}_b(\tau)$ one should always choose $\Sigma = (\sigma, \sigma, \dots, \sigma)$ and $\Sigma^* = (\overline{\sigma}, \overline{\sigma}, \dots, \overline{\sigma})$ if one is only interested in a low L_2 discrepancy of $\widetilde{\mathcal{H}}_{b,n}^{\Sigma}$.

We would like to derive results for the simplest case $\sigma = id$. To this end, we need to compute c_b^{id}. This is easily possible with aid of Proposition A.1 in the Appendix, which yields $c_b^{id} = \frac{b^2}{360} + \frac{1}{24} - \frac{2}{45b^2}$. This leads to the following corollary.

Corollary 3.19. Let $\Sigma \in \{id, \tau_b\}^n$ for some $n \in \mathbb{N}$. Then we have

$$\left(L_2(\widetilde{\mathcal{H}}_{b,n}^{\Sigma})\right)^2 = n\left(\frac{b^2}{360} + \frac{1}{24} - \frac{2}{45b^2}\right) + \frac{11}{8} + \frac{1}{b^n} + \frac{1 - 9 \cdot (-1)^b}{144b^{2n}}.$$

Remark 3.20. We remark that for $b = 2$ the formula given in Corollary 3.19 recovers Theorem 3.13. From [30, Corollary 4] we have

$$\min_{\Sigma \in \{id, \tau_b\}^n} \left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = n\left(\frac{b^2}{240} + \frac{1}{72} - \frac{13}{720b^2}\right) + \mathcal{O}(1).$$

This means that in the case $\Sigma \in \{id, \tau_b\}^n$, symmetrizing yields asymptotically a lower L_2 discrepancy than mere digit scrambling for $b \geq 5$.

Our proof is based on techniques developed and employed in several papers such as [11, 26, 27, 29, 30, 31]. The formalism we use to verify Theorem 3.17 is rather complicated and leads to several technical proofs. We therefore would like to proceed in the following way: In the subsequent paragraph, we present the high level structure of the proof, where we try to avoid as many technicalities as possible. This subsection gives the reader the basic idea of the proof. The details of the proof can be find afterwards in an extra paragraph.

The basic steps of the proof Throughout this section, we write $\Delta_1(t_1,t_2)$ for the discrepancy function of $\mathcal{H}_{b,n}^{\Sigma}$ and $\Delta_2(t_1,t_2)$ for the discrepancy function of $\mathcal{H}_{b,n}^{\Sigma^*}$. The first steps of the proof are similar to those of Theorem 3.13. With the definition of the L_2 discrepancy and Lemma 2.9 we obtain

$$\begin{aligned}(L_2(\widetilde{\mathcal{H}}_{b,n}^{\Sigma}))^2 =& \int_0^1\int_0^1 (\Delta(t_1,t_2,\widetilde{\mathcal{H}}_{b,n}^{\Sigma}))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\ =& \int_0^1\int_0^1 (\Delta_1(t_1,t_2))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 + \int_0^1\int_0^1 (\Delta_2(t_1,t_2))^2 \,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &+ 2\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ =&(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 + (L_2(\mathcal{H}_{b,n}^{\Sigma^*}))^2 + 2\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2. \quad (3.12)\end{aligned}$$

At this point, we make use of Theorem 2.2, which yields

$$\begin{aligned}&(L_2(\mathcal{H}_{b,n}^{\Sigma}))^2 + (L_2(\mathcal{H}_{b,n}^{\Sigma^*}))^2 \\ &\qquad =\left(\Phi_b^{\sigma}\right)^2(2n^2+8l^2-2n-8ln) + 2n\Phi_b^{\sigma,(2)} + \frac{3}{4} + \frac{1}{2b^n} - \frac{1}{36b^{2n}}.\end{aligned}$$

Here we regarded the obvious fact that Σ^* contains $n-l$ entries equal to id whenever Σ contains l of such entries. We examine the last integral in (3.12) and therefore regard Remark 2.8 to write

$$\begin{aligned}&\int_0^1\int_0^1 \Delta_1(t_1,t_2)\Delta_2(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \int_0^1\int_0^1 (\Delta_1(t_1(n),t_2(n)) + b^n(t_1(n)t_2(n)-t_1t_2)) \\ &\quad \times (\Delta_2(t_1(n),t_2(n)) + b^n(t_1(n)t_2(n)-t_1t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \int_0^1\int_0^1 \Delta_1(t_1(n),t_2(n))\Delta_2(t_1(n),t_2(n))\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &\quad + b^n\int_0^1\int_0^1 \Delta_1(t_1(n),t_2(n))(t_1(n)t_2(n)-t_1t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &\quad + b^n\int_0^1\int_0^1 \Delta_2(t_1(n),t_2(n))(t_1(n)t_2(n)-t_1t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &\quad + b^{2n}\int_0^1\int_0^1 (t_1(n)t_2(n)-t_1t_2)^2\,\mathrm{d}t_1\,\mathrm{d}t_2 =: \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4.\end{aligned}$$

From the proof of [31, Theorem 2] we already know that

$$\Sigma_4 = \frac{25}{72} + \frac{1}{4b^n} + \frac{1}{72b^{2n}}$$

and

$$\Sigma_2 = (2l-n)\left(\frac{1}{2}-\frac{1}{4b^n}\right)\Phi_b^{\sigma}.$$

By replacing l by $n-l$ in the result for Σ_2 we obtain

$$\Sigma_3 = (2(n-l)-n)\left(\frac{1}{2}-\frac{1}{4b^n}\right)\Phi_b^{\sigma}$$

and therefore $\Sigma_2+\Sigma_3=0$. It remains to evaluate Σ_1. In the following, we do nothing else but inserting Theorem 2.7 for $\Delta_1(\lambda/b^n, M/b^n)$ and $\Delta_2(\lambda/b^n, M/b^n)$, and then separating those indices $i \in \{1, \ldots, n\}$ where $\sigma_i = \sigma$ from those where $\sigma_i = \overline{\sigma}$. We have

$$\begin{aligned}
\Sigma_1 =& \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \Delta_1\left(\frac{\lambda}{b^n}, \frac{M}{b^n}\right) \Delta_2\left(\frac{\lambda}{b^n}, \frac{M}{b^n}\right) \\
=& \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{i=1}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\sigma_i}\left(\frac{M}{b^i}\right)\right)\left(\sum_{j=1}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\sigma_j^*}\left(\frac{M}{b^j}\right)\right) \\
=& \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{\substack{i=1\\ \sigma_i=\sigma}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\sigma}\left(\frac{M}{b^i}\right) + \sum_{\substack{i=1\\ \sigma_i=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\overline{\sigma}}\left(\frac{M}{b^i}\right)\right) \\
&\times \left(\sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\overline{\sigma}}\left(\frac{M}{b^j}\right) + \sum_{\substack{j=1\\ \sigma_j=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\sigma}\left(\frac{M}{b^j}\right)\right) \\
=& \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{\substack{i=1\\ \sigma_i=\sigma}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\sigma}\left(\frac{M}{b^i}\right)\right)\left(\sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\overline{\sigma}}\left(\frac{M}{b^j}\right)\right) \\
&+ \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{\substack{i=1\\ \sigma_i=\sigma}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\sigma}\left(\frac{M}{b^i}\right)\right)\left(\sum_{\substack{j=1\\ \sigma_j=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\sigma}\left(\frac{M}{b^j}\right)\right) \\
&+ \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{\substack{i=1\\ \sigma_i=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\overline{\sigma}}\left(\frac{M}{b^i}\right)\right)\left(\sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\overline{\sigma}}\left(\frac{M}{b^j}\right)\right) \\
&+ \frac{1}{b^{2n}} \sum_{\lambda,M=1}^{b^n} \left(\sum_{\substack{i=1\\ \sigma_i=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma)}^{\overline{\sigma}}\left(\frac{M}{b^i}\right)\right)\left(\sum_{\substack{j=1\\ \sigma_j=\overline{\sigma}}}^{n} \psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\sigma}\left(\frac{M}{b^j}\right)\right) \\
=&: S_1 + S_2 + S_3 + S_4.
\end{aligned}$$

Now, for the first time in this proof, we have to deal with the functions $\psi_{b,h}^{\sigma}$ which appear in Theorem 2.7. First, we only need results that have already been proven in previous papers. The proofs of the following auxiliary results can be found in [30, Lemma 2], [31, Lemma 3] and [31, Lemma 4], respectively.

Lemma 3.21. *Let $1 \leq N \leq b^n$, $1 \leq i < j \leq n$ and $\sigma \in \mathfrak{S}_b$. Then we have for $\sigma_i, \sigma_j \in \{\sigma, \overline{\sigma}\}$*

$$\sum_{\lambda=1}^{b^n} \psi_{b,\varepsilon_i(\lambda,M,\Sigma_1)}^{\sigma_i}\left(\frac{M}{b^i}\right) \psi_{b,\varepsilon_j(\lambda,M,\Sigma_2)}^{\sigma_j}\left(\frac{M}{b^j}\right) = b^{n-2}\psi_b^{\sigma_i}\left(\frac{M}{b^i}\right)\psi_b^{\sigma_j}\left(\frac{M}{b^j}\right),$$

where $\Sigma_1, \Sigma_2 \in \{\sigma, \overline{\sigma}\}^n$ may be different. Then, for $1 \leq i < j \leq n$ and an arbitrary permutation $\sigma \in \mathfrak{S}_b$, we have

$$\sum_{N=1}^{b^n} \psi_b^{\sigma}\left(\frac{M}{b^i}\right)\psi_b^{\sigma}\left(\frac{M}{b^j}\right) = b^{n+2}\left(\Phi_b^{\sigma}\right)^2.$$

Finally, for $\sigma \in \mathfrak{S}_b$, $\overline{\sigma} = \tau_b \circ \sigma$ and any $h \in \{0, \ldots, b-1\}$, we also have the relations

$$\psi_{b,h}^{\overline{\sigma}} = -\psi_{b,b-h}^{\sigma}$$

and, as a result from that, $\psi_b^{\sigma} = -\psi_b^{\overline{\sigma}}$ *and* $\psi_b^{\sigma,(2)} = \psi_b^{\overline{\sigma},(2)}$.

We change the summation order and use the statements of Lemma 3.21 to compute

$$S_2 = \frac{1}{b^{2n}} \sum_{\substack{i,j=1\\ \sigma_i=\sigma\\ \sigma_j=\overline{\sigma}}}^{n} \sum_{N=1}^{b^n} b^{n-2} \psi_b^{\sigma}\left(\frac{M}{b^i}\right) \psi_b^{\sigma}\left(\frac{M}{b^j}\right) = \frac{1}{b^{2n}} b^{n-2} \sum_{\substack{i,j=1\\ \sigma_i=\sigma\\ \sigma_j=\overline{\sigma}}}^{n} b^{n+2}\left(\Phi_b^{\sigma}\right)^2 = l(n-l)\left(\Phi_b^{\sigma}\right)^2.$$

Similarly, we show $S_3 = l(n-l)\left(\Phi_b^{\sigma}\right)^2 = S_2$. To evaluate S_1, we have to distinguish the cases where $i \neq j$ and where $i = j$. The first case can be treated analogously to S_2 and S_3. Hence,

$$\begin{aligned} S_1 =& \frac{1}{b^{2n}} \sum_{\substack{i,j=1\\ \sigma_i,\sigma_j=\sigma\\ i\neq j}}^{n} \sum_{N=1}^{b^n} b^{n-2} \psi_b^{\sigma}\left(\frac{M}{b^i}\right) \psi_b^{\overline{\sigma}}\left(\frac{M}{b^j}\right) \\ &+ \frac{1}{b^{2n}} \sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right) \\ =& -\left(\Phi_b^{\sigma}\right)^2 l(l-1) + \frac{1}{b^{2n}} \sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right). \end{aligned}$$

In the same way we show

$$S_4 = -\left(\Phi_b^{\sigma}\right)^2 (n-l)(n-l-1) + \frac{1}{b^{2n}} \sum_{\substack{j=1\\ \sigma_j=\overline{\sigma}}}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right).$$

From Lemma 3.22 and the proof of Lemma 3.25 we observe that

$$\sum_{\lambda,M=1}^{b^n} \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right) = \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right).$$

Summarizing, we have

$$\Sigma_1 = \left(\Phi_b^{\sigma}\right)^2 (-n^2 - 4l^2 + n + 4ln) + \frac{1}{b^{2n}} \sum_{j=1}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)$$

and thus, by putting all results together, we arrive at

$$\begin{aligned} \left(L_2(\widetilde{\mathcal{H}}^{\Sigma}_{b,n})\right)^2 =& 2n\Phi_b^{\sigma,(2)} + \frac{13}{9} + \frac{1}{b^n} \\ &+ \frac{2}{b^{2n}} \sum_{j=1}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right). \end{aligned} \tag{3.13}$$

We observe that the remaining step to finally prove Theorem 3.17 is the evaluation of the expression

$$\frac{2}{b^{2n}} \sum_{j=1}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right). \tag{3.14}$$

This is the most difficult and technical part of the proof, and all the lemmas we present in the following paragraph aim at calculating this term. The final result is stated in Lemma 3.25. Inserting the formula given in this lemma (and in Remark 3.26) into (3.13) completes the proof of Theorem 3.17. □

The details of the proof To guide the reader through the proofs in this subsection, we explain the basic ideas in a few lines preceeding the corresponding lemma, respectively.

Lemma 3.22 is the only lemma where we need the somewhat complicated definition of the numbers $\varepsilon_j(\lambda, M, \Sigma)$ appearing in Theorem 2.7. The proof of this lemma may appear extremely technical on first look, but in fact we only apply basic combinatorial considerations. The main concern is to investigate for which integers $\lambda \in \{1, \dots, b^n\}$ the numbers $\varepsilon_j(\lambda, M, \Sigma)$ and $\varepsilon_j(\lambda, M, \Sigma^*)$ take certain values $h, h+1 \in \{0, \dots, b-1\}$ simultaneously.

Lemma 3.22. *Let $\sigma \in \mathfrak{S}_b$. For all $1 \leq M \leq b^n$ and $1 \leq j \leq n-1$ we have*

$$\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)$$
$$= \begin{cases} b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right) + b^{j-1}(b^{n-j} - 1 - 2\nu_j(M,\Sigma))\left(\widetilde{\psi}^{\sigma}_{b,1}\left(\frac{M}{b^j}\right) - \widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \nu_j(M,\Sigma) < \frac{b^{n-j}-1}{2}, \\ b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right) + b^{j-1}(2\nu_j(M,\Sigma) + 1 - b^{n-j})\left(\widetilde{\psi}^{\sigma}_{b,2}\left(\frac{M}{b^j}\right) - \widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \nu_j(M,\Sigma) \geq \frac{b^{n-j}-1}{2}. \end{cases}$$

If $j = n$, then we have

$$\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_n(\lambda,M,\Sigma)}\left(\frac{M}{b^n}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_n(\lambda,M,\Sigma^*)}\left(\frac{M}{b^n}\right) = b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^n}\right).$$

Proof. The case $M = b^n$ is trivial since then the left- and the right-hand-sides of the above equality are zero. We therefore assume $1 \leq M < b^n$ now. We show the case $j = n$ first. Since $\varepsilon_n(\lambda, M, \Sigma) = \varepsilon_n(\lambda, M, \Sigma^*) = \lambda_n$ by definition, we can write

$$\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_n(\lambda,M,\Sigma)}\left(\frac{M}{b^n}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_n(\lambda,M,\Sigma^*)}\left(\frac{M}{b^n}\right)$$
$$= \sum_{h=0}^{b-1} \psi^{\sigma}_{b,h}\left(\frac{M}{b^n}\right) \psi^{\overline{\sigma}}_{b,h}\left(\frac{M}{b^n}\right) \sum_{\substack{\lambda=1 \\ \lambda_n = h}}^{b^n} 1 = b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^n}\right).$$

We fix $j \in \{1, \dots, n-1\}$, $M \in \{1, \dots, b^n - 1\}$ and $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. We have to distinguish between two cases. Let us first assume that $\nu_j(M, \Sigma) < \nu_j(M, \Sigma^*)$. Then we can either have

$$\varepsilon_j(\lambda, M, \Sigma) = \varepsilon_j(\lambda, M, \Sigma^*) \text{ or } \varepsilon_j(\lambda, M, \Sigma) = \varepsilon_j(\lambda, M, \Sigma^*) + 1.$$

We count the number of Λ_{j-1} such that $\varepsilon_j(\lambda, M, \Sigma) = \varepsilon_j(\lambda, M, \Sigma^*) = h$ for any $h \in \{0, \dots, b-1\}$. For $h \in \{1, \dots, b-1\}$ these Λ_{j-1} are given by $\nu_j(M,\Sigma) + (h-1)b + z$ for $z \in \{\nu_j(M, \Sigma^*) - \nu_j(M, \Sigma) + 1, \dots, b^{n-j}\}$ and for $h = 0$ the corresponding Λ_{j-1} are $0, \dots, \nu_j(M, \Sigma)$ and $\nu_j(M, \Sigma^*) + (b-1)b^{n-j} + 1, \dots, b^{n-j+1} - 1$. Hence for all $h \in \{0, \dots, b-1\}$ we have $b^{n-j} - (\nu_j(M, \Sigma^*) - \nu_j(M, \Sigma))$ values for Λ_{j-1} such that $\varepsilon_j(\lambda, M, \Sigma) = \varepsilon_j(\lambda, M, \Sigma^*) = h$. Since there are always b^{j-1} elements $\lambda \in \{1, \dots, b^n\}$ with the same Λ_{j-1} we have proven

$$\sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=\varepsilon_j(\lambda,M,\Sigma^*)=h\}}}^{b^n} 1 = b^{j-1}(b^{n-j} - (\nu_j(M, \Sigma^*) - \nu_j(M, \Sigma))). \tag{3.15}$$

For $h \in \{0, \dots, b-2\}$, we have $\varepsilon_j(\lambda, M, \Sigma) = h + 1 = \varepsilon_j(\lambda, M, \Sigma^*) + 1$ for Λ_{j-1} of the form $\nu_j(M, \Sigma) + hb + z$ for $z \in \{1, \dots, \nu_j(M, \Sigma^*) - \nu_j(M, \Sigma)\}$. Hence we have

$$\sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=h+1,\, \varepsilon_j(\lambda,M,\Sigma^*)=h\}}}^{b^n} 1 = b^{j-1}(\nu_j(M, \Sigma^*) - \nu_j(M, \Sigma)) \tag{3.16}$$

for all $h \in \{0, \dots, b-2\}$. Here we simply neglect the also possible case $\varepsilon_j(\lambda, M, \Sigma) = 0, \varepsilon_j(\lambda, M, \Sigma^*) = b - 1$, since the corresponding summands in the sum

$$\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)$$

are zero anyway. In the second case $\nu_j(M, \Sigma) \geq \nu_j(M, \Sigma^*)$ we only have the possibilities

$$\varepsilon_j(\lambda, M, \Sigma) = \varepsilon_j(\lambda, M, \Sigma^*) \ \text{ or } \ \varepsilon_j(\lambda, M, \Sigma) + 1 = \varepsilon_j(\lambda, M, \Sigma^*).$$

Apart from that, the situation is quite the same as in the first case and we have

$$\sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=\varepsilon_j(\lambda,M,\Sigma^*)=h\}}}^{b^n} 1 = b^{j-1}(b^{n-j} - (\nu_j(M, \Sigma) - \nu_j(M, \Sigma^*)))$$

for all $h \in \{0, \dots, b-1\}$ and

$$\sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=h,\, \varepsilon_j(\lambda,M,\Sigma^*)=h+1\}}}^{b^n} 1 = b^{j-1}(\nu_j(M, \Sigma) - \nu_j(M, \Sigma^*))$$

for all $h \in \{0, \dots, b-2\}$. Next we prove the relation $\nu_j(M, \Sigma^*) = b^{n-j} - 1 - \nu_j(M, \Sigma)$. From the definition of $\nu_j(M, \Sigma^*)$ we find

$$\begin{aligned}
\nu_j(M, \Sigma^*) =& \sigma^*_{j+1}(M_{j+1})b^{n-j-1} + \dots + \sigma^*_{n-1}(M_{n-1})b + \sigma^*_n(M_n) \\
=& (b - 1 - \sigma_{j+1}(M_{j+1}))b^{n-j-1} + \dots + (b - 1 - \sigma_{n-1}(M_{n-1}))b \\
&+ (b - 1 - \sigma_n(M_n)) \\
=& (b-1)(b^{n-j-1} + \dots + b + 1) - \nu_j(M, \Sigma) = b^{n-j} - 1 - \nu_j(M, \Sigma).
\end{aligned}$$

This identity yields the equivalence of $\nu_j(M, \Sigma) < \nu_j(M, \Sigma^*)$ and $\nu_j(M, \Sigma) < \frac{b^{n-j}-1}{2}$ as well as the equivalence of $\nu_j(M, \Sigma) \geq \nu_j(M, \Sigma^*)$ and $\nu_j(M, \Sigma) \geq \frac{b^{n-j}-1}{2}$. Now in the case $\nu_j(M, \Sigma) < \frac{b^{n-j}-1}{2}$ we find

$$\begin{aligned}
&\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right) \\
&\qquad = \sum_{h=0}^{b-1} \psi^{\sigma}_{b,h}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,h}\left(\frac{M}{b^j}\right) \sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=\varepsilon_j(\lambda,M,\Sigma^*)=h\}}}^{b^n} 1 \\
&\qquad\quad + \sum_{h=0}^{b-2} \psi^{\sigma}_{b,h+1}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,h}\left(\frac{M}{b^j}\right) \sum_{\substack{\lambda=1 \\ \{\lambda\colon \varepsilon_j(\lambda,M,\Sigma)=h+1,\varepsilon_j(\lambda,M,\Sigma^*)=h\}}}^{b^n} 1.
\end{aligned}$$

Using (3.15), (3.16) and the definition of $\widetilde{\psi}_b^\sigma$ and $\widetilde{\psi}_{b,1}^\sigma$, this leads to

$$\begin{aligned}
&\sum_{\lambda=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\\
=&b^{j-1}(b^{n-j}-(\nu_j(M,\Sigma^*)-\nu_j(M,\Sigma)))\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)\\
&+b^{j-1}(\nu_j(M,\Sigma^*)-\nu_j(M,\Sigma))\widetilde{\psi}_{b,1}^\sigma\left(\frac{M}{b^j}\right)\\
=&b^{n-1}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)+b^{j-1}(\nu_j(M,\Sigma^*)-\nu_j(M,\Sigma))\left(\widetilde{\psi}_{b,1}^\sigma\left(\frac{M}{b^j}\right)-\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)\right).
\end{aligned}$$

By applying the above relation between $\nu_j(M,\Sigma)$ and $\nu_j(M,\Sigma^*)$ we find

$$\nu_j(M,\Sigma^*)-\nu_j(M,\Sigma)=b^{n-j}-1-2\nu_j(M,\Sigma),$$

which yields the claim of this lemma in the case $\nu_j(M,\Sigma)<\frac{b^{n-j}-1}{2}$. The other case can be completed analogously. □

We are now concerned with the task to compute sums of the form $\sum_{M=1}^{b^j}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)$ (and analogous sums for $\widetilde{\psi}_{b,1}^\sigma$ and $\widetilde{\psi}_{b,2}^\sigma$). Let us first consider such sums for the special case $\sigma=id$, since in this case the functions $\psi_{b,h}^{id}$, which we introduced in Definition 2.6, can be written down in a simple way. The rest of the proof of the subsequent Lemma 3.23 contains evaluations of elementary sums and integrals.

Lemma 3.23. *For all $j\in\{1,\dots,n\}$ we have*

$$\begin{aligned}
\sum_{M=1}^{b^j}\widetilde{\psi}_b^{id}\left(\frac{M}{b^j}\right)&=b^j\left(\int_0^1\widetilde{\psi}_b^{id}(x)\,\mathrm{d}x+\frac{A_b(j,id)}{2}\right),\\
\sum_{M=1}^{b^j}\widetilde{\psi}_{b,1}^{id}\left(\frac{M}{b^j}\right)&=b^j\left(\int_0^1\widetilde{\psi}_{b,1}^{id}(x)\,\mathrm{d}x+\frac{\overline{A}_b(j,id)}{2}\right),\\
\sum_{M=1}^{b^j}\widetilde{\psi}_{b,2}^{id}\left(\frac{M}{b^j}\right)&=b^j\left(\int_0^1\widetilde{\psi}_{b,2}^{id}(x)\,\mathrm{d}x+\frac{\overline{A}_b(j,id)}{2}\right),
\end{aligned}$$

where we have

$$A_b(j,id)=\begin{cases}-\frac{1}{36b^{2j}}(b^3+2b) & \text{if } b \text{ is even,}\\ -\frac{1}{36b^{2j}}(b^3-b) & \text{if } b \text{ is odd.}\end{cases}$$

and

$$\overline{A}_b(j,id)=\begin{cases}-\frac{1}{36b^{2j}}(b^3-4b) & \text{if } b \text{ is even,}\\ -\frac{1}{36b^{2j}}(b^3-b) & \text{if } b \text{ is odd.}\end{cases}$$

Proof. We use the fact that

$$\psi_{b,h}^{id}(x)=\begin{cases}(b-h)x & \text{if } x\in\left[0,\frac{h}{b}\right],\\ h(1-x) & \text{if } x\in\left[\frac{h}{b},1\right],\end{cases}$$

and

$$\psi_{b,h}^{\tau_b}(x)=\begin{cases}-hx & \text{if } x\in\left[0,\frac{b-h}{b}\right],\\ (b-h)x-(b-h) & \text{if } x\in\left[\frac{b-h}{b},1\right],\end{cases}$$

which was already mentioned in [30]. Let $x \in [k/b, (k+1)/b]$. Then we have $x \in [0, h/b]$ for $h \in \{k+1, \dots, b-1\}$ and $x \in [h/b, 1]$ for $h \in \{0, \dots, k\}$. We have $x \in [0, (b-h)/b]$ for $h \in \{0, \dots, b-k-1\}$ and $x \in [(b-h)/b, 1]$ for $h \in \{b-k, \dots, b-1\}$. We distinguish two cases:

1. Let $k \leq (b-1)/2$. Then we have $k \leq b-k-1$ and therefore we can write

$$\begin{aligned}
\widetilde{\psi}_b^{id}(x) &= \sum_{h=0}^{k} h(1-x)(-hx) + \sum_{h=k+1}^{b-k-1} (b-h)x \cdot (-hx) \\
&\quad + \sum_{h=b-k}^{b-1} (b-h)x \cdot ((b-h)x - (b-h)) \\
&= \left(bk^2 + bk - \frac{b^3}{6} + \frac{b}{6} \right) x^2 - \left(\frac{2k^3}{3} + k^2 + \frac{k}{3} \right) x =: P_k(x).
\end{aligned}$$

2. Let $k > (b-1)/2$. Then we have $b-k-1 < k$ and therefore we can write

$$\begin{aligned}
\widetilde{\psi}_b^{id}(x) &= \sum_{h=0}^{b-k-1} h(1-x)(-hx) + \sum_{h=b-k}^{k} h(1-x) \cdot ((b-h)x - (b-h)) \\
&\quad + \sum_{h=k+1}^{b-1} (b-h)x \cdot ((b-h)x - (b-h)) \\
&= \left(bk^2 - 2b^2k + bk + \frac{5b^3}{6} - b^2 + \frac{b}{6} \right) x^2 \\
&\quad + \left(2b^2k - \frac{2k^3}{3} - k^2 - \frac{k}{3} - b^3 + b^2 \right) x \\
&\quad + \frac{2k^3}{3} - bk^2 + k^2 - bk + \frac{k}{3} + \frac{b^3}{6} - \frac{b}{6} =: Q_k(x).
\end{aligned}$$

Now we have to consider even and odd bases b separately. For even b we find

$$\int_0^1 \widetilde{\psi}_b^{id}(x) \, \mathrm{d}x = \sum_{k=0}^{\frac{b}{2}-1} \int_{\frac{k}{b}}^{\frac{k+1}{b}} P_k(x) \, \mathrm{d}x + \sum_{k=\frac{b}{2}}^{b-1} \int_{\frac{k}{b}}^{\frac{k+1}{b}} Q_k(x) \, \mathrm{d}x = -\frac{1}{90b} - \frac{7b^3}{720}$$

and

$$\begin{aligned}
\sum_{M=1}^{b^j} \widetilde{\psi}_b^{id}\left(\frac{M}{b^j}\right) &= \sum_{k=0}^{\frac{b}{2}-1} \sum_{M=kb^{j-1}+1}^{(k+1)b^{j-1}} P_k\left(\frac{M}{b^j}\right) + \sum_{k=\frac{b}{2}}^{b-1} \sum_{M=kb^{j-1}+1}^{(k+1)b^{j-1}} Q_k\left(\frac{M}{b^j}\right) \\
&= -\frac{1}{72b^j}(b^3 + 2b) + b^j \left(-\frac{1}{90b} - \frac{7b^3}{720} \right)
\end{aligned}$$

whereas for odd bases b we compute analogously

$$\int_0^1 \widetilde{\psi}_b^{id}(x) \, \mathrm{d}x = \sum_{k=0}^{\frac{b-1}{2}} \int_{\frac{k}{b}}^{\frac{k+1}{b}} P_k(x) \, \mathrm{d}x + \sum_{k=\frac{b+1}{2}}^{b-1} \int_{\frac{k}{b}}^{\frac{k+1}{b}} Q_k(x) \, \mathrm{d}x = \frac{7}{720b} - \frac{7b^3}{720}$$

and

$$\sum_{M=1}^{b^j} \widetilde{\psi}_b^{id}\left(\frac{M}{b^j}\right) = \sum_{k=0}^{\frac{b-1}{2}} \sum_{M=kb^{j-1}+1}^{(k+1)b^{j-1}} P_k\left(\frac{M}{b^j}\right) + \sum_{k=\frac{b+1}{2}}^{b-1} \sum_{M=kb^{j-1}+1}^{(k+1)b^{j-1}} Q_k\left(\frac{M}{b^j}\right)$$
$$= -\frac{1}{72b^j}(b^3-b) + b^j\left(\frac{7}{720b} - \frac{7b^3}{720}\right).$$

It is straightforward now to derive the claimed formula for $\sum_{M=1}^{b^j} \widetilde{\psi}_b^{id}\left(\frac{M}{b^j}\right)$. Since the proofs of the other two identities may be executed analogously, we omit them at this point. □

The next lemma generalizes Lemma 3.23 to arbitrary permutations $\sigma \in \mathfrak{S}_b$. The main idea of the proof is to reduce the case of general permutations σ to the case where $\sigma = id$. The latter case has been analyzed in the previous lemma already. We advise the reader to consult also the proof of [31, Lemma 4], since we follow closely the ideas there.

Lemma 3.24. *Let $\sigma \in \mathfrak{S}_b$. For all $j \in \{1, \dots, n\}$ we have*

$$\begin{aligned}
\sum_{M=1}^{b^j} \widetilde{\psi}_b^{\sigma}\left(\frac{M}{b^j}\right) &= b^j\left(\int_0^1 \widetilde{\psi}_b^{\sigma}(x)\,\mathrm{d}x + \frac{A_b(j,id)}{2}\right),\\
\sum_{M=1}^{b^j} \widetilde{\psi}_{b,1}^{\sigma}\left(\frac{M}{b^j}\right) &= b^j\left(\int_0^1 \widetilde{\psi}_{b,1}^{\sigma}(x)\,\mathrm{d}x + \frac{\overline{A}_b(j,id)}{2}\right),\\
\sum_{M=1}^{b^j} \widetilde{\psi}_{b,2}^{\sigma}\left(\frac{M}{b^j}\right) &= b^j\left(\int_0^1 \widetilde{\psi}_{b,2}^{\sigma}(x)\,\mathrm{d}x + \frac{\overline{A}_b(j,id)}{2}\right),
\end{aligned}$$

where $A_b(j,id)$ and $\overline{A}_b(j,id)$ are as defined in Lemma 3.23.

Proof. It has been shown in the proof of [31, Lemma 4], by applying Simpson's quadrature rule, that

$$\sum_{M=1}^{b^j} \psi_b^{\sigma,(2)}\left(\frac{M}{b^j}\right) = b^j\left(\int_0^1 \psi_b^{\sigma,(2)}(x)\,\mathrm{d}x + \frac{A_b(j,\sigma)}{2}\right)$$

with

$$A_b(j,\sigma) = \frac{1}{6b^{2j}}\sum_{k=1}^{b}\left(\left(\psi_b^{\sigma,(2)}\right)'\left(\frac{k}{b}-0\right) - \left(\psi_b^{\sigma,(2)}\right)'\left(\frac{k}{b}+0\right)\right),$$

where here and later on by $f'(x-0)$ we mean the left-derivative and by $f'(x+0)$ the right-derivative of the function f at x . The only properties of $\psi_b^{\sigma,(2)}$ the authors needed to show this identity are the fact that $\psi_b^{\sigma,(2)}$ is quadratic on intervals $[k/b,(k+1)/b)$ as well as the 1-periodicity of this function. Since $\widetilde{\psi}_b^{\sigma}$ has these two properties as well, an analogous relation is also true for $\widetilde{\psi}_b^{\sigma}$. Now we need the definition $\widetilde{\psi}_b^{\sigma} = \sum_{h=0}^{b-1} \psi_{b,h}^{\sigma}\psi_{b,h}^{\overline{\sigma}}$ to deduce

$$
\begin{aligned}
&\sum_{k=1}^{b} \left(\widetilde{\psi}_b^{\sigma}\right)' \left(\frac{k}{b} - 0\right) - \left(\widetilde{\psi}_b^{\sigma}\right)' \left(\frac{k}{b} + 0\right) \\
&= \sum_{k=1}^{b} \sum_{h=0}^{b-1} \Bigg\{ \psi_{b,h}^{\sigma}\left(\frac{k}{b}\right) \left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{k}{b} - 0\right) + \psi_{b,h}^{\overline{\sigma}}\left(\frac{k}{b}\right) \left(\psi_{b,h}^{\sigma}\right)' \left(\frac{k}{b} - 0\right) \\
&\qquad - \psi_{b,h}^{\sigma}\left(\frac{k}{b}\right) \left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{k}{b} + 0\right) - \psi_{b,h}^{\overline{\sigma}}\left(\frac{k}{b}\right) \left(\psi_{b,h}^{\sigma}\right)' \left(\frac{k}{b} + 0\right) \Bigg\} \\
&= \sum_{h=0}^{b-1} \sum_{k=1}^{b} \psi_{b,h}^{\sigma}\left(\frac{k}{b}\right) \left(\left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{k-1}{b} + 0\right) - \left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{k}{b} + 0\right)\right) \\
&\quad + \sum_{h=0}^{b-1} \sum_{k=1}^{b} \psi_{b,h}^{\overline{\sigma}}\left(\frac{k}{b}\right) \left(\left(\psi_{b,h}^{\sigma}\right)' \left(\frac{k-1}{b} + 0\right) - \left(\psi_{b,h}^{\sigma}\right)' \left(\frac{k}{b} + 0\right)\right) = \sum_{h=0}^{b-1} \sum_{k=1}^{b} (S_1 + S_2).
\end{aligned}
$$

We define $f_{h,k} := \left(\psi_{b,h}^{\sigma}\right)' \left(\frac{k}{b} + 0\right)$ and $\overline{f}_{h,k} := \left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{k}{b} + 0\right)$. From the linearity of $\psi_{b,h}^{\sigma}$ and $\psi_{b,h}^{\overline{\sigma}}$ on $[k/b, (k+1)/b)$ we have $\psi_{b,h}^{\sigma}(k/b) = \int_0^{k/b} \left(\psi_{b,h}^{\sigma}\right)'(x) \,\mathrm{d}x = \frac{1}{b} \sum_{l=0}^{k-1} f_{h,l}$ and also $\psi_{b,h}^{\overline{\sigma}}(k/b) = \frac{1}{b} \sum_{l=0}^{k-1} \overline{f}_{h,l}$. For $k = b$, this yields $\sum_{l=0}^{b-1} f_{h,l} = \sum_{l=0}^{b-1} \overline{f}_{h,l} = 0$. Hence for every $h \in \{0, \dots, b-1\}$ we have

$$
\sum_{k=1}^{b} S_1 = \frac{1}{b} \sum_{l=0}^{b-1} f_{h,l} \sum_{k=l+1}^{b} (\overline{f}_{h,k-1} - \overline{f}_{h,k}) = \frac{1}{b} \sum_{l=0}^{b-1} f_{h,l} \overline{f}_{h,l}
$$

and analogously

$$
\sum_{k=1}^{b} S_2 = \frac{1}{b} \sum_{l=0}^{b-1} \overline{f}_{h,l} \sum_{k=l+1}^{b} (f_{h,k-1} - f_{h,k}) = \frac{1}{b} \sum_{l=0}^{b-1} \overline{f}_{h,l} f_{h,l} = \sum_{k=1}^{b} S_1.
$$

Finally we conclude

$$
\begin{aligned}
A_b(j,\sigma) &= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1} \sum_{l=0}^{b-1} f_{h,l} \overline{f}_{h,l} = \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1} \sum_{l=0}^{b-1} \left(\left(\psi_{b,h}^{\sigma}\right)' \left(\frac{l}{b} + 0\right) \left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{l}{b} + 0\right)\right) \\
&= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1} \sum_{l=0}^{b-1} \left(\left(\psi_{b,h}^{id}\right)' \left(\frac{\sigma(l)}{b} + 0\right) \left(\psi_{b,h}^{\tau_b}\right)' \left(\frac{\sigma(l)}{b} + 0\right)\right) \\
&= \frac{1}{3b^{2j+1}} \sum_{h=0}^{b-1} \sum_{l=0}^{b-1} \left(\left(\psi_{b,h}^{id}\right)' \left(\frac{l}{b} + 0\right) \left(\psi_{b,h}^{\tau_b}\right)' \left(\frac{l}{b} + 0\right)\right) = A_b(j, id),
\end{aligned}
$$

where we used the relations

$$
\left(\psi_{b,h}^{\sigma}\right)' \left(\frac{l}{b} + 0\right) = \left(\psi_{b,h}^{id}\right)' \left(\frac{\sigma(l)}{b} + 0\right)
$$

and

$$
\left(\psi_{b,h}^{\overline{\sigma}}\right)' \left(\frac{l}{b} + 0\right) = \left(\psi_{b,h}^{\tau_b}\right)' \left(\frac{\sigma(l)}{b} + 0\right).
$$

They follow both directly from the definition of $\psi_{b,h}^{\sigma}$. The first relation has also been used in [29, 31]. The proof of the first claim of this lemma is complete. Since the other two identities may be proven completely analogously, we omit an explicit proof. □

Now we are ready to show the main lemma of this section. We will combine Lemmas 3.22, 3.23 and 3.24 to obtain this result.

Lemma 3.25. *Let $\sigma \in \mathfrak{S}_b$. Then we have for even bases b*

$$\begin{aligned}&\frac{2}{b^{2n}}\sum_{j=1}^{n}\sum_{\lambda,M=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\\&=n\left(\widetilde{\Phi}^{\sigma}_b+\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,1}+\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,2}\right)+\left(\widetilde{\Phi}^{\sigma}_b-\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,1}-\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,2}\right)-\frac{1}{36}-\frac{1}{18b^{2n}}\end{aligned}$$

and for odd bases b

$$\begin{aligned}&\frac{2}{b^{2n}}\sum_{j=1}^{n}\sum_{\lambda,M=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\\&=n\left(\widetilde{\Phi}^{\sigma}_b+\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,1}+\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,2}\right)+\left(-\frac{1}{36}+\frac{b^2}{b^2-1}\left(\widetilde{\Phi}^{\sigma}_b-\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,1}-\frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,2}\right)\right)\left(1-\frac{1}{b^{2n}}\right).\end{aligned}$$

Proof. At first we remark that for $M = M_n b^{n-1} + \cdots + M_1$ the number $\nu_j(M,\Sigma)$ depends only on the digits $M_{j+1},\dots,M_n$, which follows directly from its definition in Theorem 2.7. On the other hand, the values of $\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)$, $\widetilde{\psi}^{\sigma}_{b,1}\left(\frac{M}{b^j}\right)$ and $\widetilde{\psi}^{\sigma}_{b,2}\left(\frac{M}{b^j}\right)$ depend only on the digits $M_1,\dots,M_j$. This can be seen from the 1-periodicity of these functions, since

$$\begin{aligned}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)&=\widetilde{\psi}^{\sigma}_b\left(\left\{\frac{M}{b^j}\right\}\right)=\widetilde{\psi}^{\sigma}_b\left(\left\{M_nb^{n-j-1}+\cdots+M_{j+1}+M_jb^{-1}+\cdots+M_1b^{-j}\right\}\right)\\&=\widetilde{\psi}^{\sigma}_b\left(M_jb^{-1}+\cdots+M_1b^{-j}\right)\end{aligned}$$

and analogously for $\widetilde{\psi}^{\sigma}_{b,1}$ and $\widetilde{\psi}^{\sigma}_{b,2}$. We set $f_b(j) := \lfloor (b^{n-j}-1)/2\rfloor$. Lemma 3.22 leads to

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{\lambda,M=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\\&=\sum_{j=1}^{n-1}\Bigg\{\sum_{\ell=0}^{f_b(j)}\sum_{\substack{M_{j+1},\dots,M_n=0\\ \nu_j(M,\Sigma)=\ell}}^{b-1}\sum_{M_1,\dots,M_j=0}^{b-1}\Bigg(b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\\&\qquad+b^{j-1}(b^{n-j}-1-2\ell)\left(\widetilde{\psi}^{\sigma}_{b,1}\left(\frac{M}{b^j}\right)-\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\right)\Bigg)\\&\quad+\sum_{\ell=f_b(j)+1}^{b^{n-j}-1}\sum_{\substack{M_{j+1},\dots,M_n=0\\ \nu_j(M,\Sigma)=\ell}}^{b-1}\sum_{M_1,\dots,M_j=0}^{b-1}\Bigg(b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\\&\qquad+b^{j-1}(2\ell+1-b^{n-j})\left(\widetilde{\psi}^{\sigma}_{b,2}\left(\frac{M}{b^j}\right)-\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\right)\Bigg)\Bigg\}\\&\quad+\sum_{M=1}^{b^n}b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^n}\right)\\&=\sum_{j=1}^{n-1}\Bigg\{\sum_{\ell=0}^{f_b(j)}\sum_{M=1}^{b^j}\left(b^{n-1}\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)+b^{j-1}(b^{n-j}-1-2\ell)\left(\widetilde{\psi}^{\sigma}_{b,1}\left(\frac{M}{b^j}\right)-\widetilde{\psi}^{\sigma}_b\left(\frac{M}{b^j}\right)\right)\right)\end{aligned}$$

$$+\sum_{\ell=f_b(j)+1}^{b^{n-j}-1}\sum_{M=1}^{b^j}\left(b^{n-1}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)+b^{j-1}(2\ell+1-b^{n-j})\left(\widetilde{\psi}_{b,2}^\sigma\left(\frac{M}{b^j}\right)-\widetilde{\psi}_b^\sigma\left(\frac{M}{b^j}\right)\right)\right)\Bigg\}$$
$$+\sum_{M=1}^{b^n}b^{n-1}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^n}\right).$$

At this point we need to treat the cases of even and odd bases b separately. Let us first consider even bases. Then we have $f_b(j)=b^{n-j}/2-1$. With Lemma 3.24 we get

$$\sum_{j=1}^{n}\sum_{\lambda,M=1}^{b^n}\psi_{b,\varepsilon_j(\lambda,M,\Sigma)}^\sigma\left(\frac{M}{b^j}\right)\psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\overline{\sigma}}\left(\frac{M}{b^j}\right)$$
$$=\sum_{j=1}^{n-1}\Bigg\{b^{2n-j-1}b^j\left(b\widetilde{\Phi}_b^\sigma+\frac{A_b(j,id)}{2}\right)$$
$$+b^{j-1}\sum_{\ell=0}^{b^{n-j}/2-1}(b^{n-j}-1-2\ell)b^j\left(b\widetilde{\Phi}_{b,1}^\sigma+\frac{\overline{A}_b(j,id)}{2}-b\widetilde{\Phi}_b^\sigma-\frac{A_b(j,id)}{2}\right)$$
$$+b^{j-1}\sum_{\ell=b^{n-j}/2}^{b^{n-j}-1}(2\ell+1-b^{n-j})b^j\left(b\widetilde{\Phi}_{b,2}^\sigma+\frac{\overline{A}_b(j,id)}{2}-b\widetilde{\Phi}_b^\sigma-\frac{A_b(j,id)}{2}\right)\Bigg\}$$
$$+\sum_{M=1}^{b^n}b^{n-1}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^n}\right)$$
$$=\sum_{j=1}^{n-1}\Bigg\{b^{2n-1}\left(b\widetilde{\Phi}_b^\sigma-\frac{1}{72b^{2j}}(b^3+2b)\right)+\frac{1}{4}b^{2n-1}\left(b\widetilde{\Phi}_{b,1}^\sigma-b\widetilde{\Phi}_b^\sigma+\frac{1}{12b^{2j-1}}\right)$$
$$+\frac{1}{4}b^{2n-1}\left(b\widetilde{\Phi}_{b,2}^\sigma-b\widetilde{\Phi}_b^\sigma+\frac{1}{12b^{2j-1}}\right)\Bigg\}+b^{2n-1}\left(b\widetilde{\Phi}_b^\sigma-\frac{1}{72b^{2n}}(b^3+2b)\right).$$

Now a straightforward calculation yields the claimed result for even bases b. For odd bases b we have $f_b(j)=(b^{n-j}-1)/2$ and hence we obtain similarly as above

$$\sum_{j=1}^{n}\sum_{\lambda,M=1}^{b^n}\psi_{b,\varepsilon_j(\lambda,M,\Sigma)}^\sigma\left(\frac{M}{b^j}\right)\psi_{b,\varepsilon_j(\lambda,M,\Sigma^*)}^{\overline{\sigma}}\left(\frac{M}{b^j}\right)$$
$$=\sum_{j=1}^{n-1}\Bigg\{b^{2n-j-1}b^j\left(b\widetilde{\Phi}_b^\sigma+\frac{A_b(j,id)}{2}\right)+b^{j-1}\sum_{\ell=0}^{(b^{n-j}-1)/2}(b^{n-j}-1-2\ell)b^j\left(b\widetilde{\Phi}_{b,1}^\sigma-b\widetilde{\Phi}_b^\sigma\right)$$
$$+b^{j-1}\sum_{\ell=(b^{n-j}+1)/2}^{b^{n-j}-1}(2\ell+1-b^{n-j})b^j\left(b\widetilde{\Phi}_{b,2}^\sigma-b\widetilde{\Phi}_b^\sigma\right)\Bigg\}+\sum_{N=1}^{b^n}b^{n-1}\widetilde{\psi}_b^\sigma\left(\frac{M}{b^n}\right)$$
$$=\sum_{j=1}^{n-1}\Bigg\{b^{2n-1}\left(b\widetilde{\Phi}_b^\sigma-\frac{1}{72b^{2j}}(b^3-b)\right)+\frac{1}{4b}(b^{2n}-b^{2j})\left(b\widetilde{\Phi}_{b,1}^\sigma-b\widetilde{\Phi}_b^\sigma\right)$$
$$+\frac{1}{4b}(b^{2n}-b^{2j})\left(b\widetilde{\Phi}_{b,2}^\sigma-b\widetilde{\Phi}_b^\sigma\right)\Bigg\}+b^{2n-1}\left(b\widetilde{\Phi}_b^\sigma-\frac{1}{72b^{2n}}(b^3-b)\right).$$

The rest of the proof is again a matter of elementary calculations. □

Remark 3.26. Proposition A.2 from the Appendix provides for $\sigma\in\mathcal{A}_b(\tau)$ the relation

$$\widetilde{\Phi}_b^\sigma-\frac{1}{2}\widetilde{\Phi}_{b,1}^\sigma-\frac{1}{2}\widetilde{\Phi}_{b,2}^\sigma=\begin{cases}-\frac{1}{24} & \text{if } b \text{ is even,}\\ -\frac{1}{24}\frac{b^2-1}{b^2} & \text{if } b \text{ is odd.}\end{cases}$$

In this case, Lemma 3.25 can be displayed in a much simpler form, namely

$$\frac{2}{b^{2n}} \sum_{j=1}^{n} \sum_{\lambda,M=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) \psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)$$
$$= n\left(\widetilde{\Phi}^{\sigma}_b + \frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,1} + \frac{1}{2}\widetilde{\Phi}^{\sigma}_{b,2}\right) - \frac{5}{72} + \frac{1 - 9\cdot(-1)^b}{144 b^{2n}}.$$

Note that we have now completed the final task and found an expression for (3.14). Inserting the formula discovered in Remark 3.26 into (3.13) completes the proof of Theorem 3.17.

Numerical results We defer all the proofs in this section to the Appendix, since they contain elementary, but very lengthy and technical calculations.
The constant c^{σ}_b which appears in Theorem 3.17 is rather hard to compute. We therefore present an alternative formula in the subsequent lemma, which is a consequence of Theorem 3.17 and Proposition A.1 from the Appendix.

Lemma 3.27. *Let $n \in \mathbb{N}$, $\sigma \in \mathcal{A}_b(\tau)$ and $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. Then we have*

$$\lim_{n\to\infty} \frac{L_2(\widetilde{\mathcal{H}}^{\Sigma}_{b,n})}{\sqrt{\log(2b^n)}} = \sqrt{\frac{c^{\sigma}_b}{\log b}},$$

where

$$c^{\sigma}_b = \frac{16 - 12b - 111b^2 + 228b^3 - 112b^4}{72b^2} - \frac{1-(-1)^b}{16b^3}$$
$$+ \frac{4}{b^3} \sum_{k_1,k_2=0}^{b-1} \max\{\sigma(k_1), \sigma(k_2)\} \left(\frac{b}{2}\Big(\max\{k_1,k_2\} + \max\{k_1+k_2, b-1\}\Big) - k_1^2 - k_1\right).$$

Now we would like to find for each base b the permutation $\sigma^{\min}_b \in \mathcal{A}_b(\tau)$ for which the constant c^{σ}_b becomes minimal. We therefore employ computer search algorithms. Note that the constant c^{σ}_b is invariant with respect to switching two complementary elements in the permutation σ. To be more precise, for a given permutation $\sigma \in \mathcal{A}_b(\tau)$ the following fact is true: Let $d \in \{0,\dots,b-1\}$. Then we define the permutation $\widehat{\sigma} \in \mathcal{A}_b(\tau)$ in the following way: For $k \in \{0,\dots,b-1\} \setminus \{d, b-1-d\}$ we set $\widehat{\sigma}(k) = \sigma(k)$ and additionally we set $\widehat{\sigma}(d) = \sigma(b-1-d)$ and $\widehat{\sigma}(b-1-d) = \sigma(d)$. Then we have $c^{\sigma}_b = c^{\widehat{\sigma}}_b$, which is the result given in Proposition A.3 from the Appendix. This rule allows us to reduce the number of permutations which we have to check significantly. We do not have to check every single permutation that is contained in $\mathcal{A}_b(\tau)$, but only those which are elements of the subset

$$\mathcal{B}_b(\tau) := \left\{\sigma \in \mathcal{A}_b(\tau) : \sigma(k) \in \left\{0,1,\dots,\left\lfloor\frac{b-1}{2}\right\rfloor\right\} \text{ for all } k \in \left\{0,1,\dots,\left\lfloor\frac{b-1}{2}\right\rfloor\right\}\right\}.$$

That means we have to check $\lfloor b/2\rfloor!$ permutations instead of $2^{\lfloor b/2\rfloor}\lfloor b/2\rfloor!$ to find the minimal value for c^{σ}_b. Our numerical investigations show that there are often several permutations $\sigma \in \mathcal{B}_b(\tau)$ where the minimal value for c^{σ}_b is attained. Table 3.1 lists for each base $b \in \{2,\dots,27\}$ one permutation $\sigma \in \mathcal{B}_b(\tau)$ where c^{σ}_b is minimal and the number g_b of permutations in $\mathcal{B}_b(\tau)$ which give the minimal value for c^{σ}_b. Then there are

$2^{\lfloor b/2 \rfloor} g_b$ permutations in $\mathcal{A}_b(\tau)$ which yield the lowest constant in each base. Of course, we also present the corresponding values for c_b^σ and $\sqrt{c_b^\sigma / \log b}$. Since the permutations in $\mathcal{B}_b(\tau)$ are completely determined by the permutation of the digits $0, 1, \ldots, \lfloor (b-1)/2 \rfloor$, we only give these partial permutations in Table 3.1. We use the usual cycle notation. For instance, the permutation $(0,1,2) \in \mathcal{B}_7(\tau)$ on the set $\{0, 1, \ldots, 6\}$ is given by $\sigma(0) = 1$, $\sigma(1) = 2$ and $\sigma(2) = 0$. The values of $\sigma(3)$, $\sigma(4)$, $\sigma(5)$ and $\sigma(6)$ can then be obtained through the relation $\sigma(6-k) = 6 - \sigma(k)$ for $k = 0, 1, 2, 3$.

b	$\sigma_b^{\min}$	g_b	c_b^σ	$\sqrt{\frac{c_b^\sigma}{\log b}}$
2	id	1	1/24	0.245178
3	id	1	5/81	0.237039
4	id	2	1/12	0.245178
5	$(0,1)$	1	29/375	0.219202
6	$(0,1)$	4	67/648	0.240220
7	$(0,1,2)$	2	2/21	0.221229
8	$(0,2,3,1)$	2	3/32	0.212330
9	$(0,1,3)$	4	26/243	0.220671
10	$(0,3,4,1)$	2	111/1000	0.219560
11	$(0,2)(1,4)$	1	415/3993	0.208189
12	$(0,3)(2,5)$	2	35/324	0.208500
13	$(0,2)(1,5)(3,4)$	1	55/507	0.205654
14	$(0,2)(1,5)(4,6)$	2	983/8232	0.212715
15	$(0,4)(2,6)$	3	236/2025	0.207450
16	$(0,5,4)(2,3,7)$	4	23/192	0.207859
17	$(0,3,5,6,4,2)(1,7)$	2	584/4913	0.204829
18	$(0,5,8,3)(1,2,7,6)$	2	241/1944	0.207101
19	$(0,5)(2,8)(4,6,7)$	2	827/6859	0.202358
20	$(0,2,4)(1,8)(3,6)(5,7,9)$	8	193/1500	0.207243
21	$(0,6)(2,9)(5,8)$	1	491/3969	0.201576
22	$(0,4,2,1,9,8,5,6,10,3,7)$	8	4219/31944	0.206708
23	$(0,6)(2,10)(4,8)(7,9)$	1	4586/36501	0.200175
24	$(0,7,11,3,5,8,1,2,10,9,6,4)$	16	343/2592	0.204055
25	$(0,4,6,8,10,7)(1,9,5,3,11,2)$	8	1234/9375	0.202218
26	$(0,7,12,5)(1,2,11,10)(3,4,9,8)$	2	2236/17576	0.198792
27	$(0,3,1,10,6,8,11,9,4,12,2,7)$	14	289/2187	0.200235

Table 3.1.: Numerical results for the full search in $\mathcal{B}_b(\tau)$

Finally, we should compare our numerical results to those in Section 5 of [31]. There the authors searched for the best permutations $\sigma \in \mathcal{A}_b(\tau)$ to obtain a minimal L_2 discrepancy of the digit scrambled Hammersley point sets $\mathcal{H}_{b,n}^\Sigma$, where $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. The authors obtained the lowest L_2 discrepancy overall in base 22; the corresponding leading constant is 0.179069... We obtain the lowest leading constant for $L_2(\widetilde{\mathcal{H}}_{b,n}^\Sigma)$ in base 26, namely 0.198792.... In general, the minimal constants of the symmetrized Hammersley point sets are slightly higher than the minimal constants of the digit scrambled Hammersley point sets in every base, at least up to base 23 (Table 1 in [31] ends after this base). The advantage of the symmetrized point sets is the fact that we do not have to care about the arrangement of σ and $\overline{\sigma}$ in Σ (see Remark 3.18). As for $L_2(\mathcal{H}_{b,n}^{\Sigma})$, this is the case if and only if $\Phi_b^\sigma = 0$ (see [31, Table 2]). Additionally, Corollary 3.19 indicates that the values of c_b^σ for "good" and "bad" permutations do not spread so much for the symmetrized point sets as it is the case for the digit scrambled point sets. Indeed, our numerical results suggest that for even bases the highest value for c_b^σ of all $\sigma \in \mathcal{B}_b(\tau)$ is always attained only for the identity and the permutation which is determined through the relations $\sigma(k) = b/2 - 1 - k$ for all $k \in \{0, \dots, b/2-1\}$ and that for odd bases $\max_{\sigma \in \mathcal{B}_b(\tau)} c_b^\sigma$ is attained if and only if $\sigma = id$.

Open Problem 3.28. We state several unsolved problems on the L_2 discrepancy of the symmetrized Hammersley point sets.

- Prove or disprove that $\max_{\sigma \in \mathcal{B}_b(\tau)} c_b^\sigma = c_b^{id}$ as conjectured above.
- Find $\min_{\sigma \in \mathcal{B}_b(\tau)} c_b^\sigma$ for bases $b \geq 28$ (maybe there is a law behind it). Investigate in particular if $\min_{\sigma \in \mathcal{B}_b(\tau)} \sqrt{c_b^\sigma/(\log b)}$ is bounded in b as suggested by Table 3.1.

3.3. L_2 discrepancy of symmetrized van der Corput sequences

In Section 2.1.2 we discussed methods to find good bounds on the constants involved in the L_2 discrepancy of the symmetrized van der Corput sequences. The result of Faure [27] on the sequence $\widetilde{\mathcal{V}}_2$ provides the smallest constant known so far. However, it is reasonable to assume that there are sequences among the huge class of symmetrized generalized van der Corput sequences $\widetilde{\mathcal{V}}_b^\sigma$ whose L_2 discrepancy has even smaller constants asymptotically. Since the approach via the diaphony, executed in the papers [11] and [56], failed to beat Faure's constant from 1990, we intend to find an exact formula for $L_{2,N}(\widetilde{\mathcal{V}}_b^\sigma)$. The aim of this section is to find a precise expression for the L_2 discrepancy of the symmetrized van der Corput sequences $\widetilde{\mathcal{V}}_b^\sigma$ for any base b and any permutation $\sigma \in \mathfrak{S}_b$ with $\sigma(0) = 0$. The following theorem gives a result for even N. To this end, we introduce the functions

$$\eta_b^\sigma := \varphi_b^{\sigma,(2)} + \widetilde{\varphi}_b^\sigma, \quad \eta_{b,1}^\sigma := \widetilde{\varphi}_{b,1}^\sigma - \widetilde{\varphi}_b^\sigma \quad \text{and} \qquad \eta_{b,2}^\sigma := \widetilde{\varphi}_{b,2}^\sigma - \widetilde{\varphi}_b^\sigma.$$

Theorem 3.29. *Let $N = 2M$ with $1 \leq M < b^n$, where $n \in \mathbb{N}$ and $b \in \mathbb{N}$, $b \geq 2$. Then we have*

$$L_{2,N}(\widetilde{\mathcal{V}}_b^\sigma) = \Bigg(\frac{2}{b} \sum_{j=1}^{n} \eta_b^\sigma \left(\frac{M}{b^j} \right) + \frac{2}{b} \sum_{\substack{j=1 \\ \nu_j(M,\Sigma) \leq \frac{b^{n-j}-1}{2}}}^{n-1} \left| 1 - \frac{2\nu_j(M,\Sigma)+1}{b^{n-j}} \right| \eta_{b,1}^\sigma \left(\frac{M}{b^j} \right)$$
$$+ \frac{2}{b} \sum_{\substack{j=1 \\ \nu_j(M,\Sigma) > \frac{b^{n-j}-1}{2}}}^{n-1} \left| 1 - \frac{2\nu_j(M,\Sigma)+1}{b^{n-j}} \right| \eta_{b,2}^\sigma \left(\frac{M}{b^j} \right) \Bigg)^{\frac{1}{2}} + \mathcal{O}(1).$$

Proof. We define $\widetilde{\mathcal{V}}_{b,N}^{\sigma}$ to be the ordered set of the first N elements of $\widetilde{\mathcal{V}}_b^{\sigma}$. Then we have

$$\widetilde{\mathcal{V}}_{b,N}^{\sigma} = (\varphi_b^{\sigma}(m))_{m=0}^{M-1} \cup (1-\varphi_b^{\sigma}(m))_{m=0}^{M-1}.$$

We consider the slightly different set

$$\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)} = (\varphi_b^{\sigma}(m))_{m=0}^{M-1} \cup (1-1/b^n-\varphi_b^{\sigma}(m))_{m=0}^{M-1} = (\varphi_b^{\sigma}(m))_{m=0}^{M-1} \cup (\varphi_b^{\overline{\sigma}}(m))_{m=0}^{M-1}.$$

Note that $\varphi_b^{\overline{\sigma}}(m)$ means here that we always apply the permutation $\overline{\sigma}=\tau_b\circ\sigma$ to the first n digits of m, even if they are zero from a certain index on. With the same arguments as used in the proof of Lemma 2.3 we find

$$|L_{2,N}(\widetilde{\mathcal{V}}_{b,N}^{\sigma}) - L_{2,N}(\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)})| \leq 1. \tag{3.17}$$

We therefore compute $L_{2,N}(\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)})$ in the following. An application of Lemma 2.9 yields

$$\Delta_N(t,\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)}) = \Delta_M(t,\mathcal{V}_b^{\sigma}) + \Delta_M(t,\varphi_b^{\overline{\sigma}}).$$

Now we derive a useful relation between the discrepancy functions of the van der Corput sequence and the Hammersley point set. Note that the generalized Hammersley point set in base b with respect to the n-tuple $\Sigma=(\sigma,\sigma,\dots,\sigma)$ may also be written as

$$\mathcal{H}_{b,n}^{\Sigma} = \left\{\left(\varphi_b^{\sigma}(m), \frac{m}{b^n}\right) : m \in \{0,1,\dots,b^n-1\}\right\}.$$

Analogously we have for the generalized Hammersley point set with respect to $\Sigma^*=(\overline{\sigma},\overline{\sigma},\dots,\overline{\sigma})$ the alternative definition

$$\mathcal{H}_{b,n}^{\Sigma^*} = \left\{\left(\varphi_b^{\overline{\sigma}}(m), \frac{m}{b^n}\right) : m \in \{0,1,\dots,b^n-1\}\right\}.$$

Recall the definition of $t(n)$ from Remark 2.8. We have

$$\begin{aligned}\Delta_M(t,\mathcal{V}_b^{\sigma}) =& A_M([0,t),\mathcal{V}_b^{\sigma}) - Mt = A_M([0,t(n)),\mathcal{V}_b^{\sigma}) - Mt \\ =& A_{b^n}\left([0,t(n))\times\left[0,\frac{M}{b^n}\right),\mathcal{H}_{b,n}^{\sigma}\right) - b^n\frac{M}{b^n}t(n) + M(t(n)-t) \\ =& \Delta_{b^n}\left(t(n),\frac{M}{b^n},\mathcal{H}_{b,n}^{\sigma}\right) + M(t(n)-t).\end{aligned}$$

(see also [57, Lemma 2]) and in the same way we find

$$\Delta_M(t,\varphi_b^{\overline{\sigma}}) = \Delta_{b^n}\left(t(n),\frac{M}{b^n},\mathcal{H}_{b,n}^{\overline{\sigma}}\right) + M(t(n)-t). \tag{3.18}$$

Now we have

$$\begin{aligned}(L_{2,N}(\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)}))^2 &= \int_0^1 (\Delta_N(t,\widetilde{\mathcal{V}}_{b,N}^{\sigma,(2)}))^2 \,\mathrm{d}t \\ &= \int_0^1 (\Delta_M(t,\mathcal{V}_b^{\sigma}))^2\,\mathrm{d}t + \int_0^1 (\Delta_M(t,\mathcal{V}_b^{\overline{\sigma}}))^2\,\mathrm{d}t + 2\int_0^1 \Delta_M(t,\mathcal{V}_b^{\sigma})\Delta_M(t,\mathcal{V}_b^{\overline{\sigma}})\,\mathrm{d}t \\ &=: S_1 + S_2 + 2S_3.\end{aligned}$$

With the above relations we have

$$\begin{aligned}S_1 =& \int_0^1 \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right) + M(t(n)-t)\right)^2 \,\mathrm{d}t \\ =& \int_0^1 \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right)\right)^2 \mathrm{d}t + 2M\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right)(t(n)-t)\,\mathrm{d}t \\ &+ M^2\int_0^1 (t(n)-t)^2\,\mathrm{d}t =: A_1 + A_2 + A_3.\end{aligned}$$

At first we have

$$A_3 = M^2 \sum_{\lambda=1}^{b^n} \int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}} \left(\frac{\lambda}{b^n} - t\right)^2 \mathrm{d}t = \frac{M^2}{3b^{2n}}.$$

To calculate A_2 we apply Theorem 2.7 and the fact that

$$\sum_{\lambda=1}^{b^n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) = b^{n-1}\psi^{\sigma}_{b}\left(\frac{M}{b^j}\right)$$

(see [31, Lemma 2]) to write

$$\begin{aligned}A_2 =& 2M\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right)(t(n)-t)\,\mathrm{d}t \\ =& 2M\sum_{\lambda=1}^{b^n} \Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right)\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\left(\frac{\lambda}{b^n}-t\right)\mathrm{d}t \\ =& \frac{M}{b^{2n}}\sum_{\lambda=1}^{b^n}\sum_{j=1}^{n} \psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right) = \frac{M}{b^{2n}}b^{n-1}\sum_{j=1}^{n}\psi^{\sigma}_{b}\left(\frac{M}{b^j}\right) = \frac{M}{b^{n+1}}\sum_{j=1}^{n}\psi^{\sigma}_{b}\left(\frac{M}{b^j}\right).\end{aligned}$$

Similarly, we compute with Lemma 3.21

$$\begin{aligned}A_1 =& \frac{1}{b^n}\sum_{\lambda=1}^{b^n}\left(\Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}^{\Sigma}_{b,n}\right)\right)^2 = \frac{1}{b^n}\sum_{\lambda=1}^{b^n}\left(\sum_{j=1}^{n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\right)^2 \\ =& \frac{1}{b^n}\sum_{\lambda=1}^{b^n}\left(\sum_{j=1}^{n}\left(\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\right)^2 + \sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}\psi^{\sigma}_{b,\varepsilon_{j_1}(\lambda,M,\Sigma)}\left(\frac{M}{b^{j_1}}\right)\psi^{\sigma}_{b,\varepsilon_{j_2}(\lambda,M,\Sigma)}\left(\frac{M}{b^{j_2}}\right)\right) \\ =& \frac{1}{b}\sum_{j=1}^{n}\psi^{\sigma,(2)}_{b}\left(\frac{M}{b^j}\right) + \frac{1}{b^2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}\psi^{\sigma}_{b}\left(\frac{M}{b^{j_1}}\right)\psi^{\sigma}_{b}\left(\frac{M}{b^{j_2}}\right).\end{aligned}$$

Now we consider S_2 and write, similarly as above,

$$\begin{aligned}S_2 =& \int_0^1 \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma^*}_{b,n}\right) + M(t(n)-t)\right)^2 \mathrm{d}t = \int_0^1 \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma^*}_{b,n}\right)\right)^2 \mathrm{d}t \\ &+ 2M\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}^{\Sigma^*}_{b,n}\right)(t(n)-t)\,\mathrm{d}t + M^2\int_0^1 (t(n)-t)^2\,\mathrm{d}t \\ =&: B_1 + B_2 + B_3.\end{aligned}$$

With the same techniques as above and the identities $\psi^{\sigma}_{b} = -\psi^{\overline{\sigma}}_{b}$ and $\psi^{\sigma,(2)}_{b} = \psi^{\overline{\sigma},(2)}_{b}$ from Lemma 3.21 one can easily check that $B_1 = A_1$, $B_2 = -A_2$ and $B_3 = A_3$.

We turn to S_3 and get

$$\begin{aligned}S_3 =& \int_0^1 \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) + M(t(n)-t)\right)\\ &\times \left(\Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right) + M(t(n)-t)\right) \mathrm{d}t\\ =& \int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right) \mathrm{d}t\\ &+ M\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right)(t(n)-t)\,\mathrm{d}t\\ &+ M\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right)(t(n)-t)\,\mathrm{d}t\\ &+ M^2\int_0^1 (t(n)-t)^2\,\mathrm{d}t = \frac{1}{b^n}\sum_{\lambda=1}^{b^n} \Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) \Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right) + \frac{M^2}{3b^{2n}}.\end{aligned}$$

Here we regarded the fact that

$$\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right)(t(n)-t)\,\mathrm{d}t = -\int_0^1 \Delta_{b^n}\left(t(n), \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right)(t(n)-t)\,\mathrm{d}t$$

(compare with the computation of A_2 and B_2, respectively). We derive

$$\begin{aligned}&\frac{1}{b^n}\sum_{\lambda=1}^{b^n} \Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) \Delta_{b^n}\left(\frac{\lambda}{b^n}, \frac{M}{b^n}, \mathcal{H}_{b,n}^{\Sigma^*}\right)\\ =&\frac{1}{b^n}\sum_{\lambda=1}^{b^n}\left(\sum_{j=1}^{n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\right)\left(\sum_{j=1}^{n}\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\right)\\ =&\frac{1}{b^n}\sum_{\lambda=1}^{b^n}\left(\sum_{j=1}^{n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right)\right.\\ &\left.+\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}\psi^{\sigma}_{b,\varepsilon_{j_1}(\lambda,M,\Sigma)}\left(\frac{M}{b^{j_1}}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_{j_2}(\lambda,M,\Sigma^*)}\left(\frac{M}{b^{j_2}}\right)\right)\\ =&\frac{1}{b^n}\sum_{j=1}^{n}\sum_{\lambda=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right) - \frac{1}{b^2}\sum_{\substack{j_1,j_2=1\\ j_1\neq j_2}}^{n}\psi^{\sigma}_{b}\left(\frac{M}{b^{j_1}}\right)\psi^{\sigma}_{b}\left(\frac{M}{b^{j_2}}\right).\end{aligned}$$

Now we put all the results together and apply Lemma 3.22 to obtain

$$\begin{aligned}&(L_{2,N}(\widetilde{\mathcal{V}}^{\sigma,(2)}_{b,N}))^2\\ &=\frac{2}{b}\sum_{j=1}^{n}\psi_b^{\sigma,(2)}\left(\frac{M}{b^j}\right) + \frac{2}{b^n}\sum_{j=1}^{n}\sum_{\lambda=1}^{b^n}\psi^{\sigma}_{b,\varepsilon_j(\lambda,M,\Sigma)}\left(\frac{M}{b^j}\right)\psi^{\overline{\sigma}}_{b,\varepsilon_j(\lambda,M,\Sigma^*)}\left(\frac{M}{b^j}\right) + \frac{4M^2}{3b^{2n}}\\ &=\frac{2}{b}\sum_{j=1}^{n}\left(\psi_b^{\sigma,(2)}\left(\frac{M}{b^j}\right) + \widetilde{\psi}_b^{\sigma}\left(\frac{M}{b^j}\right)\right)\\ &\quad+\frac{2}{b}\sum_{\substack{j=1\\ \nu_j(M,\Sigma)\leq\frac{b^{n-j}-1}{2}}}^{n-1}\left(1-\frac{2\nu_j(M,\Sigma)+1}{b^{n-j}}\right)\left(\widetilde{\psi}^{\sigma}_{b,1}\left(\frac{M}{b^j}\right)-\widetilde{\psi}^{\sigma}_{b}\left(\frac{M}{b^j}\right)\right)\\ &\quad+\frac{2}{b}\sum_{\substack{j=1\\ \nu_j(M,\Sigma)>\frac{b^{n-j}-1}{2}}}^{n-1}\left(\frac{2\nu_j(M,\Sigma)+1}{b^{n-j}}-1\right)\left(\widetilde{\psi}^{\sigma}_{b,2}\left(\frac{M}{b^j}\right)-\widetilde{\psi}^{\sigma}_{b}\left(\frac{M}{b^j}\right)\right) + \frac{4M^2}{3b^{2n}}.\end{aligned}$$

Considering (3.17) completes the proof. □

The following lemma shows that it is not necessary to extend Theorem 3.29 to odd N in order to obtain results on the asymptotic behaviour of $L_{2,N}(\mathcal{V}_b^\sigma)$, since the difference between the L_2 discrepancies of the first N and the first $N+1$ elements of a sequence $\mathcal{S}$ is always small.

Lemma 3.30. *Let $L_{2,N}(\mathcal{S})$ be the L_2 discrepancy of an arbitrary sequence $\mathcal{S} = (x_n)_{n\geq 0}$ in $[0,1)$. For all $N \in \mathbb{N}$ we have*

$$|L_{2,N+1}(\mathcal{S}) - L_{2,N}(\mathcal{S})| \leq 1.$$

Proof. We consider a fixed subinterval $[0,t)$ of the unit interval $[0,1]$. Since $\Delta_N(t,\mathcal{S}) = A_N([0,t),\mathcal{S}) - Nt$ and $\Delta_{N+1}(t,\mathcal{S}) = A_{N+1}([0,t),\mathcal{S}) - (N+1)t$ and since we obviously have $A_N([0,t),\mathcal{S}) \leq A_{N+1}([0,t),\mathcal{S}) \leq A_N([0,t),\mathcal{S}) + 1$, we have

$$\Delta_N(t,\mathcal{S}) - t \leq \Delta_{N+1}(t,\mathcal{S}) \leq \Delta_N(t,\mathcal{S}) + 1 - t.$$

We derive $\left|\Delta_{N+1}(t,\mathcal{S}) - \Delta_N(t,\mathcal{S})\right| \leq 1$ and apply the inequality $||x| - |y|| \leq |x-y|$ to obtain

$$\left||\Delta_{N+1}(t,\mathcal{S})| - |\Delta_N(t,\mathcal{S})|\right| \leq 1$$

and therefore $|\Delta_{N+1}(t,\mathcal{S})| \leq |\Delta_N(t,\mathcal{S})| + 1$. Now with the Minkowski's inequality for the L_2 norm this yields

$$\begin{aligned} L_{2,N+1}(\mathcal{S}) =& \|\Delta_{N+1}(\cdot,\mathcal{S})\|_{L_2([0,1))} \leq \|\Delta_N(\cdot,\mathcal{S})\|_{L_2([0,1))} + 1 \\ &\leq \|\Delta_N(\cdot,\mathcal{S})\|_{L_2([0,1))} + 1 = L_{2,N}(\mathcal{S}) + 1. \end{aligned}$$

Analogously we can show $L_{2,N}(\mathcal{S}) \leq L_{2,N+1}(\mathcal{S}) + 1$, which completes the proof. □

Example 3.31. We would like to recover (2.18) from Theorem 3.29. Let $b = 2$ and $\sigma = id$. It is easy to check that $\eta_2^{id} = 0$ and $\eta_{2,1}^{id} = \eta_{2,2}^{id} = \|\cdot\|^2$. We consider the expression

$$\begin{aligned} \mathcal{M}_{2,id}(M) :=& \sum_{j=1}^{n} \eta_2^{id}\left(\frac{M}{2^j}\right) + \sum_{\substack{j=1 \\ \nu_j(M,\Sigma)\leq \frac{2^{n-j}-1}{2}}}^{n-1} \left|1 - \frac{2\nu_j(M,\Sigma)+1}{2^{n-j}}\right| \eta_{2,1}^{id}\left(\frac{M}{2^j}\right) \\ &+ \sum_{\substack{j=1 \\ \nu_j(M,\Sigma)> \frac{2^{n-j}-1}{2}}}^{n-1} \left|1 - \frac{2\nu_j(M,\Sigma)+1}{2^{n-j}}\right| \eta_{2,2}^{id}\left(\frac{M}{2^j}\right), \end{aligned}$$

which we can simplify to

$$\begin{aligned} \mathcal{M}_{2,id}(M) =& \sum_{\substack{j=1 \\ \nu_j(M,\Sigma)\leq \frac{2^{n-j}-1}{2}}}^{n-1} \left|1 - \frac{2\nu_j(M,\Sigma)}{2^{n-j}}\right| \left\|\frac{M}{2^j}\right\|^2 \\ &+ \sum_{\substack{j=1 \\ \nu_j(M,\Sigma)> \frac{2^{n-j}-1}{2}}}^{n-1} \left|1 - \frac{2\nu_j(M,\Sigma)}{2^{n-j}}\right| \left\|\frac{M}{2^j}\right\|^2 + \mathcal{O}(1). \end{aligned}$$

Since $\nu_j(M,\Sigma) = 2^{n-j}\{2^j\varphi_2(M)\}$, we obtain further

$$\begin{aligned}\mathcal{M}_{2,id}(M) = &\sum_{\substack{j=1\\ \{2^j\varphi_2(M)\}<\frac{1}{2}}}^{n-1} \left(1-2\|2^j\varphi_2(M)\|\right)\left\|\frac{M}{2^j}\right\|^2 \\ &+ \sum_{\substack{j=1\\ \{2^j\varphi_2(M)\}\geq\frac{1}{2}}}^{n-1} \left|1-2(1-\|2^j\varphi_2(M)\|)\right|\left\|\frac{M}{2^j}\right\|^2 + \mathcal{O}(1) \\ =&\sum_{j=1}^{n-1}(1-2\|2^j\varphi_2(M)\|)\left\|\frac{M}{2^j}\right\|^2 + \mathcal{O}(1).\end{aligned}$$

Finally, since $N=2M$ and $\varphi_2(M) = 2\varphi_2(2M)$, we find

$$\begin{aligned}\mathcal{M}_{2,id}(N) &= \sum_{j=1}^{n-1}(1-2\|2^{j+1}\varphi_2(N)\|)\left\|\frac{N}{2^{j+1}}\right\|^2 + \mathcal{O}(1) \\ &= \sum_{j=2}^{n}(1-2\|2^{j}\varphi_2(N)\|)\left\|\frac{N}{2^{j}}\right\|^2 + \mathcal{O}(1),\end{aligned}$$

where we may include the summand for $j=1$, since it is zero anyway. We have indeed recovered Faure's formula in a slightly less precise form (and only for even N).

We would like to derive several more specific results from Theorem 3.29. More precisely, we prove that Faure's best upper bound on the quantity

$$\inf_{\mathcal{S}\in[0,1)^{\mathbb{N}}} \limsup_{N\to\infty} \frac{L_{2,N}(\mathcal{S})}{\sqrt{\log N}}$$

(see (2.19)) can not be improved by considering the symmetrized van der Corput sequences $\widetilde{\mathcal{V}}_3$, $\widetilde{\mathcal{V}}_4$, $\widetilde{\mathcal{V}}_5$ and $\widetilde{\mathcal{V}}_6$. To this end, we prove lower bounds on $L_{2,N}(\widetilde{\mathcal{V}}_3)$, $L_{2,N}(\widetilde{\mathcal{V}}_4)$, $L_{2,N}(\widetilde{\mathcal{V}}_5)$ and $L_{2,N}(\widetilde{\mathcal{V}}_6)$, respectively. Therefore we evaluate the expression

$$\begin{aligned}\mathcal{M}_{b,\sigma}(M) :=&\frac{2}{b}\sum_{j=1}^{n}\eta_b^{\sigma}\left(\frac{M}{b^j}\right) + \frac{2}{b}\sum_{\substack{j=1\\ \nu_j(M,\Sigma)\leq\frac{b^{n-j}-1}{2}}}^{n-1}\left|1-\frac{2\nu_j(M,\Sigma)+1}{b^{n-j}}\right|\eta_{b,1}^{\sigma}\left(\frac{M}{b^j}\right) \\ &+ \frac{2}{b}\sum_{\substack{j=1\\ \nu_j(M,\Sigma)>\frac{b^{n-j}-1}{2}}}^{n-1}\left|1-\frac{2\nu_j(M,\Sigma)+1}{b^{n-j}}\right|\eta_{b,2}^{\sigma}\left(\frac{M}{b^j}\right).\end{aligned}$$

for those numbers $1\leq M\leq b^n$, for which we conjecture that the above expression is maximized.

- Let $b=3$ and $\sigma=id$. Computations with a computer algebra system lead to the conjecture that
$$\max_{1\leq M<3^n}\mathcal{M}_{3,id}(M)$$
is attained for $n=2k$ at $M_3(n)$ such that $M_3(n)/3^n = 0.0101\dots01$ (k times) in 3-ary representation. By inserting these numbers $M_3(n)$ into $\mathcal{M}_{3,id}(M)$, we obtain by Theorem 3.29
$$L_{2,2M_3(n)}(\widetilde{\mathcal{V}}_3) = \left(\frac{43}{384}n\right)^{\frac{1}{2}} + \mathcal{O}(1).$$

Thus, we have

$$L_{2,N}(\widetilde{\mathcal{V}}_3) \geq \left(\frac{43}{384\log 3}\right)^{\frac{1}{2}} \sqrt{\log N} + \mathcal{O}(1)$$

for infinitely many N and hence

$$l_2(\widetilde{\mathcal{V}}_3) = \limsup_{N\to\infty} \frac{L_{2,N}(\widetilde{\mathcal{V}}_3)}{\sqrt{\log N}} \geq \left(\frac{43}{384\log 3}\right)^{\frac{1}{2}} = 0.319261\ldots \tag{3.19}$$

We conjecture that we even have equality in (3.19). Note that this lower bound is already larger than the constant given in (2.19), and hence the sequence $\widetilde{\mathcal{V}}_3$ has certainly a higher asymptotic L_2 discrepancy than $\widetilde{\mathcal{V}}_2$.

- Now we choose $b = 4$ and $\sigma = id$. We conjecture that $\max_{1\leq M<4^n} \mathcal{M}_{4,id}(M)$ is attained for $n = 3k$ at $M_4(n)$ such that $M_4(n)/4^n = 0.011011\ldots011$ (k times) in 4-ary representation. Based on these integers we find

$$l_2(\widetilde{\mathcal{V}}_4) \geq \left(\frac{116288}{750141\log 4}\right)^{\frac{1}{2}} = 0.334402\ldots \tag{3.20}$$

  We conjecture again the equality in (3.20).

- For $b = 5$ and $\sigma = id$ we find that $\max_{1\leq M<5^n} \mathcal{M}_{5,id}(M)$ is probably attained at $M_5(n)$ such that $M_5(n)/5^n = 0.111\ldots111$ (n times) (in 5-ary representation). These integers lead to the lower bound

$$l_2(\widetilde{\mathcal{V}}_5) \geq \left(\frac{9}{40\log 5}\right)^{\frac{1}{2}} = 0.373899\ldots,$$

  where we conjecture this lower bound to be the exact value of the limes superior again.

- For $b = 6$ and $\sigma = id$ the maximum $\max_{1\leq M<6^n} \mathcal{M}_{6,id}(M)$ is probably attained at $M_6(n)$ such that $M_6(n)/6^n = 0.444\ldots444$ (n times) (in 6-ary representation). These integers lead to the lower bound

$$l_2(\widetilde{\mathcal{V}}_6) \geq \left(\frac{104}{375\log 6}\right)^{\frac{1}{2}} = 0.393424\ldots$$

  and again we conjecture this to be the exact value of the limes superior.

Open problems Although we provided a precise formula for $L_{2,N}(\widetilde{\mathcal{V}}_b^\sigma)$ in this section, it remains an open question how to extract exact values for $l_2(\widetilde{\mathcal{V}}_b^\sigma)$ (or at least good upper bounds) from it and how to determine those permutations which lead to an L_2 discrepancy as low as possible. We propose the following open problems:

Open Problem 3.32. The following problems are still unsolved:

- Prove Faure's conjecture on $l_2(\widetilde{\mathcal{V}}_2)$; i.e.

$$l_2(\widetilde{\mathcal{V}}_2) = \left(\frac{421}{6750\log 2}\right)^{\frac{1}{2}} = 0.299969\ldots$$

- Prove or disprove the above conjectures on $l_2(\widetilde{\mathcal{V}}_b)$ for $b = 3, 4, 5, 6$ and try to find a general formula for $l_2(\widetilde{\mathcal{V}}_b)$ for all bases $b \geq 2$. In particular, investigate whether $l_2(\widetilde{\mathcal{V}}_b)$ increases as b increases.

- Improve (2.19) by finding a base b and a permutation $\sigma \in \mathfrak{S}_b$ such that $l_2(\widetilde{\mathcal{V}}_b^\sigma) < l_2(\widetilde{\mathcal{V}}_2)$. It appears to be reasonable to try sequences $\widetilde{\mathcal{V}}_b$ for which it is known that their non-symmetrized versions $\mathcal{V}_b^\sigma$ have low diaphony. Examples of such sequences can be found in [11] and [56]. Note that from [11] we know that the sequences $\mathcal{V}_b^\sigma$ in bases $b = 3, 4, 5, 6$ have higher diaphony than $\mathcal{V}_2$ for all permutations. In base 7 there exists a permutation such that $\mathcal{V}_7^\sigma$ has a smaller diaphony than the dyadic van der Corput sequence, which is given by $\sigma = (1, 3)(2, 5)$. Since we assume that smaller bases are easier to handle than larger ones, we suggest to investigate $l_2(\widetilde{\mathcal{V}}_7^\sigma)$ for this particular permutation to begin with.

4. Optimal L_p discrepancy rate and beyond

4.1. Generalized and symmetrized Hammersley point sets

One of the central questions of interest in this thesis is to find conditions on the digital shift $\boldsymbol{\sigma} \in \{0,1\}^n$ or on the tuple $\Sigma \in \{\sigma, \overline{\sigma}\}^n$ which guarantee the optimal order of L_p discrepancy of the corresponding generalized Hammersley point sets. Section 3.1.1 was a first step towards a solution of this problem. It turned out however that a proof based on the exact formula of the discrepancy function from Theorem 2.5 yields exact discrepancy results for certain shifts, whereas this approach seems to be useless in order to characterize good digital shifts in general. The reasons for this drawback are twofold:

1. The fact that the discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ depends only on the number of zero digits in $\boldsymbol{\sigma}$ seems to be a peculiarity of the L_2 discrepancy. This means that there exists no simple general formula for $L_p(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ for an arbitrary shift in the style of Theorem 2.1, from which one can read off desired conditions on the shifts for low discrepancy.
2. It seems hardly possible to prove an exact formula for $L_p(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ in the first place, since the combinatorial aspects of the proof would be too hard to handle.

Surprisingly, there exist tools from harmonic analysis which provide quite simple possibilities to investigate the L_p discrepancy of point sets and sequences. These tools include Haar functions, Littlewood-Paley theory and embedding theorems between Besov and Triebel-Lizorkin spaces of dominating mixed smoothness, as they have been outlined in Section 2.3. We will exploit these tools throughout this section and also the subsequent one.

4.1.1. Optimal order of L_p discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ and $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$

In this section we demonstrate the general method how to prove upper bounds on the L_p discrepancy of point sets on the example of digit shifted Hammersley point sets based on Proposition 2.10. The outline of the proof is taken from a joint work with Hinrichs and Pillichshammer [39]. We rely heavily on estimates of the Haar coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$ of $\Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ from [36, Theorem 3.1]. Note that Hinrichs worked with the normalized version of the discrepancy function; here however we state a non-normalized version of his result.

Lemma 4.1 ([36, Theorem 3.1]). *Let $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$. Then*

(i) if $j_1 + j_2 < n - 1$ and $j_1, j_2 \geq 0$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-n-2}$.

(ii) *if* $j_1 + j_2 \geq n - 1$ *and* $0 \leq j_1, j_2 \leq n$ *then* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \leq 2^{-j_1-j_2-1}$ *and* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{n-2j_1-2j_2-4}$ *for all but at most* 2^n *coefficients* $\mu_{\boldsymbol{j},\boldsymbol{m}}$ *with* $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ *(the latter appears if there is no point of* $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ *in the interior of* $I_{\boldsymbol{j},\boldsymbol{m}}$*).*

(iii) *if* $j_1 \geq n$ *or* $j_2 \geq n$ *then* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{n-2j_1-2j_2-4}$.

Now let $\boldsymbol{j} = (-1, k)$ *or* $\boldsymbol{j} = (k, -1)$ *with* $k \in \mathbb{N}_0$*. Then*

(iv) *if* $k < n$ *then* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \leq 2^{-k}$.

(v) *if* $k \geq n$ *then* $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{n-2k-3}$.

Finally, if $l = |\{j \ : \ \sigma_j = 0\}|$ *as in Section 2.1 then*

(vi) $\mu_{(-1,-1),(0,0)} = \frac{1}{8}(2l + 4 - n) + 2^{-n-2}$.

We use these bounds on the Haar coefficients to show the following result, which classifies all the digital shifts $\boldsymbol{\sigma}$ such that the L_p discrepancy of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ is of optimal order according to the lower bound by Roth and Schmidt. It is remarkable that the condition on $\boldsymbol{\sigma}$ is the same for all $p \in [1, \infty)$. Consequently, the given condition is the same that was found in (2.6) to assure the optimal order of L_2 discrepancy for the digit shifted Hammersley point set.

Theorem 4.2. *Let* $p \in (1, \infty)$*. Let* $\boldsymbol{\sigma} \in \{0,1\}^n$ *and* $l = |\{j \ : \ \sigma_j = 0\}|$*. The* L_p *discrepancy of the two-dimensional digit shifted Hammersley point set satisfies*

$$L_p(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) \lesssim_p \sqrt{\log N}$$

if and only if $|2l - n| \lesssim_p \sqrt{n}$*.*

Remark 4.3. It follows from the monotonicity of the L_p norm that $|2l - n| \lesssim \sqrt{n}$ also implies $L_1(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) \lesssim \sqrt{\log N}$, which is best possible according to the result of Halász (1.14).

Proof. First we show the sufficiency of the condition. Using Proposition 2.10 from Section 2.3.2 (the Littlewood-Paley inequality) with $f = \Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ we have

$$\begin{aligned} L_{p,N}(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) &= \|\Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma}))\|_{L_p([0,1)^2)} \\ &\lesssim_p \|S(\Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma})))\|_{L_p([0,1)^2)} \\ &= \left\| \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right)^{1/2} \right\|_{L_p([0,1)^2)} \\ &= \left\| \sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} 2^{2|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)}^{1/2} \\ &\leq \left(\sum_{\boldsymbol{j} \in \mathbb{N}_{-1}^2} 2^{2|\boldsymbol{j}|} \left\| \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \right)^{1/2}, \end{aligned}$$

where we used Minkowski's inequality for the $L_{p/2}$ norm. Hence, in order to prove the result it suffices to show that

$$\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^2} 2^{2|\boldsymbol{j}|} \left\| \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \lesssim n. \tag{4.1}$$

To this end we split the sum over the $\boldsymbol{j}$'s into several parts and apply Lemma 4.1:

- $\boldsymbol{j} \in \mathbb{N}_0^2$ such that $|\boldsymbol{j}| < n-1$: According to (i) of Lemma 4.1 we have

$$\begin{aligned}\sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^2\\|\boldsymbol{j}|<n-1}} 2^{2|\boldsymbol{j}|} \left\| \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} &= \sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^2\\|\boldsymbol{j}|<n-1}} 2^{2|\boldsymbol{j}|} 2^{-2n-4} \left\| \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \\ &= \sum_{\substack{\boldsymbol{j}\in\mathbb{N}_0^2\\|\boldsymbol{j}|<n-1}} 2^{2|\boldsymbol{j}|} 2^{-2n-4} \\ &\lesssim \frac{1}{2^{2n}} \sum_{k=0}^{n-2} 2^{2k} \underbrace{\sum_{\substack{j_1,j_2=0\\j_1+j_2=k}}^{\infty} 1}_{=k+1\leq n-1} \lesssim n.\end{aligned}$$

  Here we used that for fixed $\boldsymbol{j}$ the intervals $I_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ form a partition of the unit square $[0,1)^2$ and hence $\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} = 1$.

- $|\boldsymbol{j}| \geq n-1$ and $0 \leq j_1, j_2 \leq n$: Let $I_{\boldsymbol{j},\boldsymbol{m}}^\circ$ denote the interior of a dyadic box $I_{\boldsymbol{j},\boldsymbol{m}}$. According to (ii) of Lemma 4.1 we have

$$\begin{aligned}&\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} \left\| \sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \\ &= \sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} \left\| \sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ \mathcal{H}_{2,n}(\boldsymbol{\sigma})\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ=\emptyset}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} + \sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ \mathcal{H}_{2,n}(\boldsymbol{\sigma})\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ\neq\emptyset}} \mu_{\boldsymbol{j},\boldsymbol{m}}^2 \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \\ &\leq \sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} 2^{2n-4|\boldsymbol{j}|-8} + \sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} 2^{-2|\boldsymbol{j}|-2} \left\| \sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ \mathcal{H}_{2,n}(\boldsymbol{\sigma})\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ\neq\emptyset}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)},\end{aligned}$$

  where we used Minkowski's inequality again. For the first sum in this estimate we have

$$\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} 2^{2n-4|\boldsymbol{j}|-8} \lesssim \sum_{k=n-1}^{2n} \frac{2^{2n}}{2^{2k}} \sum_{\substack{j_1,j_2=0\\j_1+j_2=k}}^{n} 1 = \sum_{k=n-1}^{2n} \frac{2^{2n}}{2^{2k}} \sum_{\substack{j_1=0\\0\leq k-j_1\leq n}}^{n} 1 \lesssim n.$$

  Now we turn to the second sum

$$\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|} 2^{-2|\boldsymbol{j}|-2} \left\| \sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}\\ \mathcal{H}_{2,n}(\boldsymbol{\sigma})\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ\neq\emptyset}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)}. \tag{4.2}$$

Note that

$$\sum_{\substack{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}} \\ \mathcal{H}_{2,n}(\boldsymbol{\sigma})\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}\neq\emptyset}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}$$

is the indicator function of a set, say $A_{\boldsymbol{j}}$, of measure at most $2^{n-|\boldsymbol{j}|}$. Hence (4.2) can be written as

$$\begin{aligned}\frac{1}{4}\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n}\left\|\mathbf{1}_{A_{\boldsymbol{j}}}\right\|_{L_{p/2}([0,1)^2)} &= \frac{1}{4}\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n}\left(\int_{[0,1]^2}\mathbf{1}_{A_{\boldsymbol{j}}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right)^{2/p} \lesssim \sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n}(2^{n-|\boldsymbol{j}|})^{2/p}\\ &= 2^{2n/p}\sum_{k=n-1}^{2n}\frac{1}{2^{2k/p}}\sum_{\substack{j_1,j_2=0\\j_1+j_2=k}}^{n}1 \lesssim 2^{2n/p}\sum_{k=n-1}^{2n}\frac{n}{2^{2k/p}}\lesssim n.\end{aligned}$$

Altogether we obtain that

$$\sum_{\substack{j_1,j_2=0\\|\boldsymbol{j}|\geq n-1}}^{n} 2^{2|\boldsymbol{j}|}\left\|\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2\,\mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}\right\|_{L_{p/2}([0,1)^2)} \lesssim n$$

as desired.

- $\boldsymbol{j}\in\mathbb{N}_0^2$, $j_1\geq n$: According to *(iii)* of Lemma 4.1 we have

$$\begin{aligned}&\sum_{j_2=0}^{\infty}\sum_{j_1=n}^{\infty}2^{2|\boldsymbol{j}|}\left\|\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mu_{\boldsymbol{j},\boldsymbol{m}}^2\,\mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}\right\|_{L_{p/2}([0,1)^2)}\\ &\quad= \sum_{j_2=0}^{\infty}\sum_{j_1=n}^{\infty}2^{2|\boldsymbol{j}|}2^{2n-4|\boldsymbol{j}|-8}\left\|\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}\mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}}\right\|_{L_{p/2}([0,1)^2)}\\ &\quad= 2^{2n}\sum_{j_2=0}^{\infty}\sum_{j_1=n}^{\infty}2^{-2|\boldsymbol{j}|-8}\lesssim 1.\end{aligned}$$

- $\boldsymbol{j}\in\mathbb{N}_0^2$, $j_2\geq n$: Analogous to the case $\boldsymbol{j}\in\mathbb{N}_0^2$, $j_1\geq n$.

- $\boldsymbol{j}=(-1,k)$ with $k\in\mathbb{N}_0$ and $0\leq k<n$: According to *(iv)* of Lemma 4.1 we have

$$\begin{aligned}&\sum_{k=0}^{n-1}2^{2k}\left\|\sum_{\boldsymbol{m}\in\mathbb{D}_{(-1,k)}}\mu_{(-1,k),\boldsymbol{m}}^2\,\mathbf{1}_{I_{(-1,k),\boldsymbol{m}}}\right\|_{L_{p/2}([0,1)^2)}\\ &\quad\leq \sum_{k=0}^{n-1}2^{2k}2^{-2k}\left\|\sum_{\boldsymbol{m}\in\mathbb{D}_{(-1,k)}}\mathbf{1}_{I_{(-1,k),\boldsymbol{m}}}\right\|_{L_{p/2}([0,1)^2)}\\ &\quad= \sum_{k=0}^{n-1}1 = n.\end{aligned}$$

- $\boldsymbol{j}=(k,-1)$ with $k\in\mathbb{N}_0$ and $0\leq k<n$: Analogous to the case $\boldsymbol{j}=(-1,k)$ with $k\in\mathbb{N}_0$ and $0\leq k<n$.

- $\boldsymbol{j} = (-1, k)$ with $k \in \mathbb{N}_0$ and $k \geq n$: According to (v) of Lemma 4.1 we have

$$\begin{aligned}
&\sum_{k=n}^{\infty} 2^{2k} \left\| \sum_{\boldsymbol{m} \in \mathbb{D}_{(-1,k)}} \mu^2_{(-1,k),\boldsymbol{m}} \mathbf{1}_{I_{(-1,k),\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \\
&= \sum_{k=n}^{\infty} 2^{2k} 2^{2n-4k-6} \left\| \sum_{\boldsymbol{m} \in \mathbb{D}_{(-1,k)}} \mathbf{1}_{I_{(-1,k),\boldsymbol{m}}} \right\|_{L_{p/2}([0,1)^2)} \\
&\lesssim 2^{2n} \sum_{k=n}^{\infty} 2^{-2k} \lesssim 1.
\end{aligned}$$

- $\boldsymbol{j} = (k, -1)$ with $k \in \mathbb{N}_0$ and $k \geq n$: Analogous to the case $\boldsymbol{j} = (-1, k)$ with $k \in \mathbb{N}_0$ and $k \geq n$.

- $\boldsymbol{j} = (-1, -1)$: According to (vi) of Lemma 4.1 we have

$$\begin{aligned}
\left\| \mu^2_{(-1,-1),(0,0)} \mathbf{1}_{[0,1]^2} \right\|_{L_{p/2}([0,1)^2)} &= \mu^2_{(-1,-1),(0,0)} \| \mathbf{1}_{[0,1]^2} \|_{L_{p/2}([0,1)^2)} \\
&= \left(\frac{1}{8}(2l + 4 - n) + 2^{-n-2} \right)^2. \qquad (4.3)
\end{aligned}$$

Now we assume that $|2l - n| \lesssim \sqrt{n}$. Then we have

$$\left\| \mu^2_{(-1,-1),(0,0)} \mathbf{1}_{[0,1]^2} \right\|_{L_{p/2}([0,1)^2)} \lesssim n.$$

Altogether this proves inequality (4.1) and therefore also the first point of Theorem 4.2. It remains to show that the condition on l is also necessary. We use again Lemma 2.10 with $f = \Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ and obtain

$$\begin{aligned}
L_{p,N}(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) &= \| \Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma})) \|_{L_p([0,1)^2)} \\
&\gtrsim_p \| S(\Delta_N(\cdot, \mathcal{H}_{2,n}(\boldsymbol{\sigma}))) \|_{L_p([0,1)^2)} \\
&= \left\| \left(\sum_{\boldsymbol{j} \in \mathbb{N}^2_{-1}} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} 2^{2|\boldsymbol{j}|} \mu^2_{\boldsymbol{j},\boldsymbol{m}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right)^{1/2} \right\|_{L_p([0,1)^2)} \\
&= \left\| \sum_{\boldsymbol{j} \in \mathbb{N}^2_{-1}} 2^{2|\boldsymbol{j}|} \sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}} \mu^2_{\boldsymbol{j},\boldsymbol{m}} \mathbf{1}_{I_{\boldsymbol{j},\boldsymbol{m}}} \right\|^{1/2}_{L_{p/2}([0,1)^2)} \\
&\gtrsim \left\| \mu^2_{(-1,-1),(0,0)} \mathbf{1}_{[0,1]^2} \right\|^{1/2}_{L_{p/2}([0,1)^2)} \\
&= \left| \frac{1}{8}(2l + 4 - n) + 2^{-n-2} \right|,
\end{aligned}$$

where the last equality follows from (4.3). From this it is evident that

$$L_{p,N}(\mathcal{H}_{2,n}(\boldsymbol{\sigma})) \lesssim_p \sqrt{\log N} \asymp \sqrt{n}$$

implies $|2l - n| \lesssim_p \sqrt{n}$. □

We employ the same method for the symmetrized Hammersley point sets $\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma})$. We use a simple trick to find bounds on the Haar coefficients $\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}$ of $\Delta_N(\cdot,\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))$.

Lemma 4.4. *Let $\boldsymbol{j}=(j_1,j_2)\in\mathbb{N}_{-1}^2$. Then in the case $\boldsymbol{j}\neq(-1,-1)$ we have*

$$|\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|\leq 2|\mu_{\boldsymbol{j},\boldsymbol{m}}| \quad \text{for all } \boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}},$$

where $\mu_{\boldsymbol{j},\boldsymbol{m}}$ are the Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$. Hence the results in Lemma 4.1 apply accordingly also to $|\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|$ (the additional factor 2 does not influence the order of magnitude in n). In the case $\boldsymbol{j}=(-1,-1)$ we have $\tilde{\mu}_{(-1,-1),(0,0)}=1+2^{-n-1}$.

Proof. By Lemma 2.9 we have

$$\Delta_{2^{n+1}}(\boldsymbol{t},\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))=\Delta_{2^n}(\boldsymbol{t},\mathcal{H}_{2,n}(\boldsymbol{\sigma}))+\Delta_{2^n}(\boldsymbol{t},\mathcal{H}_{2,n}(\boldsymbol{\sigma}^*)).$$

Regarding the linearity of integration, we obtain

$$\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}=\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}+\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}^*}, \tag{4.4}$$

where here we write $\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}$ for the the Haar coefficients of the discrepancy function of $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ in order to stress the dependence on the digit shift $\boldsymbol{\sigma}$ and accordingly for $\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}^*}$. Then the triangle inequality yields

$$|\tilde{\mu}_{\boldsymbol{j},\boldsymbol{m}}|\leq|\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}|+|\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}^*}|.$$

We analyze the case $\boldsymbol{j}\neq(-1,-1)$. We note that the identities and upper bounds for $|\mu_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{\sigma}}|$ in Lemma 4.1 do not depend on the shift $\boldsymbol{\sigma}$ and therefore we get our desired results in this case directly from this lemma. In case that $\boldsymbol{j}=(-1,-1)$ we observe that the shift $\boldsymbol{\sigma}^*$ has $n-l$ zero entries if $\boldsymbol{\sigma}$ has l zero entries, and thus the result in this case follows immediately from (4.4) and Lemma 4.1. □

Theorem 4.5. *Let $p\in[1,\infty)$. Independently of $\boldsymbol{\sigma}\in\{0,1\}^n$ the two-dimensional symmetrized digit shifted Hammersley point set satisfies*

$$L_{p,N}(\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))\lesssim_p\sqrt{\log N}.$$

Proof. It suffices to consider $p>1$. Since the absolute values of the Haar coefficients of $\Delta_N(\cdot,\widetilde{\mathcal{H}}_{2,n}(\boldsymbol{\sigma}))$ are equal or less than two times the absolute values of the Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n}(\boldsymbol{\sigma}))$ and since $\tilde{\mu}_{(-1,-1),(0,0)}$ is bounded in n for every digital shift $\boldsymbol{\sigma}$, the proof of this theorem follows the same lines as the proof of Theorem 4.2. □

Theorem 4.5 states that we do not need any conditions on $\boldsymbol{\sigma}$ at all to assure the optimal order of L_p discrepancy for the symmetrized L_p discrepancy. This is in accordance to the results of Proinov on the L_2 discrepancy and our exact formula as stated in Theorem 3.13.

4.1.2. Generalizations to arbitrary bases and discrepancy in spaces with dominating mixed smoothness

In the previous section we were able to fully classify the digital shifts for which

$$L_{p,N}(\mathcal{H}_{2,n}(\boldsymbol{\sigma}))=\mathcal{O}(\sqrt{\log N}).$$

We observed that the required condition on $\boldsymbol{\sigma}$ is the same that follows already from the exact formula of Kritzer and Pillichhammer for the L_2 discrepancy. It is therefore reasonable to assume that the conditions on Σ to achieve the optimal order for $L_{p,N}(\mathcal{H}_{b,n}^{\Sigma})$ match the corresponding conditions for the L_2 discrepancy. We have stated these conditions in the lines after Theorem 2.2. In the dyadic case from the previous section, we found that the only delicate Haar coefficient is the one where $\boldsymbol{j}=(-1,-1)$, whereas the other coefficients have the desired order independently of $\boldsymbol{\sigma}$. We hope to find the same in the b-adic case and therefore investigate the said first Haar coefficient $\mu_{(-1,-1),(0,0),(1,1)}(\Delta_N(\cdot,\mathcal{H}_{b,n}^{\Sigma}))$ in the following lemma.

Lemma 4.6. *Let $n\in\mathbb{N}$, $\sigma\in\mathfrak{S}_b$, $\Sigma=(\sigma_1,\dots,\sigma_n)\in\{\sigma,\overline{\sigma}\}^n$ and $l=|\{i\in\{1,\dots,n\}:\sigma_i=\sigma\}|$ as defined in Section 2.1. Then we have*

$$\mu_{(-1,-1),(0,0),(1,1)}(\Delta_N(\cdot,\mathcal{H}_{b,n}^{\Sigma}))=(n-2l)\left(\frac{(b-1)^2}{4b}-\frac{1}{b^2}\sum_{a=0}^{b-1}\sigma(a)a\right)+\frac{1}{2}+\frac{1}{4N},$$

where $N=b^n$ is the number of elements in $\mathcal{H}_{b,n}^{\Sigma}$.

Proof. We denote the points of $\mathcal{H}_{b,n}^{\Sigma}$ by $\{\boldsymbol{x}_0,\dots,\boldsymbol{x}_{N-1}\}$, where $\boldsymbol{x}_r=(x_r^{(1)},x_r^{(2)})$ for all $r\in\{0,1,\dots,N-1\}$. We have

$$\begin{aligned}\mu_{(-1,-1),(0,0),(1,1)}&=\int_0^1\int_0^1\Delta_N(t_1,t_2,\mathcal{H}_{b,n}^{\Sigma})\,\mathrm{d}t_1\,\mathrm{d}t_2\\&=\sum_{r=0}^{N-1}\int_0^1\int_0^1\mathbf{1}_{[0,t_1)\times[0,t_2)}(\boldsymbol{x}_r)\,\mathrm{d}t_1\,\mathrm{d}t_2-N\int_0^1\int_0^1 t_1t_2\,\mathrm{d}t_1\,\mathrm{d}t_2\\&=\sum_{r=0}^{N-1}(1-x_r^{(1)})(1-x_r^{(2)})-\frac{N}{4}\\&=\frac{3N}{4}-\sum_{r=0}^{N-1}x_r^{(1)}-\sum_{r=0}^{N-1}x_r^{(2)}+\sum_{r=0}^{N-1}x_r^{(1)}x_r^{(2)}\\&=\frac{3N}{4}-2\sum_{r=0}^{N-1}\frac{r}{N}+\sum_{r=0}^{N-1}x_r^{(1)}x_r^{(2)}\\&=\frac{3N}{4}-(N-1)+\sum_{r=0}^{N-1}x_r^{(1)}x_r^{(2)}\\&=-\frac{N}{4}+1+\sum_{r=0}^{N-1}x_r^{(1)}x_r^{(2)}.\end{aligned}\tag{4.5}$$

We regarded the obvious fact that $\sum_{r=0}^{N-1}x_r^{(1)}=\sum_{r=0}^{N-1}x_r^{(2)}=\sum_{r=0}^{N-1}\frac{r}{N}$. It remains to investigate the sum $S:=\sum_{r=0}^{N-1}x_r^{(1)}x_r^{(2)}$. We have

$$S=\sum_{a_1,\dots,a_n=0}^{b-1}\left(\frac{\sigma_n(a_n)}{b}+\dots+\frac{\sigma_1(a_1)}{b^n}\right)\left(\frac{a_1}{b}+\dots+\frac{a_n}{b^n}\right)=\sum_{a_1,\dots,a_n=0}^{b-1}\sum_{k_1,k_2=1}^{n}\frac{\sigma_{k_1}(a_{k_1})a_{k_2}}{b^{n+1-k_1}b^{k_2}}.$$

Next we distinguish between the cases where $k_1=k_2=k$ and where $k_1\neq k_2$ and change the orders of the sums, which results in

$$S=\underbrace{\sum_{k=1}^{n}b^{n-1}\sum_{a_k=0}^{b-1}\frac{\sigma_k(a_k)a_k}{b^{n+1}}}_{S_1}+\underbrace{\sum_{\substack{k_1,k_2=1\\k_1\neq k_2}}^{n}b^{n-2}\sum_{a_{k_1},a_{k_2}=0}^{b-1}\frac{\sigma_{k_1}(a_{k_1})a_{k_2}}{b^{n+1-k_1}b^{k_2}}}_{S_2}.$$

The factors b^{n-1} and b^{n-2} come from the fact that the summands of S_1 and S_2 only depend on the digit a_k or on the digits a_{k_1} and a_{k_2}, respectively, and hence the sums over the remaining digits give a factor b each. Now we have

$$\begin{aligned} S_1 &= \frac{1}{b^2}\sum_{\substack{k=1\\ \sigma_k=\sigma}}^{n}\sum_{a_k=0}^{b-1}\sigma(a_k)a_k + \frac{1}{b^2}\sum_{\substack{k=1\\ \sigma_k=\overline{\sigma}}}^{n}\sum_{a_k=0}^{b-1}\overline{\sigma}(a_k)a_k \\ &= \frac{l}{b^2}\sum_{a=0}^{b-1}\sigma(a)a + \frac{(n-l)}{b^2}\sum_{a=0}^{b-1}\overline{\sigma}(a)a \\ &= \frac{l}{b^2}\sum_{a=0}^{b-1}\sigma(a)a + \frac{(n-l)}{b^2}\sum_{a=0}^{b-1}(b-1-\sigma(a))a \\ &= \frac{l-(n-l)}{b^2}\sum_{a=0}^{b-1}\sigma(a)a + \frac{n-l}{b^2}(b-1)\sum_{a=0}^{b-1}a \\ &= \frac{2l-n}{b^2}\sum_{a=0}^{b-1}\sigma(a)a + \frac{n-l}{2b}(b-1)^2 \end{aligned}$$

and

$$\begin{aligned} S_2 &= \frac{1}{b^3}\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{n} b^{k_1-k_2}\left(\sum_{a_{k_1}=0}^{b-1}\sigma_{k_1}(a_{k_1})\right)\left(\sum_{a_{k_2}=0}^{b-1}a_{k_2}\right) \\ &= \frac{1}{b^3}\left(\frac{b(b-1)}{2}\right)^2\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{n} b^{k_1-k_2}. \end{aligned}$$

straightforward algebra yields

$$\sum_{\substack{k_1,k_2=1\\ k_1\neq k_2}}^{n} b^{k_1-k_2} = \sum_{k_1,k_2=1}^{n} b^{k_1-k_2} - \sum_{k=1}^{n}1 = \frac{b}{(b-1)^2}(b^n+b^{-n}-2)-n,$$

which leads to

$$S_2 = \frac{1}{4}\left(\frac{1}{N}+N-2-\frac{(b-1)^2}{b}n\right).$$

Altogether we find

$$\begin{aligned} S &= \frac{2l-n}{b^2}\sum_{a=0}^{b-1}\sigma(a)a + \frac{n-l}{2b}(b-1)^2 + \frac{1}{4}\left(\frac{1}{N}+N-2-\frac{(b-1)^2}{b}n\right) \\ &= (n-2l)\left(\frac{(b-1)^2}{4b} - \frac{1}{b^2}\sum_{a=0}^{b-1}\sigma(a)a\right) + \frac{1}{4N} + \frac{N}{4} - \frac{1}{2}. \end{aligned}$$

Inserting this into (4.5) yields the formula for $\mu_{(-1,-1),(0,0),(1,1)}$. □

Remark 4.7. We note that the Haar coefficient $\mu_{(-1,-1),(0,0),(1,1)}$ is of order $\log N$ in general. It can be reduced to the order $(\log N)^{\frac{1}{q}}$ (for $1 \leq q \leq \infty$) however by either choosing l such that $|2l-n| = \mathcal{O}(n^{\frac{1}{q}})$ or by choosing the permutation σ such that

$$\frac{1}{b}\sum_{a=0}^{b-1}\sigma(a)a = \frac{(b-1)^2}{4}.$$

In the latter case we even have $\mu_{(-1,-1),(0,0),(1,1)} = \mathcal{O}(1)$. We remark that these are exactly the conditions which appear in the subsequent Theorem 4.17 to assure the optimal $S_{p,q}^r B$-discrepancy for the digit scrambled Hammersley point set for $r=0$.

Remark 4.8. Lemma 4.6 can also be proven with aid of Theorem 2.7. We have

$$\begin{aligned}\int_0^1 \Delta_N(\boldsymbol{t},\mathcal{H}_{b,n}^\sigma)\,\mathrm{d}\boldsymbol{t} &= \sum_{\lambda,M=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{M-1}{b^n}}^{\frac{M}{b^n}}\left(\Delta_N\left(\frac{\lambda}{b^n},\frac{M}{b^n},\mathcal{H}_{b,n}^\sigma\right)+b^n\left(\frac{\lambda M}{b^{2n}}-t_1t_2\right)\right)\mathrm{d}\boldsymbol{t}\\ &=\frac{1}{b^{2n}}\sum_{\lambda,M=1}^{b^n}\sum_{j=1}^{n}\varphi_{b,\varepsilon_j}^{\sigma_j}\left(\frac{M}{b^n}\right)+b^n\sum_{\lambda,M=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{M-1}{b^n}}^{\frac{M}{b^n}}\left(\frac{\lambda M}{b^{2n}}-t_1t_2\right)\mathrm{d}\boldsymbol{t}\\ &=\sum_{j=1}^{n}\Phi_b^{\sigma_j}+\frac{1}{2}+\frac{1}{4b^n},\end{aligned}$$

where we considered the identities $\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_j}^{\sigma}\left(\frac{M}{b^n}\right)=b^{n-1}\varphi_b^\sigma\left(\frac{M}{b^n}\right)$ (see [31, Lemma 2]) and $\sum_{M=1}^{b^n}\varphi_b^\sigma\left(\frac{M}{b^n}\right)=b^{n+1}\Phi_b^\sigma$ (see [31, Lemma 4]). We conclude (regarding $\Phi_b^{\overline{\sigma}}=-\Phi_b^\sigma$)

$$\sum_{j=1}^{n}\Phi_b^{\sigma_j}=\sum_{\substack{j=1\\ \sigma_j=\sigma}}^{n}\Phi_b^\sigma+\sum_{\substack{j=1\\ \sigma_j=\overline{\sigma}}}^{n}\Phi_b^{\overline{\sigma}}=l\Phi_b^\sigma-(n-l)\Phi_b^\sigma=(2l-n)\Phi_b^\sigma,$$

which yields

$$\int_0^1 \Delta_N(\boldsymbol{t},\mathcal{H}_{b,n}^\sigma)\,\mathrm{d}\boldsymbol{t}=(2l-n)\Phi_b^\sigma+\frac{1}{2}+\frac{1}{4b^n}.$$

We recover Lemma 4.6 by considering [31, Lemma 5]. However, in order to keep the computations in Chapter 4 independent of Faure's apparatus, we have given a direct proof of this lemma above.

Remark 4.9. Lemma 4.6 leads to a new proof of the formula for the L_1 discrepancy of the classical Hammersley point set $\mathcal{H}_{b,n}$. It is known that the discrepancy function of $\mathcal{H}_{b,n}$ is nonnegative on the whole unit square; i.e. $\Delta_N(\boldsymbol{t},\mathcal{H}_{b,n})\geq 0$ for all $\boldsymbol{t}\in[0,1)^2$. This follows from Theorem 2.7 and Remark 2.8. As a result we have

$$L_{1,N}(\mathcal{H}_{b,n})=\int_{[0,1)^2}\Delta_N(\boldsymbol{t},\mathcal{H}_{b,n})\,\mathrm{d}\boldsymbol{t}=\mu_{(-1,-1),(0,0),(1,1)}(\Delta_N(\cdot,\mathcal{H}_{b,n})).$$

Choosing $\sigma=id$ and $l=n$ in Lemma 4.6 leads to

$$\mu_{(-1,-1),(0,0),(1,1)}(\Delta_N(\cdot,\mathcal{H}_{b,n}))=n\frac{b^2-1}{12b}+\frac{1}{2}+\frac{1}{4b^n}$$

and we recover (2.3). Of course, the proof of this result would be simpler by restricting to $\sigma=id$ and $l=n$ in the proof of Lemma 4.6 in the first place.

It is necessary to have a look on the remaining Haar coefficients where $\boldsymbol{j}\neq(-1,-1)$. To this end, we can mainly rely on Markhasin's computations in [50] and [52], where he studied the case $\Sigma\in\{id,\tau_b\}^n$. To start with, we state a lemma that can also be found in [50, Lemma 4.2, 4.3]. We state an one-dimensional version of this lemma together with its proof in Lemma 4.34.

Lemma 4.10. *Let $f(\boldsymbol{t})=b^n t_1 t_2$ for $\boldsymbol{t}=(t_1,t_2)\in[0,1)^2$. Let $\boldsymbol{j}\in\mathbb{N}_{-1}^2$, $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$, $\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}$ and let $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f)$ be the b-adic Haar coefficient of f. Then*

1. *If* $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$*, then*

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f) = \frac{b^{n-2j_1-2j_2-2}}{\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_2} - 1\right)}.$$

2. *If* $\boldsymbol{j} = (j_1, -1)$ *or* $\boldsymbol{j} = (-1, j_2)$ *with* $j_1 \in \mathbb{N}_0$ *or* $j_2 \in \mathbb{N}_0$*, then*

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(f) = \frac{1}{2}\frac{b^{n-2j_i-1}}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_i} - 1} \text{ with } i = 1 \text{ or } i = 2, \text{ respectively.}$$

Let now $\boldsymbol{z} = (z_1, z_2) \in [0,1)^2$ *and* $g(\boldsymbol{t}) = \mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{z})$ *for* $\boldsymbol{t} = (t_1, t_2) \in [0,1)^2$*. Let* $\boldsymbol{j} \in \mathbb{N}_{-1}^2$*,* $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$*,* $\boldsymbol{\ell} \in \mathbb{B}_{\boldsymbol{j}}$ *and let* $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(g)$ *be the b-adic Haar coefficient of g. Then* $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}} = 0$ *whenever* $\boldsymbol{z}$ *is not contained in the interior of the b-adic interval* $I_{\boldsymbol{j},\boldsymbol{m}}$*. If* $\boldsymbol{z}$ *is contained in the interior of* $I_{\boldsymbol{j},\boldsymbol{m}}$*, then*

1. *If* $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$*, then there is a* $\boldsymbol{k} = (k_1, k_2) \in \{0, 1, \ldots, b-1\}^2$ *such that* $\boldsymbol{z}$ *is contained in* $I_{\boldsymbol{j},\boldsymbol{m}}^{\boldsymbol{k}}$ *(see Section 2.3). Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(g) = b^{-j_1-j_2-2}\left((bm_1 + k_1 - b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_1\ell_1} - \sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_1\ell_1}\right) \times$$
$$\times \left((bm_2 + k_2 - b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_2\ell_2} - \sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_2\ell_2}\right).$$

2. *If* $\boldsymbol{j} = (j_1, -1)$ *with* $j_1 \in \mathbb{N}_0$*, then there is a* $k_1 \in \{0, 1, \ldots, b-1\}$ *such that* $\boldsymbol{z}$ *is contained in* $I_{\boldsymbol{j},\boldsymbol{m}}^{(k_1,-1)}$*. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(g) = b^{-j_1-1}(1 - z_2)\left((bm_1 + k_1 - b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_1\ell_1} - \sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_1\ell_1}\right).$$

3. *If* $\boldsymbol{j} = (-1, j_2)$ *with* $j_2 \in \mathbb{N}_0$*, then there is a* $k_2 \in \{0, 1, \ldots, b-1\}$ *such that* $\boldsymbol{z}$ *is contained in* $I_{\boldsymbol{j},\boldsymbol{m}}^{(-1,k_2)}$*. Then*

$$\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}(g) = b^{-j_2-1}(1 - z_1)\left((bm_2 + k_2 - b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_2\ell_2} - \sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_2\ell_2}\right).$$

Lemma 4.11. *Let* $\boldsymbol{j} \in \mathbb{N}_0^2$ *such that* $j_1 + j_2 < n - 1$*,* $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ *and* $\boldsymbol{\ell} \in \mathbb{B}_{\boldsymbol{j}}$*. Then*

$$\sum_{\boldsymbol{z} \in \mathcal{H}_{b,n}^{\Sigma} \cap I_{\boldsymbol{j},\boldsymbol{m}}}\left((bm_1 + k_1 - b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_1\ell_1} - \sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_1\ell_1}\right) \times$$
$$\times \left((bm_2 + k_2 - b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_2\ell_2} - \sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r_2\ell_2}\right)$$
$$= \frac{b^{n-j_1-j_2}}{\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_2} - 1\right)} \pm b^{j_1+j_2-n}\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}p(k_1)\ell_1}\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k_2\ell_2},$$

where $p(k_1) = k_1$ *or* $p(k_1) = -k_1 - 1$ *depending on* j_1 *and where the sign depends on* j_2*.*

Proof. With the very same argumentation as in the proof of [50, Lemma 4.4] or [52, Lemma 4.10], we can show that

$$\sum_{\boldsymbol{z}\in\mathcal{H}^{\Sigma}_{b,n}\cap I_{\boldsymbol{j},\boldsymbol{m}}}\left((bm_1+k_1-b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}-\sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_1\ell_1}\right)\times$$

$$\times\left((bm_2+k_2-b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}-\sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_2\ell_2}\right)$$

$$=\frac{b^{n-j_1-j_2}}{\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right)\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right)}$$

$$+\underbrace{b^{j_1+j_2-n}\sum_{k_1=0}^{b-1}a_{n-j_1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}\sum_{k_2=0}^{b-1}\sigma_{j_2+1}(a_{j_2+1})\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}}_{S},$$

where $\sigma_{n-j_1}(a_{n-j_1})=k_1$ and $a_{j_2+1}=k_2$. We analyse the expression S. We have

$$S=b^{j_1+j_2-n}\underbrace{\sum_{k_1=0}^{b-1}\sigma^{-1}_{n-j_1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}}_{S_1}\underbrace{\sum_{k_2=0}^{b-1}\sigma_{j_2+1}(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}}_{S_2}.$$

We have to distinguish the cases $\sigma_{n-j_1}=\sigma$ and $\sigma_{n-j_1}=\overline{\sigma}$ as well as the cases $\sigma_{j_2+1}=\sigma$ and $\sigma_{j_2+1}=\overline{\sigma}$, respectively. The case $\sigma_{n-j_1}=\sigma$ leads to $S_1=\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}$, whereas $\sigma_{n-j_1}=\overline{\sigma}$ yields

$$S_1=\sum_{k_1=0}^{b-1}\sigma^{-1}(b-1-k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}=\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}(b-1-k_1)\ell_1}=\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}(-1-k_1)\ell_1}.$$

Combining these results, we have $S_1=\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}p(k_1)\ell_1}$, where $p(k_1)=k_1$ if $\sigma_{n-j_1}=\sigma$ or $p(k_1)=-k_1-1$ if $\sigma_{n-j_1}=\overline{\sigma}$. Hence, $p(k_1)$ depends only on j_1. The case $\sigma_{j_2+1}=\sigma$ yields $S_2=\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}$, whereas $\sigma_{j_2+1}=\overline{\sigma}$ leads to

$$S_2=\sum_{k_2=0}^{b-1}(b-1-\sigma(k_2))\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}=-\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2},$$

and therefore we have $S_2=\pm\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}$, where the sign depends only on j_2. The proof is complete. □

Lemma 4.12. *Let $\boldsymbol{j}=(j_1,-1)$ such that $j_1\in\mathbb{N}_0$ with $j_1<n-1$, $\boldsymbol{m}=(m_1,0)$ with $m_1\in\mathbb{D}_{j_1}$ and $\boldsymbol{\ell}=(\ell_1,1)$ with $\ell_1\in\mathbb{B}_{j_1}$. Then*

$$\sum_{\boldsymbol{z}\in\mathcal{H}^{\Sigma}_{b,n}\cap I_{\boldsymbol{j},\boldsymbol{m}}}\left((bm_1+k_1-b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}-\sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_1\ell_1}\right)(1-z_2)$$

$$=\frac{b^{n-j_1}(1-2\varepsilon)+b}{2\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right)}+\frac{b^{-1}-b^{j_1-n}}{2}\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}p(k_1)\ell_1}$$

$$+\frac{b^{-1}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1}\left(\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}p(k_1)\ell_1}-\frac{b(b-1)}{2}\right),$$

where ε is a positive real number depending on j_1 and m_1 which satisfies $\varepsilon b^{n-j_1}\leq b$ and where $p(k_1)=k_1$ or $p(k_1)=-k_1-1$ depending on j_1.

Proof. With the very same argumentation as in the proof of [50, Lemma 4.10] or [52, Lemma 4.17], we can show that

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathcal{H}_{b,n}^{\Sigma}\cap I_{\boldsymbol{j},\boldsymbol{m}}}&\left((bm_1+k_1-b^{j_1+1}z_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}-\sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_1\ell_1}\right)(1-z_2)\\&=\frac{b^{n-j_1}(1-2\varepsilon)+b}{2\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right)}+\underbrace{\sum_{h=1}^{b^{n-j_1-1}}hb^{j_1-n+1}b^{j_1-n}\sum_{k_1=0}^{b-1}a_{n-j_1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}}_{T_1}\\&\quad+\underbrace{\sum_{h=0}^{b^{n-j_1-1}-1}b^{j_1-n}\sum_{k_1=0}^{b-1}a_{n-j_1}\sum_{r_1=0}^{k_1-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_1\ell_1}}_{T_2},\end{aligned}$$

where $\sigma_{n-j_1}(a_{n-j_1})=k_1$. Analogously as in the proof of Lemma 4.11, we find

$$T_1=\frac{b^{-1}-b^{j_1-n}}{2}\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}p(k_1)\ell_1},$$

where the value of $p(k_1)$ depends only on j_1. We also obtain

$$\begin{aligned}T_2=&\frac{1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1}b^{n-j_1-1}b^{j_1-n}\sum_{k_1=0}^{b-1}a_{n-j_1}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}-1\right)\\=&\frac{b^{-1}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1}\left(\sum_{k_1=0}^{b-1}a_{n-j_1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_1\ell_1}-\frac{b(b-1)}{2}\right)\\=&\frac{b^{-1}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1}\left(\sum_{k_1=0}^{b-1}\sigma^{-1}(k_1)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}p(k_1)\ell_1}-\frac{b(b-1)}{2}\right).\end{aligned}$$

The proof is complete. ☐

Lemma 4.13. *Let $\boldsymbol{j}=(-1,j_2)$ such that $j_2\in\mathbb{N}_0$ with $j_2<n-1$, $\boldsymbol{m}=(0,m_2)$ with $m_2\in\mathbb{D}_{j_2}$ and $\boldsymbol{\ell}=(1,\ell_2)$ with $\ell_2\in\mathbb{B}_{j_2}$. Then*

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathcal{H}_{b,n}^{\Sigma}\cap I_{\boldsymbol{j},\boldsymbol{m}}}&(1-z_1)\left((bm_2+k_2-b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}-\sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_2\ell_2}\right)\\&=\frac{b^{n-j_2}(1-2\varepsilon)+b}{2\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right)}\pm\frac{b^{-1}-b^{j_2-n}}{2}\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}\\&\quad+\frac{b^{-1}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1}\left(\pm\sum_{k_2=0}^{b-1}\sigma(k_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}-\frac{b(b-1)}{2}\right),\end{aligned}$$

where ε' is a positive real number depending on j_2 and m_2 which satisfies $\varepsilon' b^{n-j_2}\leq b$ and where the signs depend only on j_2.

Proof. This fact follows from

$$\sum_{\boldsymbol{z}\in\mathcal{H}_{b,n}^{\Sigma}\cap I_{\boldsymbol{j},\boldsymbol{m}}}(1-z_1)\left((bm_2+k_2-b^{j_2+1}z_2)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}-\sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r_2\ell_2}\right)$$

$$
\begin{aligned}
=&\frac{b^{n-j_2}(1-2\varepsilon')+b}{2\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right)}+\sum_{h=1}^{b^{n-j_2-1}} h b^{j_2-n+1} b^{j_2-n}\sum_{k_2=0}^{b-1} a_{j_2+1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}\\
&+\sum_{h=0}^{b^{n-j_2-1}-1} b^{j_2-n}\sum_{k_2=0}^{b-1}\sigma_{j_2+1}(a_{j_2+1})\sum_{r_2=0}^{k_2-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_2\ell_2}
\end{aligned}
$$

and the relation $a_{j_2+1}=k_2$. The argumentation is very similar to the proofs of [50, Lemma 4.10], [52, Lemma 4.17], Lemma 4.11 and Lemma 4.12. □

Lemma 4.14. *Let $\boldsymbol{j}\in\mathbb{N}_{-1}^2$, $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$, $\boldsymbol{\ell}\in\mathbb{B}_{\boldsymbol{j}}$ and $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}$ be the b-adic Haar coefficients of the discrepancy function of $\mathcal{H}_{b,n}^{\Sigma}$. We recall the definition $|\boldsymbol{j}|=\max\{0,j_1\}+\max\{0,j_2\}$. Then*

1. *if $\boldsymbol{j}\in\mathbb{N}_0^2$ and $|\boldsymbol{j}|<n-1$, then*
$$|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|\leq\left(\frac{b-1}{2}\right)^2 b^{-n}\lesssim b^{-n},$$
2. *if $\boldsymbol{j}\in\mathbb{N}_0^2$, $|\boldsymbol{j}|\geq n-1$ and $j_1,j_2\leq n$, then $|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|\lesssim b^{-|\boldsymbol{j}|}$ and*
$$|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|=\frac{b^{n-2|\boldsymbol{j}|-2}}{\left|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right|\left|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right|}\lesssim b^{n-2|\boldsymbol{j}|}$$
for all but b^n coefficients $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}$,
3. *if $\boldsymbol{j}\in\mathbb{N}_0^2$ and $j_1\geq n$ or $j_2\geq n$, then*
$$|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|=\frac{b^{n-2|\boldsymbol{j}|-2}}{\left|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right|\left|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right|}\lesssim b^{n-2|\boldsymbol{j}|},$$
4. *if $\boldsymbol{j}=(j_1,-1)$ or $\boldsymbol{j}=(-1,j_2)$ with $j_1\in\mathbb{N}_0$, $j_1<n$ or $j_1\in\mathbb{N}_0$, $j_2<n$ respectively, then we have*
$$|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|\leq(b^2-1)b^{-j_i}\lesssim b^{-|\boldsymbol{j}|}$$
for $i=1$ and $i=2$, respectively,
5. *if $\boldsymbol{j}=(j_1,-1)$ or $\boldsymbol{j}=(-1,j_2)$ with $j_1\in\mathbb{N}_0$, $j_1\geq n$ or $j_2\geq n$ respectively, then we have*
$$|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|=\frac{1}{2}\frac{b^{n-2j_i-1}}{\left|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_i}-1\right|}\lesssim b^{n-2|\boldsymbol{j}|}$$
for $i=1$ and $i=2$, respectively.

Proof. Point (2) can be verified analogously as [50, Proposition 5.1, (ii)] or [52, Proposition 4.18, (ii)]. Point (3) and Point (5) follow from Lemma 4.10 and the fact that there are no points contained in the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$ for $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ if $j_1\geq n$ or $j_2\geq n$. For the verification of Point (1) we use Lemma 4.10 and Lemma 4.11 and obtain

$$
\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}=b^{-j_1-j_2-2}\left(\frac{b^{n-j_1-j_2}}{\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_1}-1\right)\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell_2}-1\right)}\right.
$$

$$
\pm b^{j_1+j_2-n} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} \sum_{k_2=0}^{b-1} \sigma(k_2) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k_2 \ell_2} \Bigg) - \frac{b^{n-2j_1-2j_2-2}}{\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_2} - 1\right)}
$$

$$
= \pm\, b^{-n-2} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} \sum_{k_2=0}^{b-1} \sigma(k_2) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k_2 \ell_2},
$$

which leads to

$$
\begin{aligned}
|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}| =& b^{-n-2} \left| \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} \right| \left| \sum_{k_2=0}^{b-1} \sigma(k_2) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k_2 \ell_2} \right| \\
\leq& b^{-n-2} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \sum_{k_2=0}^{b-1} \sigma(k_2) = b^{-n-2} \left(\frac{b(b-1)}{2} \right)^2 = \left(\frac{b-1}{2} \right)^2 b^{-n} \lesssim b^{-n}
\end{aligned}
$$

as claimed, since with k also $\sigma^{-1}(k)$ and $\sigma(k)$ runs through $\{0, 1, \ldots, b-1\}$, respectively. We turn to the case that $\boldsymbol{j} = (j_1, -1)$ with $j_1 \in \mathbb{N}_0$, $j_1 < n$ and therefore regard Lemma 4.10 and Lemma 4.12. We have

$$
\begin{aligned}
\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}} =& b^{-j_1-1} \left(\frac{b^{n-j_1}(1-2\varepsilon) + b}{2\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)} + \frac{b^{-1} - b^{j_1-n}}{2} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} \right. \\
&\left. + \frac{b^{-1}}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1} \left(\sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} - \frac{b(b-1)}{2} \right) \right) - \frac{b^{n-2j_1-1}}{2\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)} \\
=& -\frac{b^{n-2j_1-1}\varepsilon}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1} + \frac{b^{-j_1}}{2\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1\right)} + \frac{b^{-j_1-2} - b^{-n-1}}{2} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} \\
&+ \frac{b^{-j_1-2}}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1} \left(\sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} p(k_1)\ell_1} - \frac{b(b-1)}{2} \right).
\end{aligned}
$$

The triangle inequality yields (since $\varepsilon b^{n-j_1} \leq b$ and $b^{-2n-1} \leq b^{n-j_1-2}$)

$$
\begin{aligned}
|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}| \leq& \frac{b^{n-2j_1-1}\varepsilon}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} + \frac{b^{-j_1}}{2|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} + \frac{b^{-j_1-2} + b^{-n-1}}{2} \sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) \\
&+ \frac{b^{-j_1-2}}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} \left(\sum_{k_1=0}^{b-1} \sigma^{-1}(k_1) + \frac{b(b-1)}{2} \right) \\
\leq& \frac{b^{-j_1}}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} + \frac{b^{-j_1}}{2|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} + \frac{b^{-j_1-2} + b^{-n-1}}{2} \frac{b(b-1)}{2} + \frac{b^{-j_1-2}}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} b(b-1) \\
\leq& \frac{5}{2} \frac{b^{-j_1}}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell_1} - 1|} + \frac{b^{-j_1}}{2} \leq \left(\frac{5}{2} \frac{b^2-1}{6} + \frac{1}{2} \right) b^{-j_1} \leq (b^2-1) b^{-j_1} \lesssim b^{-|\boldsymbol{j}|},
\end{aligned}
$$

where we used Lemma 4.31. The case $(-1, j_2)$ can be handled completely analogously. □

In order to prove results on the discrepancy of the symmetrized Hammersley point sets we also need upper bounds on the absolute values of the Haar coefficients $\tilde{\mu}^{\Sigma}_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}} = \langle D_{\widetilde{N}}(\cdot, \widetilde{\mathcal{H}}^{\Sigma}_{b,n}), h_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}} \rangle$. Here, $\widetilde{N} = 2b^n$ denotes the number of elements of $\widetilde{\mathcal{H}}^{\Sigma}_{b,n}$.

Lemma 4.15. *Let* $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_{-1}^2$. *Then in the case* $\boldsymbol{j} \neq (-1,-1)$ *we have*

$$|\tilde{\mu}^{\Sigma}_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}| \leq 2|\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}| \quad \textit{for all } \boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}, \boldsymbol{\ell} \in \mathbb{B}_{\boldsymbol{j}},$$

where the coefficients $\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}$ *refer to* $\Delta_N(\cdot, \mathcal{H}^{\Sigma}_{b,n})$. *Hence the results in Lemma 4.14 apply accordingly also to* $|\tilde{\mu}^{\Sigma}_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|$ *(up to a factor 2). In the case* $\boldsymbol{j} = (-1,-1)$ *we have*

$$\tilde{\mu}^{\Sigma}_{(-1,-1),(0,0),(1,1)} = 1 + \frac{1}{2b^n} = 1 + \frac{1}{\widetilde{N}}.$$

Proof. The proof is basically the same as for Lemma 4.4 and follows in a similar manner from Lemma 4.6 and Lemma 4.14. □

Remark 4.16. The Haar coefficient $\mu_{(-1,-1),(0,0),(1,1)}(\cdot, \Delta_N(\mathcal{H}^{\Sigma}_{b,n}))$ does not depend on the position of the components in Σ, but only on the number of σ-entries and $\overline{\sigma}$-entries, respectively. Therefore Lemma 4.15 is also true for every point set of the form $\mathcal{H}^{\Sigma_1}_{b,n} \cup \mathcal{H}^{\Sigma_2}_{b,n}$, where $\Sigma_1, \Sigma_2 \in \{\sigma, \overline{\sigma}\}^n$ and where Σ_1 has l entries equal to σ and Σ_2 has $n-l$ of such entries for any $l \in \{0, 1, \dots, n\}$. The proof of the subsequent Theorem 4.18 therefore works also for point sets of this kind. However, these point sets are in general not symmetrized in the sense of (2.2).

Now we have collected all the tools to show the following theorems.

Theorem 4.17. *Let* $1 \leq p, q \leq \infty$ *and* $0 \leq r < \frac{1}{p}$. *Then for any integer* $b \geq 2$ *and* $r = 0$ *we have*

$$\left\| \Delta_N(\cdot, \mathcal{H}^{\Sigma}_{b,n}) \right\|_{S^r_{p,q}B([0,1)^2)} \lesssim (\log N)^{\frac{1}{q}}$$

if and only if $|2l - n| = \mathcal{O}(n^{\frac{1}{q}})$ *or* $\frac{1}{b}\sum_{a=0}^{b-1} \sigma(a)a = \frac{(b-1)^2}{4}$. *For* $0 < r < \frac{1}{p}$ *we have*

$$\left\| \Delta_N(\cdot, \mathcal{H}^{\Sigma}_{b,n}) \right\|_{S^r_{p,q}B([0,1)^2)} \lesssim N^r (\log N)^{\frac{1}{q}}$$

independently of Σ.

Proof. For $\boldsymbol{j} \neq (-1,-1)$ the bounds on the Haar coefficients of the digit scrambled Hammersley point set we found in Lemma 4.14 are of the same order of magnitude in N as the bounds given in [50, Proposition 5.1]. By separating the sum over $\boldsymbol{j}$ in the estimate

$$\left\| \Delta_N(\cdot, \mathcal{H}^{\Sigma}_{b,n}) \right\|_{S^r_{p,q}B([0,1)^2)} \lesssim \left(\sum_{\boldsymbol{j} \in \mathbb{N}^2_{-1}} b^{(j_1+j_2)\left(r - \frac{1}{p} + 1\right)q} \left(\sum_{\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}, \boldsymbol{\ell} \in \mathbb{B}_{\boldsymbol{j}}} |\mu_{\boldsymbol{j},\boldsymbol{m},\boldsymbol{\ell}}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

in six parts according to Lemma 4.14, we have by [50] that all these parts (expect the summand where $\boldsymbol{j} = (-1,-1)$) achieve the order $N^r (\log N)^{\frac{1}{q}}$ for all $0 \leq r < \frac{1}{p}$. Let us now consider the case $\boldsymbol{j} = (-1,-1)$. If $r = 0$, we have $|\mu_{(-1,-1),(0,0),(1,1)}| \lesssim (\log N)^{\frac{1}{q}}$ if and only if Σ is such that $|2l - n| = \mathcal{O}(n^{\frac{1}{q}})$ or $\frac{1}{b}\sum_{a=0}^{b-1} \sigma(a)a = \frac{(b-1)^2}{4}$ (see Remark 4.7). However, if $0 < r < \frac{1}{p}$, we have $|\mu_{(-1,-1),(0,0),(1,1)}| \lesssim \log N \lesssim N^r \lesssim N^r (\log N)^{\frac{1}{q}}$ for all tuples $\Sigma \in \{\sigma, \overline{\sigma}\}^n$. □

Theorem 4.18. *Let $1 \leq p,q \leq \infty$ and $0 \leq r < \frac{1}{p}$. Then for any integer $b \geq 2$ we have*

$$\left\| \Delta_{\widetilde{N}}(\widetilde{\mathcal{H}}_{b,n}^{\Sigma}) \right\|_{S_{p,q}^r B([0,1)^2)} \lesssim \widetilde{N}^r (\log \widetilde{N})^{\frac{1}{q}}$$

independently of Σ.

Proof. The proof is obvious, since the bounds on the Haar coefficients of $\Delta_{\widetilde{N}}(\cdot, \widetilde{\mathcal{H}}_{b,n}^{\Sigma})$ are (up to a constant factor 2) the same as for $\Delta_N(\mathcal{H}_{b,n}^{\Sigma}, \cdot)$, except for the coefficient $\mu_{(-1,-1),(0,0),(1,1)}$, which is of order 1 independently of Σ. We therefore have $|\mu_{(-1,-1),(0,0),(1,1)}| \lesssim \widetilde{N}^r(\log \widetilde{N})^{\frac{1}{q}}$ for all $0 \leq r < \frac{1}{p}$ independently of Σ and we can refer to the proof of [50, Theorem 1.1] again. □

Corollary 4.19. We have the following estimates of the L_p discrepancy for $p \in [1, \infty)$ and all $b \geq 2$:

- $\left\| \Delta_N(\cdot, \mathcal{H}_{b,n}^{\Sigma}) \right\|_{L_p([0,1)^2)} \lesssim \sqrt{\log N}$, if and only if

$$|2l - n| = \mathcal{O}(\sqrt{n}) \text{ or } \frac{1}{b}\sum_{a=0}^{b-1} \sigma(a)a = \frac{(b-1)^2}{4},$$

- $\left\| \Delta_{\widetilde{N}}(\cdot, \widetilde{\mathcal{H}}_{b,n}^{\Sigma}) \right\|_{L_p([0,1)^2)} \lesssim \sqrt{\log N}$ independently of Σ.

Proof. The results can be obtained from Theorem 4.18 via the embeddings (2.24) or by a direct application of Proposition 2.12. □

We observe that the conditions for the optimal order of L_p discrepancy match again the corresponding conditions for L_2 discrepancy which were known before.

4.1.3. Optimal discrepancy rate in spaces with negative smoothness

Statement of the result Recall that Triebel [69, 70] could show that for all $1 \leq p,q \leq \infty$ and $r \in \mathbb{R}$ satisfying $\frac{1}{p} - 1 < r < \frac{1}{p}$ and $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$ we have for any N-element point set $\mathcal{P}$, $N \geq 2$, in $[0,1)^2$ that its discrepancy function satisfies

$$\|\Delta_N(\cdot, \mathcal{P})\|_{S_{p,q}^r B([0,1)^2)} \gtrsim N^{r-1}(\log N)^{\frac{1}{q}}.$$

Hinrichs showed in [36] that this lower bound is sharp for $1 \leq p,q \leq \infty$ and $0 \leq r < \frac{1}{p}$. He used digit shifted Hammersley point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ to obtain this result. It follows from his proof that these point sets can not be used to close the gap also for the parameter range $1/p - 1 < r < 0$. It remained an open problem to find a point set which closes this gap also for $1/p - 1 < r < 0$. This problem was again mentioned in [37, Problem 3] (here also for higher dimensions) and [72, Remark 6.8]. Recall that also Chen-Skriganov point sets and higher order digital nets, which achieve the best possible discrepancy rate in Besov spaces in arbitrary dimension, work only if $r \geq 0$, see [51, 52, 53]. It is the aim of this section to show that this problem can be solved in dimension two by applying some simple modifications to the point sets $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$. For our purposes, we need a new definition of symmetrized Hammersley point sets. To this end we fix $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ and introduce three connected point sets by

$$\mathcal{H}_{2,n}^{(1)}(\boldsymbol{\sigma}) \quad := \quad \{(x, 1-y) | (x,y) \in \mathcal{H}_{2,n}(\boldsymbol{\sigma})\},$$

$$\mathcal{H}_{2,n}^{(2)}(\boldsymbol{\sigma}) := \{(1-x,y)|(x,y)\in\mathcal{H}_{2,n}(\boldsymbol{\sigma})\},$$
$$\mathcal{H}_{2,n}^{(3)}(\boldsymbol{\sigma}) := \{(1-x,1-y)|(x,y)\in\mathcal{H}_{2,n}(\boldsymbol{\sigma})\}.$$

We set $\mathcal{H}_{2,n}^{\mathrm{sym}}(\boldsymbol{\sigma}) := \mathcal{H}_{2,n}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(1)}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(2)}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(3)}(\boldsymbol{\sigma})$ and call $\mathcal{H}_{2,n}^{\mathrm{sym}}(\boldsymbol{\sigma})$ a doubly symmetrized Hammersley type point set. This is because $\mathcal{H}_{2,n}^{(1)}(\boldsymbol{\sigma})$ is obtained from $\mathcal{H}_{2,n}(\boldsymbol{\sigma})$ by reflecting it at the line $y=1/2$, and then $\mathcal{H}_{2,n}^{(2)}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(3)}(\boldsymbol{\sigma})$ is the reflection of $\mathcal{H}_{2,n}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(1)}(\boldsymbol{\sigma})$ at the line $x=1/2$. The point set $\mathcal{H}_{2,n}^{\mathrm{sym}}(\boldsymbol{\sigma})$ has $N=2^{n+2}$ elements, where some points might coincide. Since the following results will all be independent of the digital shift, we will consequently omit the $\boldsymbol{\sigma}$ and write $\mathcal{H}_{2,n}$, $\mathcal{H}_{2,n}^{(1)}$, $\mathcal{H}_{2,n}^{(2)}$, $\mathcal{H}_{2,n}^{(3)}$ and $\mathcal{H}_{2,n}^{\mathrm{sym}}$ throughout this section. With the point sets $\mathcal{H}_{2,n}^{\mathrm{sym}}$ we have the following result.

Theorem 4.20. *Let $1 \le p,q \le \infty$ and $r \in \mathbb{R}$ such that $1/p-1<r<1/p$. Then the point sets $\mathcal{H}_{2,n}^{\mathrm{sym}}$ in $[0,1)^2$ with $N=2^{n+1}$ elements satisfy*

$$\|\Delta_N(\cdot,\mathcal{H}_{2,n}^{\mathrm{sym}})\|_{S_{p,q}^rB([0,1)^2)} \lesssim N^{r-1}(\log N)^{1/q}.$$

We would like to stress again that our result improves on [36, Theorem 1.1] in the sense that we extended the range for the smoothness parameter r to negative values.

We will follow the same approach as Hinrichs and first estimate the Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n}^{\mathrm{sym}})$ and then apply Proposition 2.11.

Proof of Theorem 4.20 To begin with, we state several auxiliary results from [36, Lemmas 3.2–3.4, 3.6]. These were the fundamental lemmas in order to prove [36, Theorem 3.1] (see also Lemma 4.1).

Lemma 4.21. *Let $f(\boldsymbol{t})=t_1t_2$ for $\boldsymbol{t}=(t_1,t_2)\in[0,1)^2$. For $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of f. Then*

(i) If $\boldsymbol{j}=(j_1,j_2)\in\mathbb{N}_0^2$ then $\langle f,h_{\boldsymbol{j},\boldsymbol{m}}\rangle = 2^{-2j_1-2j_2-4}$.

(ii) If $\boldsymbol{j}=(-1,k)$ or $\boldsymbol{j}=(k,-1)$ with $k\in\mathbb{N}_0$ then $\langle f,h_{\boldsymbol{j},\boldsymbol{m}}\rangle = -2^{-2k-3}$.

Lemma 4.22. *Fix $\boldsymbol{z}=(z_1,z_2)\in[0,1)^2$ and let $f(\boldsymbol{t})=\mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{z})$ for $\boldsymbol{t}=(t_1,t_2)\in[0,1)^2$. For $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ and $\boldsymbol{m}=(m_1,m_2)\in\mathbb{D}_{\boldsymbol{j}}$ let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of f. Then $\mu_{\boldsymbol{j},\boldsymbol{m}}=0$ whenever $\boldsymbol{z}\notin I_{\boldsymbol{j},\boldsymbol{m}}^\circ$, where $I_{\boldsymbol{j},\boldsymbol{m}}^\circ$ denotes the interior of $I_{\boldsymbol{j},\boldsymbol{m}}$. If $\boldsymbol{z}\in I_{\boldsymbol{j},\boldsymbol{m}}^\circ$ then*

(i) If $\boldsymbol{j}=(j_1,j_2)\in\mathbb{N}_0^2$ then

$$\langle f,h_{\boldsymbol{j},\boldsymbol{m}}\rangle = 2^{-(j_1+j_2+2)}(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|).$$

(ii) If $\boldsymbol{j}=(-1,k)$ with $k\in\mathbb{N}_0$ then $\langle f,h_{\boldsymbol{j},\boldsymbol{m}}\rangle = -2^{-(k+1)}(1-z_1)(1-|2m_2+1-2^{k+1}z_2|)$.

(iii) If $\boldsymbol{j}=(k,-1)$ with $k\in\mathbb{N}_0$ then $\langle f,h_{\boldsymbol{j},\boldsymbol{m}}\rangle = -2^{-(k+1)}(1-z_2)(1-|2m_1+1-2^{k+1}z_1|)$.

Lemma 4.23. *Let $\mathcal{H}_{2,n}$ be a shifted Hammersley point set with 2^n points. Let $\boldsymbol{j}=(j_1,j_2)\in\mathbb{N}_0^2$ and $\boldsymbol{m}=(m_1,m_2)\in\mathbb{D}_{\boldsymbol{j}}$. Then, if $j_1+j_2<n$,*

$$\sum_{\boldsymbol{z}\in\mathcal{H}_{2,n}\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ}(1-|2m_1+1-2^{j_1+1}z_1|) = \sum_{\boldsymbol{z}\in\mathcal{H}_{2,n}\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ}(1-|2m_2+1-2^{j_2+1}z_2|) = 2^{n-j_1-j_2-1}$$

and, if $j_1+j_2<n-1$,

$$\sum_{\boldsymbol{z}\in\mathcal{H}_{2,n}\cap I_{\boldsymbol{j},\boldsymbol{m}}^\circ}(1-|2m_1+1-2^{j_1+1}z_1|)(1-|2m_2+1-2^{j_2+1}z_2|) = 2^{n-j_1-j_2-2}+2^{j_1+j_2-n}.$$

Now we are ready to compute the Haar coefficients of the discrepancy function of $\mathcal{H}_{2,n}^{\mathrm{sym}}$.

Proposition 4.24. *Let $\mathcal{H}_{2,n}^{\mathrm{sym}}$ be a doubly symmetrized Hammersley type point set with $N = 2^{n+2}$ elements and let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of $\Delta_N(\cdot, \mathcal{H}_{2,n}^{\mathrm{sym}})$ for $\boldsymbol{j} \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{D}_{\boldsymbol{j}}$.*
Let $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$. Then

(i) if $j_1 + j_2 < n - 1$ and $j_1, j_2 \geq 0$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{-n}$.

(ii) if $j_1 + j_2 \geq n - 1$ and $0 \leq j_1, j_2 \leq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \leq 2^{-j_1-j_2+2}$ and $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{n-2j_1-2j_2-2}$ for all but at most 2^{n+2} coefficients $\mu_{\boldsymbol{j},\boldsymbol{m}}$ with $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$.

(iii) if $j_1 \geq n$ or $j_2 \geq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = 2^{n-2j_1-2j_2-2}$.

Now let $\boldsymbol{j} = (-1, k)$ or $\boldsymbol{j} = (k, -1)$ with $k \in \mathbb{N}_0$. Then

(iv) if $k < n$ then $\mu_{\boldsymbol{j},\boldsymbol{m}} = 0$.

(v) if $k \geq n$ then $|\mu_{\boldsymbol{j},\boldsymbol{m}}| = -2^{n-2k-1}$.

Finally,

(vi) $\mu_{(-1,-1),(0,0)} = 0$.

Proof. The cases (iii) and (v) follow from the fact that no elements of $\mathcal{H}_{2,n}^{\mathrm{sym}}$ are contained in the interior of a dyadic box $I_{(j_1,j_2),\boldsymbol{m}}$ if $j_1 \geq n$ or $j_2 \geq n$, together with Lemma 4.21.
We consider the case (ii). For a fixed $\boldsymbol{j} = (j_1, j_2)$ the interiors of the dyadic boxes $I_{\boldsymbol{j},\boldsymbol{m}}$ for $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ are mutually disjunct and at most 2^{n+2} of these boxes can contain points from $\mathcal{H}_{2,n}^{\mathrm{sym}}$. We have $\mu_{\boldsymbol{j},\boldsymbol{m}} = 2^{n-2j_1-2j_2-4}$ if the corresponding box $I_{\boldsymbol{j},\boldsymbol{m}}$ is empty. The other boxes contain at most 8 points (because the volume of $I_{\boldsymbol{j},\boldsymbol{m}}$ is at most $2^{-(n-1)}$ due to the condition $j_1+j_2 \geq n-1$ and because of the net property of $\mathcal{H}_{2,n}$ and its connected point sets). Together with the first part of Lemma 4.22 and the triangle inequality this yields $|\mu_{\boldsymbol{j},\boldsymbol{m}}| \leq 8 \cdot 2^{-(j_1+j_2+2)} + 2^{n-2j_1-2j_2-2} \leq 2^{-j_1-j_2+2}$.
The case (vi) can be seen as follows:

$$\begin{aligned}
\mu_{(-1,-1),(0,0)} &= \int_0^1 \int_0^1 \Delta_N(t_1, t_2, \mathcal{H}_{2,n}^{\mathrm{sym}})\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \sum_{\boldsymbol{z} \in \mathcal{H}_{2,n}^{\mathrm{sym}}} \int_{z_1}^1 \int_{z_2}^1 1\,\mathrm{d}t_1\,\mathrm{d}t_2 - 2^{n+2} \int_0^1 \int_0^1 t_1 t_2\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \sum_{\boldsymbol{z} \in \mathcal{H}_{2,n}^{\mathrm{sym}}} (1 - z_1)(1 - z_2) - 2^n \\
&= \sum_{(x,y) \in \mathcal{H}_{2,n}} [(1-x)(1-y) + (1-x)y + x(1-y) + xy] - 2^n \\
&= \sum_{(x,y) \in \mathcal{H}_{2,n}} 1 - 2^n = 0.
\end{aligned}$$

To show the claim in (iv) for the case $\boldsymbol{j} = (k, -1)$ with $k \in \mathbb{N}_0$, $k < n$, we have to consider the expression

$$S := \sum_{\boldsymbol{z} \in \mathcal{H}_{2,n}^{\mathrm{sym}} \cap I^{\circ}_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1} z_1|)(1 - z_2)$$

for any $m_1 \in \{0, \ldots, 2^k - 1\}$. We can write

$$\begin{aligned}
S = &\sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}x|)(1-y) \\
&+ \sum_{(x,1-y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}x|)y \\
&+ \sum_{(1-x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}(1-x)|)(1-y) \\
&+ \sum_{(1-x,1-y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}(1-x)|)y \\
= &\sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}x|) \\
&+ \sum_{(1-x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}} (1 - |2m_1 + 1 - 2^{k+1}(1-x)|) =: S_1 + S_2,
\end{aligned}$$

where we used the trivial equivalences $(x, y) \in \mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}$ if and only if $(x, 1-y) \in \mathcal{H}_{2,n} \cap I^\circ_{(k,-1),(m_1,0)}$ as well as $(1-x, y) \in \mathcal{H}_{2,n} \cap I^\circ_{(k,-1),(m_1,0)}$ if and only if $(1-x, 1-y) \in \mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(m_1,0)}$ in the last step. Since the interval $I^\circ_{(k,-1),(m_1,0)}$ is the same as $I^\circ_{(k,0),(m_1,0)}$, we obtain $S_1 = 2^{n-k-1}$ from the first part of Lemma 4.23. To evaluate S_2 we observe that

$$1 - x \in I^\circ_{k,m_1} \Leftrightarrow \frac{m_1}{2^k} < 1 - x < \frac{m_1 + 1}{2^k} \Leftrightarrow \frac{2^k - 1 - m_1}{2^k} < x < \frac{2^k - m_1}{2^k} \Leftrightarrow x \in I^\circ_{k,\widetilde{m}_1},$$

where we set $\widetilde{m}_1 = 2^k - 1 - m_1$. This yields the equivalence of $(1 - x, y) \in \mathcal{H}_{2,n} \cap I^\circ_{(k,-1),(m_1,0)}$ and $(x, y) \in \mathcal{H}_{2,n} \cap I^\circ_{(k,-1),(\widetilde{m}_1,0)}$. We also find

$$\begin{aligned}
|2m_1 + 1 - 2^{k+1}(1 - x)| =&|2(m_1 + 1 - 2^k) - 1 + 2^{k+1}x| \\
=&| - 2\widetilde{m}_1 - 1 + 2^{k+1}x| = |2\widetilde{m}_1 + 1 - 2^{k+1}x|
\end{aligned}$$

and hence we obtain

$$S_2 = \sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{(k,-1),(\widetilde{m}_1,0)}} (1 - |2\widetilde{m}_1 + 1 - 2^{k+1}x|) = 2^{n-k-1},$$

where we regarded the first part of Lemma 4.23 again. Altogether, we have

$$\mu_{(k,-1),(m_1,0)} = -2^{-(k+1)}(S_1 + S_2) - (-2^{n+2-(2k+3)}) = -2^{-(k+1)}2^{n-k} + 2^{n-2k-1} = 0$$

with Lemmas 4.21 and 4.22, and this part of the proposition is verified. It is clear that the result for $\mu_{(-1,k),(0,m_2)}$ if $k < n$ can be shown analogously.
Finally, we prove (i) and therefore have to analyze the sum

$$T := \sum_{\boldsymbol{z}\in\widetilde{\mathcal{H}}_{2,n}\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_1 + 1 - 2^{j_1+1}z_1|)(1 - |2m_2 + 1 - 2^{j_2+1}z_2|),$$

where $\boldsymbol{j} = (j_1, j_2) \in \mathbb{N}_0^2$ with $j_1 + j_2 < n - 1$. We have

$$T = \sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}} (1 - |2m_1 + 1 - 2^{j_1+1}x|)(1 - |2m_2 + 1 - 2^{j_2+1}y|)$$

$$
\begin{aligned}
&+ \sum_{(x,1-y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}x|)(1-|2m_2+1-2^{j_2+1}(1-y)|)\\
&+ \sum_{(1-x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}(1-x)|)(1-|2m_2+1-2^{j_2+1}y|)\\
&+ \sum_{(1-x,1-y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},\boldsymbol{m}}} (1-|2m_1+1-2^{j_1+1}(1-x)|)(1-|2m_2+1-2^{j_2+1}(1-y)|)\\
=&:T_1+T_2+T_3+T_4.
\end{aligned}
$$

We obtain directly from the second part of Lemma 4.23 that $T_1 = 2^{n-j_1-j_2-2} + 2^{j_1+j_2-n}$. With the same arguments as in the proof of point (iv) we can show

$$
\begin{aligned}
T_2 &= \sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},(m_1,\widetilde{m}_2)}} (1-|2m_1+1-2^{j_1+1}x|)(1-|2\widetilde{m}_2+1-2^{j_2+1}y|),\\
T_3 &= \sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},(\widetilde{m}_1,m_2)}} (1-|2\widetilde{m}_1+1-2^{j_1+1}x|)(1-|2m_2+1-2^{j_2+1}y|),\\
T_4 &= \sum_{(x,y)\in\mathcal{H}_{2,n}\cap I^\circ_{\boldsymbol{j},(\widetilde{m}_1,\widetilde{m}_2)}} (1-|2\widetilde{m}_1+1-2^{j_1+1}x|)(1-|2\widetilde{m}_2+1-2^{j_2+1}y|),
\end{aligned}
$$

where $\widetilde{m}_i = 2^{j_i}-1-m_i$ for $i\in\{1,2\}$. But from this and Lemma 4.23 we see that $T_2=T_3=T_4=T_1$ and together with Lemma 4.21 and Lemma 4.22

$$
\begin{aligned}
\mu_{\boldsymbol{j},\boldsymbol{m}} =& 2^{-j_1-j_2-2}(T_1+T_2+T_3+T_4) - 2^{n-2j_1-2j_2-2}\\
=& 2^{-j_1-j_2-2}(2^{n-j_1-j_2}+2^{j_1+j_2-n+2}) - 2^{n-2j_1-2j_2-2} = 2^{-n}
\end{aligned}
$$

as claimed. The proof of the proposition is complete. □

Now we are ready to give the proof of Theorem 4.20.

Proof of Theorem 4.20. We choose any doubly symmetrized Hammersley type point set $\mathcal{H}_{2,n}^{\rm sym}$ (we do not have to specify the digital shift $\boldsymbol{\sigma}$). For $\boldsymbol{j}\in\mathbb{N}_{-1}^2$ and $\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}$ let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the Haar coefficients of the discrepancy function of $\mathcal{H}_{2,n}^{\rm sym}$. According to Proposition 2.11, it suffices to show that for all p,q,r satisfying the conditions in Theorem 4.20 we have

$$
\left(\sum_{\boldsymbol{j}\in\mathbb{N}_{-1}^2} 2^{(j_1+j_2)\left(r-\frac{1}{p}+1\right)q}\left(\sum_{\boldsymbol{m}\in\mathbb{D}_{\boldsymbol{j}}}|\mu_{\boldsymbol{j},\boldsymbol{m}}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \lesssim 2^{nr}n^{1/q}. \tag{4.6}
$$

This yields

$$
\|\Delta_N(\cdot,\mathcal{H}_{2,n}^{\rm sym})\|_{S_{p,q}^rB([0,1)^2)} \lesssim 2^{-2r}2^{(n+2)r}(n+2)^{1/q} \lesssim N^r(\log N)^{1/q}.
$$

To verify (4.6), we split the sum over $\boldsymbol{j}$ in six cases according to Proposition 4.24 (and thereby applying Minkowski's inequality). We remark that the cases (i), (ii), (iii) and (v) have already been treated in [36, Section 4], since in these cases the bounds on the Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n}^{\rm sym})$ are (basically) the same as those for the Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n})$. In all cases Hinrichs obtained an upper bound of the form $c2^{nr}n^{1/q}$ with c independent of n for the whole parameter range $1/p-1<r<1/p$. The only cases where the condition $r\geq 0$ was necessary were (iv) and (vi). However, the symmetrization of $\mathcal{H}_{2,n}$ has the effect that the corresponding Haar coefficients of $\Delta_N(\cdot,\mathcal{H}_{2,n}^{\rm sym})$ in these two cases vanish, and the result follows. □

Remark 4.25. Consider the point set $\mathcal{H}_{2,n}(\boldsymbol{\sigma}) \cup \mathcal{H}_{2,n}^{(1)}(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \{0,1\}^n$. For $\boldsymbol{j} \in \mathbb{N}_{-1}^2$ and $\boldsymbol{m} \in \mathbb{D}_{\boldsymbol{j}}$ let $\mu_{\boldsymbol{j},\boldsymbol{m}}$ be the corresponding Haar coefficients. Then one can show that $\mu_{(-1,-1),(0,0)} = \frac{1}{2}$ and $\mu_{(-1,k),(0,m_2)} = -2^{-(2k+2)} + 2^{-(n+1)}T_k$ for $k \in \mathbb{N}_0$, $k < n$. Here, $T_k = 1$ if $\sigma_{k+1} = 0$ and $T_k = -1$ if $\sigma_{k+1} = 1$. Hence, the proof of Theorem 4.20 does not work for this class of point sets. This is the reason why we have to deal with the doubly symmetrized Hammersley type point sets.

Discrepancy in further function spaces and numerical integration As pointed out in Section 2.3.4 one can easily deduce results on the discrepancy of point sets in Triebel-Lizorkin spaces from the discrepancy estimates in Besov spaces. From the first embedding in (2.24) together with Theorem 4.20 we obtain

Corollary 4.26. Let $1 \leq p, q < \infty$ and $\frac{1}{\max\{p,q\}} - 1 < r < \frac{1}{\max\{p,q\}}$. Then the point sets $\mathcal{H}_{2,n}^{\mathrm{sym}}$ in $[0,1)^2$ with $N = 2^{n+1}$ elements satisfy

$$\|\Delta_N(\cdot, \mathcal{H}_{2,n}^{\mathrm{sym}})\|_{S_{p,q}^r F([0,1)^2)} \lesssim N^{r-1}(\log N)^{1/q}.$$

This corollary improves on [50, Theorem 6.1], where Hammersley type point sets in arbitrary base $b \geq 2$ have been considered, by extending again the range of r to negative values. There exist corresponding lower bounds for the norm of the discrepancy function in Triebel-Lizorkin spaces for $\frac{1}{\min\{p,q\}} - 1 < r < \frac{1}{p}$ as shown in [52, Corollary 4.2]. This follows from the lower bounds on the discrepancy in Besov spaces as stated in Section 2.4, together with the second embedding in (2.24).

By choosing $q = 2$ in Corollary 4.26 we obtain an analogous result on Sobolev spaces. Further, from the fact that $S_p^0 H([0,1)^2) = L_p([0,1)^2)$ we derive from Corollary 4.26 that the doubly symmetrized Hammersley type point sets achieve an L_p discrepancy of order $\sqrt{\log N}$ for all $p \in [1,\infty)$, which is best possible in the sense of (1.13). This however is not so surprising, since in Theorem 4.5 (in conjunction with Lemma 2.3) we have shown that already a Davenport type symmetrization of $\mathcal{H}_{2,n}$ achieves the best possible rate of L_p discrepancy for all $p \in [1,\infty)$, i.e. $L_p(\mathcal{H}_{2,n} \cup \mathcal{H}_{2,n}^{(1)}) \lesssim \sqrt{\log N}$. By different means as employed here, a certain type of symmetrized Hammersley point sets in a prime base b has been studied by Goda [32, Theorem 24], which matches our construction of $\mathcal{H}_{2,n}^{\mathrm{sym}}$ for $b = 2$. We notice that the construction of point sets with the optimal rate of discrepancy in Besov, Triebel-Lizorkin or Sobolev spaces with negative smoothness is even more subtle than to find point sets with the optimal order of L_p discrepancy.

We would like to add a few words concerning errors of quasi-Monte Carlo methods for numerical integration in spaces with dominating mixed smoothness. For a function f in a normed space F of functions on $[0,1)^2$ we would like to approximate the integral $I(f) := \int_{[0,1)^2} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ by a quasi-Monte Carlo algorithm $Q_N(\mathcal{P}, f) = \frac{1}{N}\sum_{k=1}^N f(\boldsymbol{x}_k)$, where $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ is a set of N points in the unit square. The minimal worst-case error of quasi-Monte Carlo algorithms with respect to a class of functions F is defined as

$$\mathrm{err}_N(F) := \inf_{|\mathcal{P}|=N} \sup_{\|f\|_F \leq 1} |I(f) - Q_N(\mathcal{P}, f)|.$$

The infimum is extended over all point sets in $[0,1)^2$ with N elements and the supremum is extended over all functions in the unit ball of F which consists of all functions with norm smaller or equal 1. We state a remarkable connection between discrepancy and integration errors in Besov spaces. Let therefore

$$\mathrm{disc}_N(S^r_{p,q}B([0,1)^2)) := \inf_{|\mathcal{P}|=N} \|\Delta_N(\cdot,\mathcal{P})\|_{S^r_{p,q}B([0,1)^2)}.$$

It is known that $S^{1-r}_{p',q'}B([0,1)^2)^{\urcorner}$, where $1/p+1/p'=1/q+1/q'=1$, is the dual space of $S^r_{p,q}B([0,1)^2)$. Here $S^{1-r}_{p',q'}B([0,1)^2)^{\urcorner}$ is the class of all functions in $S^{1-r}_{p',q'}B([0,1)^2)$ with zero boundary on the upper and right boundary line. Let $1 \leq p,q \leq \infty$ ($q<\infty$ if $p=1$ and $q>1$ if $p=\infty$) and $1/p<r<1/p+1$. Then we have for every integer $N \geq 2$

$$\mathrm{err}_N(S^r_{p,q}B([0,1)^2)^{\urcorner}) \asymp N^{-1}\mathrm{disc}_N(S^{1-r}_{p',q'}B([0,1)^2)), \tag{4.7}$$

which follows from [69, Theorem 6.11]. This relation leads to the following result:

Theorem 4.27. *Let $1 \leq p,q \leq \infty$ ($q<\infty$ if $p=1$ and $q>1$ if $p=\infty$) and $1/p<r<1+1/p$. Then for $N=2^{n+1}$ with $n\in\mathbb{N}$ we have*

$$\mathrm{err}_N(S^r_{p,q}B([0,1)^2)^{\urcorner}) \lesssim N^{-r}(\log N)^{1-1/q}.$$

Proof. From (4.7) we have

$$\mathrm{err}_N(S^r_{p,q}B([0,1)^2)^{\urcorner}) \lesssim N^{-1}\mathrm{disc}_N(S^{1-r}_{p',q'}B([0,1)^2))$$

for $1/p<r<1+1/p$. Theorem 4.20 yields further

$$N^{-1}\mathrm{disc}_N(S^{1-r}_{p',q'}B([0,1)^2)) \lesssim N^{1-r-1}(\log N)^{1/q'} = N^{-r}(\log N)^{1-1/q}$$

for $1/p'-1<1-r<1/p'$. The last condition on r is equivalent to $1/p<r<1+1/p$ and the result follows. □

We remark that there exists a corresponding lower bound on $\mathrm{err}_N(S^r_{p,q}B([0,1)^2)$ which shows that the rate of convergence in this theorem is optimal. For the smaller parameter range $1/p<r<1$ the assertion in Theorem 4.27 has already been found in [52, Theorem 5.6] (for arbitrary dimensions). It has been shown in [72, Corollary 6.4] that the result in Theorem 4.27 can be extended to $1/p<r<2$, if one does not restrict to quasi-Monte Carlo rules but chooses more general cubature rules. In the same smoothness range quasi-Monte-Carlo rules based on Chen-Skriganov point sets can achieve the optimal order for periodic functions in every dimension, see [38]. In [71] is has been proven that Frolov's cubature rules (which are in general not of quasi-Monte Carlo type) achieve the optimal convergence rate even for all $r>1/p$ in all dimensions and also in the non-periodic setting. Our new result shows that in the two-dimensional case at least for $1<r<1+1/p$ the optimal rate of convergence can also be achieved with quasi-Monte Carlo rules (based on doubly symmetrized Hammersley type point sets). With similar arguments as above we obtain an analogous result on integration errors in Triebel-Lizorkin spaces (and hence in Sobolov spaces).

Corollary 4.28. Let $1 \leq p,q \leq \infty$ and $1/\min\{p,q\}<r<1+1/\min\{p,q\}$. Then for $N=2^{n+1}$ with $n\in\mathbb{N}$ we have

$$\mathrm{err}_N(S^r_{p,q}F([0,1)^2)^{\urcorner}) \lesssim N^{-r}(\log N)^{1-1/q}.$$

Proof. This result is a consequence of the second embedding in (2.24), which implicates

$$\mathrm{err}_N(S^r_{p,q}F([0,1)^2)^{\urcorner}) \leq \mathrm{err}_N(S^r_{\min\{p,q\},q}B([0,1)^2)^{\urcorner}),$$

and Theorem 4.27. □

4.2. Symmetrized van der Corput sequences

4.2.1. Optimal order of L_p discrepancy of $\widetilde{\mathcal{V}}_b^\sigma$

We consider the generalized van der Corput sequences and their symmetrized variants. We will prove that the sequences $\mathcal{V}_b^\sigma$ do not achieve the optimal order of L_p discrepancy for any $p \in (1,\infty)$ and any base b and permutation σ. However, the symmetrized sequences $\widetilde{\mathcal{V}}_b^\sigma$ overcome this defect for every b and σ. This is stated more precisely in the following theorem.

Theorem 4.29. *Let $b \geq 2$ be an integer and $\sigma \in \mathfrak{S}_b$ such that $\sigma(0) = 0$. Then we have*

$$L_{p,N}(\mathcal{V}_b^\sigma) \gtrsim \log N$$

for all $p \in (1,\infty)$ and infinitely many $N \in \mathbb{N}$ and

$$L_{p,N}(\widetilde{\mathcal{V}}_b^\sigma) \lesssim \sqrt{\log N}$$

for all $p \in [1,\infty)$ and all $N \geq 2$.

Recall that $L_{\infty,N}(\mathcal{V}_b^\sigma) \lesssim \log N$, which yields $L_{p,N}(\mathcal{V}_b^\sigma) \asymp \log N$ together with Theorem 4.29.

The proof is again based on the estimation of the corresponding Haar coefficients. To show the result on the L_p discrepancy of the symmetrized Hammersley point sets, we need to estimate all the Haar coefficients. We consider here the general case where the base b and the permutation σ may be chosen arbitrarily. The calculations are less technical if one restricts to the case $b = 2$, for which we refer to [45]. We write $\mu_{j,m,\ell}^N := \mu_{j,m,\ell}(\Delta_N(\cdot, \mathcal{V}_b^\sigma))$ and $\tilde{\mu}_{j,m,\ell}^N := \mu_{j,m,\ell}(\Delta_N(\cdot, \widetilde{\mathcal{V}}_b^\sigma))$, where the upper index N denotes that we consider the first N elements of $\mathcal{V}_b^\sigma$ and $\widetilde{\mathcal{V}}_b^\sigma$, respectively. At first we need several lemmas.

In the following, we will consequently omit the lower index b and the upper index σ in the radical inverse function φ_b^σ, since we will always consider an arbitrary but fixed base b and permutation σ. We will further use the shorthand $\varphi^{(-1)} := \varphi_b^{\sigma^{-1}}$, where σ^{-1} is the inverse permutation of σ. Recall the definitions of $I_{j,m}$ and $I_{j,m}^k$ from Section 2.3.1. We denote by $I_{j,m}^\circ$ and $I_{j,m}^{k,\circ}$ the interior of the intervals $I_{j,m}$ and $I_{j,m}^k$, respectively.

Lemma 4.30. *The following relations hold for the radical inverse function φ:*

1. $\varphi(b^j w) = \frac{1}{b^j}\varphi(w)$ *for all* $j, w \in \mathbb{N}_0$,
2. $\varphi(b^j \varphi^{(-1)}(m)) = \frac{m}{b^j}$ *for all* $j \in \mathbb{N}_0$ *and* $m \in \{0, \dots, b^j - 1\}$,
3. $\varphi(n) \in I_{j,m}$ *if and only if* $n = b^j\varphi^{(-1)}(m) + b^j w$ *for some* $w \in \mathbb{N}_0$, *especially* $\varphi(n) \in I_{j,m}^\circ$ *if and only if* $n = b^j\varphi^{(-1)}(m) + b^j w$ *for some* $w \in \mathbb{N}$,
4. $\varphi(n) \in I_{j,m}^k$ *for some* $k \in \{0, 1, \dots, b-1\}$ *if and only if* $n = b^{j+1}\varphi^{(-1)}(bm + k) + b^{j+1}w = b^j\varphi^{(-1)}(m) + b^j k + b^{j+1}w$ *for some* $w \in \mathbb{N}_0$, *especially* $\varphi(n) \in I_{j,m}^{k,\circ}$ *if and only if* $n = b^j\varphi^{(-1)}(m) + b^j k + b^{j+1}w$ *for some* $w \in \mathbb{N}$.

Proof. 1. Let $w = \sum_{i=0}^{k} w_i b^i$, where $w_i \in \{0, 1, \ldots, b-1\}$ for all $i \in \{0, \ldots, k\}$. Then $\varphi(b^j w) = \varphi(\sum_{i=0}^{k} \sigma(w_i) b^{i+j}) = \sum_{i=0}^{k} \sigma(w_i) b^{-i-j-1} = \frac{1}{b^j} \sum_{i=0}^{k} \sigma(w_i) b^{-i-1} = \frac{1}{b^j} \varphi(w)$.

2. Since $0 \leq m \leq b^j - 1$, m has a b-ary representation of the form $m = \sum_{i=0}^{j-1} m_i b^i$, where $m_i \in \{0, 1, \ldots, b-1\}$ for all $i \in \{0, \ldots, j-1\}$. Then $b^j \varphi^{(-1)}(m) = \sum_{i=0}^{j-1} \sigma^{-1}(m_i) b^{j-i-1}$ and therefore $\varphi(b^j \varphi^{(-1)}(m)) = \sum_{i=0}^{j-1} \sigma(\sigma^{-1}(m_i)) b^{-(j-i-1)-1} = \frac{1}{b^j} \sum_{i=0}^{j-1} m_i b^i = \frac{m}{b^j}$.

3. Again we have $m = \sum_{i=0}^{j-1} m_i b^i$. Let $n = \sum_{i \geq 0} n_i b^i$, where $n_i \in \{0, 1, \ldots, b-1\}$ and only finitely many n_i are non-zero. We have

$$\frac{m}{b^j} = \frac{m_{j-1}}{b} + \frac{m_{j-2}}{b^2} + \cdots + \frac{m_0}{b^j}$$

and

$$\varphi(n) = \frac{\sigma(n_0)}{b} + \frac{\sigma(n_1)}{b^2} + \cdots + \frac{\sigma(n_{j-1})}{b^j} + \sum_{i \geq j} \frac{\sigma(n_i)}{b^{i+1}}.$$

It is evident that $\varphi(n) \in I_{j,m}$ if and only if $n_0 = \sigma^{-1}(m_{j-1})$, $n_1 = \sigma^{-1}(m_{j-2})$ and so on up to $n_{j-1} = \sigma^{-1}(m_0)$, whereas the digits n_i for $i \geq j$ may be chosen arbitrarily. We conclude that $\varphi(n) \in I_{j,m}$ if and only if

$$\begin{aligned} n =& \sigma^{-1}(m_{j-1}) + \sigma^{-1}(m_{j-2}) b + \cdots + \sigma^{-1}(m_0) b^{j-1} + \sum_{i \geq j} n_i b^i \\ =& b^j \left(\frac{\sigma^{-1}(m_0)}{b} + \cdots + \frac{\sigma^{-1}(m_{j-2})}{b^{j-1}} + \frac{\sigma^{-1}(m_{j-1})}{b^j} \right) + b^j \sum_{i \geq 0} n_{i+j} b^i \\ =& b^j \varphi^{(-1)}(m) + b^j \sum_{i \geq 0} n_{i+j} b^i, \end{aligned}$$

where $w := \sum_{i \geq 0} n_{i+j} b^i$ may run through all integers in $\mathbb{N}_0$. For $w = 0$ we have $\varphi(n) = \varphi(b^j \varphi^{(-1)}(m)) = \frac{m}{b^j} \notin I_{j,m}^{\circ}$, whereas for $w \geq 1$ we have $\varphi(n) = \frac{m}{b^j} + \frac{\varphi(w)}{b^j} \in I_{j,m}^{\circ}$, which proves the additional claim in item 3.

4. This follows from 3.

□

The next lemma contains several formulas for exponential expressions which will occur in diverse parts of our proofs.

Lemma 4.31. *The following equalities and inequalities hold for all integers $b \geq 2$:*

1. $\sum_{k=0}^{b-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} = 0$ *for all* $\ell \in \{1, \ldots, b-1\}$,

2. $\sum_{\ell=1}^{b-1} \frac{1}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} \ell} - 1|^2} = \frac{b^2 - 1}{12}$,

3. $\frac{1}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} \ell} - 1|} \leq \frac{2}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} \ell} - 1|^2} \lesssim_b 1$ *for all* $\ell \in \{1, \ldots, b-1\}$.

Proof. The first item is a well-known result and can be verified by applying the formula for finite geometric sums. The proof of the second item can be found in [21, Corollary A.23] or in [52, Proposition 3.5]. The last item can be shown directly with aid of the triangle inequality and 2., because

$$\frac{1}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|}=\frac{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}\leq\frac{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}|+|1|}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}=\frac{2}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}$$
$$\leq\sum_{\ell=1}^{b-1}\frac{2}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}=\frac{b^2-1}{6}\lesssim_b 1.$$

The proof is complete. □

We start with the computation of the first Haar coefficient $\tilde{\mu}^N_{-1,0,1}$:

Lemma 4.32. *The Haar coefficient $\tilde{\mu}^N_{-1,0,1}$ of the discrepancy function $\Delta_N(\cdot,\widetilde{\mathcal{V}}_b)$ satisfies*

$$|\tilde{\mu}^N_{-1,0,1}|=\begin{cases}0 & \text{if } N=2M,\\ \left|\frac{1}{2}-\varphi(M)\right|\leq\frac{1}{2} & \text{if } N=2M+1.\end{cases}$$

Proof. For $\widetilde{\mathcal{V}}_b^\sigma=(x_n)_{n\geq 0}$ we have

$$\begin{aligned}\tilde{\mu}^N_{-1,0,1}&=\int_0^1\Delta_N(t,\widetilde{\mathcal{V}}_b^\sigma)\,\mathrm{d}t=\int_0^1\left(\sum_{n=0}^{N-1}\mathbf{1}_{[0,t)}(x_n)-Nt\right)\mathrm{d}t\\&=\sum_{n=0}^{N-1}\int_0^1\mathbf{1}_{[0,t)}(x_n)\,\mathrm{d}t-\frac{N}{2}=\sum_{n=0}^{N-1}(1-x_n)-\frac{N}{2}=\frac{N}{2}-\sum_{n=0}^{N-1}x_n.\end{aligned}$$

We therefore have to investigate the sum $\sum_{n=0}^{N-1}x_n$. If $N=2M$, then we have

$$\sum_{n=0}^{2M-1}x_n=\sum_{m=0}^{M-1}\varphi(m)+\sum_{m=0}^{M-1}(1-\varphi(m))=M=\frac{N}{2}.$$

For $N=2M+1$, we find

$$\sum_{n=0}^{2M}x_n=\sum_{n=0}^{2M-1}x_n+\varphi(M)=M+\varphi(M)=\frac{N-1}{2}+\varphi\left(M\right).$$

This leads to the desired result. □

In the following, let $\tilde{\mu}^N_{j,m,\ell}$, $\mu^{N,\varphi}_{j,m,\ell}$ and $\mu^{N,1-\varphi}_{j,m,\ell}$ be the Haar coefficients of the discrepancy function of the first N elements of the sequences $\widetilde{\mathcal{V}}_b^\sigma$, $(\varphi(n))_{n\geq 0}$ and $(1-\varphi(n))_{n\geq 0}$, respectively.

Lemma 4.33. *For all $j\in\mathbb{N}_0$, $m\in\mathbb{D}_j$ and $\ell\in\mathbb{B}_j$ we have*

$$|\tilde{\mu}^N_{j,m,\ell}|\leq\begin{cases}|\mu^{M,\varphi}_{j,m,\ell}|+|\mu^{M,1-\varphi}_{j,m,\ell}| & \text{if } N=2M,\\ |\mu^{M+1,\varphi}_{j,m,\ell}|+|\mu^{M,1-\varphi}_{j,m,\ell}| & \text{if } N=2M+1.\end{cases}$$

Proof. The proof is simple and similar to the proofs of Lemma 2.9 and Lemma 4.4. □

We proceed with the calculation of the Haar coefficients of the discrepancy function in the case $j\in\mathbb{N}_0$ and prove the following general lemma first.

Lemma 4.34. *Let $j \in \mathbb{N}_0$, $m \in \mathbb{D}_j$ and $\ell \in \mathbb{B}_j$. Then for the linear part $f(t) = Nt$ of the discrepancy function we have*

$$\mu_{j,m,\ell}(f) = \frac{Nb^{-2j-1}}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell} - 1}$$

and for the counting part $g(t) = \sum_{n=0}^{N-1} \mathbf{1}_{[0,t)}(x_n)$ we have

$$\mu_{j,m,\ell}(g) = b^{-j-1} \sum_{k=0}^{b-1} \sum_{\substack{n=0 \\ x_n \in I_{j,m}^k,\, x_n \neq \frac{m}{b^j}}}^{N-1} \left(\left(bm + k - b^{j+1}x_n \right) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r\ell} \right),$$

where the last sum in the brackets is empty for $k = 0$.

Proof. The assertion on $\mu_{j,m,\ell}(f)$ may be verified by simple integration. The Haar coefficients of g are given by

$$\mu_{j,m,\ell}(g) = \int_0^1 \left(\sum_{n=0}^{N-1} \mathbf{1}_{[0,t)}(x_n) h_{j,m,\ell}(t) \right) \mathrm{d}t = \sum_{n=0}^{N-1} \underbrace{\int_0^1 \mathbf{1}_{[0,t)}(x_n) h_{j,m,\ell}(t)\, \mathrm{d}t}_{\mathcal{I}_n}.$$

It is obvious that $\mathcal{I}_n = 0$ in case that $x_n \notin I_{j,m}$ or $x_n = \frac{m}{b^j}$. Now we assume that $x_n \in I_{j,m}^k$ for some $k = 0, 1, \ldots, b-1$. Then we have

$$\begin{aligned}
\mathcal{I}_n &= \int_{x_n}^{\frac{m}{b_j} + \frac{k+1}{b^{j+1}}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k\ell}\, \mathrm{d}t + \sum_{r=k+1}^{b-1} \int_{I_{j,m}^r} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r\ell}\, \mathrm{d}t \\
&= b^{-j-1} \left(\left(bm + k + 1 - b^{j+1}x_n \right) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k\ell} + \sum_{r=k+1}^{b-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r\ell} \right) \\
&= b^{-j-1} \left(\left(bm + k + 1 - b^{j+1}x_n \right) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k\ell} - \sum_{r=0}^{k} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r\ell} \right) \\
&= b^{-j-1} \left(\left(bm + k - b^{j+1}x_n \right) \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b}r\ell} \right)
\end{aligned}$$

and the proof of this lemma is done. □

Now we are ready to show a central lemma.

Lemma 4.35. *We have*

$$|\mu_{j,m,\ell}^{N,\varphi}| \leq \frac{1}{b^j} \frac{9}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell} - 1|^2} \quad \textit{and} \quad |\mu_{j,m,\ell}^{N,1-\varphi}| \leq \frac{1}{b^j} \frac{15}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell} - 1|^2}$$

for all $0 \leq j < \lceil \log_b N \rceil$ and

$$|\mu_{j,m,\ell}^{N,\varphi}| = |\mu_{j,m,\ell}^{N,1-\varphi}| = \frac{Nb^{-2j-1}}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell} - 1|}$$

for all $j \geq \lceil \log_b N \rceil$.

Proof. We start with $x_n = \varphi(n)$ and therefore employ Lemma 4.30. It allows us to display the sum

$$\sum_{k=0}^{b-1} \sum_{\substack{n=0 \\ \varphi(n)\in I_{j,m}^k}}^{N-1} \left(\left(bm + k - b^{j+1}\varphi(n) \right) \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \right),$$

which appears in Lemma 4.34, as

$$\sum_{k=0}^{b-1} \sum_{w=0}^{A(k)} \left(\left(bm + k - b^{j+1}\varphi\left(b^j\varphi^{(-1)}(m) + b^j k + b^{j+1}w \right) \right) \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \right). \tag{4.8}$$

We may include the case $\varphi(n) = \frac{m}{b^j}$ since the corresponding summand is zero anyway. In the above expression, $A(k) := \left\lfloor \frac{N-1}{b^{j+1}} - \frac{\varphi^{(-1)}(m)}{b} - \frac{k}{b} \right\rfloor$. We choose this value for the upper index of the sum, since we have to ensure that the conditions $0 \leq n \leq N-1$ and $n = b^j\varphi^{(-1)}(m) + b^j k + b^{j+1}w$ are fulfilled simultaneously. With the aid of part 1 and 2 of Lemma 4.30 we obtain

$$\varphi\left(b^j\varphi^{(-1)}(m) + b^j k + b^{j+1}w \right) = \frac{m}{b^j} + \frac{1}{b^j}\varphi(k+bw) = \frac{m}{b^j} + \frac{1}{b^j}\left(\frac{k}{b} + \frac{1}{b}\varphi(w) \right),$$

the expression (4.8) can be simplified to

$$-\sum_{k=0}^{b-1} \sum_{w=0}^{A(k)} \left(\varphi(w)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \right) =: S.$$

Next we notice that $A(k)$ can only take two possible values, namely $A(k) = \tilde{A}$ or $A(k) = \tilde{A} - 1$, where $\tilde{A} = \left\lfloor \frac{N-1}{b^{j+1}} - \frac{\varphi^{(-1)}(m)}{b} \right\rfloor$. We assume that $k_0 \in \{1, \ldots, b\}$ is such that $A(k) = \tilde{A}$ for $k \in \{0, \ldots, k_0 - 1\}$ and, in case that $k_0 < b$, $A(k) = \tilde{A} - 1$ for $k \in \{k_0, \ldots, b-1\}$. Hence, we can write

$$S = -\underbrace{\sum_{k=0}^{k_0-1} \left(\varphi(\tilde{A})\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \right)}_{S_1} - \underbrace{\sum_{k=0}^{b-1} \sum_{w=0}^{\tilde{A}-1} \left(\varphi(w)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \right)}_{S_2}.$$

We intend to simplify S_2 and therefore change the order of the sums to obtain

$$\begin{aligned} S_2 &= \sum_{w=0}^{\tilde{A}-1} \varphi(w) \sum_{k=0}^{b-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} + \sum_{w=0}^{\tilde{A}-1} \sum_{k=0}^{b-1} \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell} \\ &= \frac{\tilde{A}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell} - 1} \sum_{k=0}^{b-1} \left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell} - 1 \right) = -\frac{b\tilde{A}}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell} - 1}. \end{aligned}$$

We combine the previous results with Lemma 4.34 to obtain

$$\mu_{j,m,\ell}^{N,\varphi} = \frac{1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell} - 1} \left(\frac{1}{b^j}\tilde{A} - Nb^{-2j-1} \right) - \frac{1}{b^{j+1}}S_1.$$

Now we take the absolute value and apply the triangle inequality. This yields

$$|\mu_{j,m,\ell}^{N,\varphi}| \leq \frac{1}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell} - 1|} \left| \frac{1}{b^j}\tilde{A} - Nb^{-2j-1} \right| + \frac{1}{b^{j+1}}|S_1|.$$

The inequalities $x-1<\lfloor x\rfloor\leq x$ for all $x\in\mathbb{R}$ yield

$$\frac{1}{b^j}\tilde{A}-Nb^{-2j-1}\leq\frac{1}{b^j}\left(\frac{N-1}{b^{j+1}}-\frac{\varphi(m)}{b}\right)-Nb^{-2j-1}=-\frac{1}{b^{2j+1}}-\frac{\varphi(m)}{b^{j+1}}<0$$

and

$$\begin{aligned}\frac{1}{b^j}\tilde{A}-Nb^{-2j-1}&\geq\frac{1}{b^j}\left(\frac{N-1}{b^{j+1}}-\frac{\varphi(m)}{b}-1\right)-Nb^{-2j-1}\\&=-\frac{1}{b^{2j+1}}-\frac{\varphi(m)}{b^{j+1}}-\frac{1}{b^j}\geq-\frac{b+2}{b^{j+1}}\geq-\frac{2}{b^j};\end{aligned}$$

thus we get

$$\left|\mu_{j,m,\ell}^{N,\varphi}\right|\leq\frac{1}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}\frac{2}{b^j}+\frac{1}{b^{j+1}}|S_1|.$$

It remains to estimate $|S_1|$. We have

$$\begin{aligned}|S_1|&\leq\varphi(\tilde{A})\left|\sum_{k=0}^{k_0-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell}\right|+\left|\sum_{k=0}^{k_0-1}\sum_{r=0}^{k-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell}\right|\\&\leq\frac{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_0\ell}-1|}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}+\left|\frac{1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1}\left(\frac{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k_0\ell}-1}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1}-k_0\right)\right|\\&\leq\frac{2}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}+\frac{2}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2}+\frac{b}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}\leq\frac{5b}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2},\end{aligned}$$

where we used Lemma 4.31 in the last step. Altogether, we have verified

$$\left|\mu_{j,m,\ell}^{N,\varphi}\right|\leq\frac{1}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}\frac{2}{b^j}+\frac{1}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2}\frac{5}{b^j}\leq\frac{1}{b^j}\frac{9}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2}.$$

This proves the first estimate of the lemma. It follows from Lemma 4.30, that there are no elements of $\{x_0,x_1,\ldots,x_{N-1}\}$ contained in the interior of $I_{j,m}$, if $b^j\varphi(m)+b^j\geq N$, which is certainly fulfilled if $j\geq\lceil\log_b N\rceil$. Therefore, in this case the counting part does not contribute to the Haar coefficient $\mu_{j,m,\ell}^{N,\varphi}$ as we have seen in the proof of Lemma 4.34, and we have

$$|\mu_{j,m,\ell}^{N,\varphi}|=\frac{Nb^{-2j-1}}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}$$

as claimed.

We investigate $\left|\mu_{j,m,\ell}^{N,1-\varphi}\right|$ and write

$$\sum_{k=0}^{b-1}\sum_{\substack{n=0\\1-\varphi(n)\in I_{j,m}^k\\1-\varphi(n)\neq\frac{m}{b^j}}}^{N-1}\left(\left(bm+k-b^{j+1}(1-\varphi(n))\right)\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}k\ell}-\sum_{r=0}^{k-1}\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}r\ell}\right)$$

as

$$
\sum_{k=0}^{b-1} \sum_{\substack{n=0 \\ 1-\varphi(n)\in I_{j,m}^{k,\circ}}}^{N-1} \left(\Big(bm + k - b^{j+1}(1-\varphi(n))\Big) \, \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right)
$$

$$
- \sum_{k=1}^{b-1} \sum_{\substack{n=0 \\ 1-\varphi(n)=\frac{m}{b^j}+\frac{k}{b^{j+1}}}}^{N-1} \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} =: T_1 - T_2.
$$

It is easily shown that $1-\varphi(n) \in I_{j,m}^{k,\circ}$ if and only if $\varphi(n) \in I_{j,b^j-m-1}^{b-k-1,\circ}$. In analogy to preceding parts of this proof we define $B(k) := \left\lfloor \frac{N-1}{b^{j+1}} - \frac{\varphi^{(-1)}(b^j-m-1)}{b} - \frac{b-k-1}{b} \right\rfloor$ as well as $\widetilde{B} := \left\lfloor \frac{N-1}{b^{j+1}} - \frac{\varphi^{(-1)}(b^j-m-1)}{b} \right\rfloor$. Let $k_0' \in \{0,\dots,b-1\}$ be such that $B(k) = \tilde{B}$ for $k \in \{k_0',\dots,b-1\}$ and, in case that $k_0' > 0$, $B(k) = \tilde{B}-1$ for $k \in \{0,\dots,k_0'-1\}$. We apply Lemma 4.30 to obtain

$$
\begin{aligned}
T_1 &= \sum_{k=0}^{b-1} \sum_{\substack{n=0 \\ \varphi(n)\in I_{j,b^j-m-1}^{b-k-1,\circ}}}^{N-1} \left(\Big(bm + k - b^{j+1}(1-\varphi(n))\Big) \, \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right) \\
&= \sum_{k=0}^{b-1} \sum_{w=1}^{B(k)} \Bigg(\Big(bm + k - b^{j+1}\Big(1-\varphi(b^{j+1}\varphi^{(-1)}(b(b^j-m-1)+b-k-1)+b^{j+1}w)\Big)\Big) \\
&\qquad\qquad \times \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} - \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \Bigg) \\
&= - \sum_{k=0}^{b-1} \sum_{w=1}^{B(k)} \left((\varphi(w)+1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right) \\
&= \underbrace{- \sum_{k=k_0'}^{b-1} \left((\varphi(\tilde{B})+1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right)}_{T_{1,1}} \underbrace{- \sum_{k=0}^{b-1} \sum_{w=1}^{\tilde{B}-1} \left((\varphi(w)+1)\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} + \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right)}_{T_{1,2}}.
\end{aligned}
$$

Similarly as above, we can show that

$$
T_{1,2} = -\frac{b(\tilde{B}-1)}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1}.
$$

Altogether, we have

$$
\mu_{j,m,\ell}^{N,1-\varphi} = \frac{1}{\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1} \left(\frac{1}{b^j}(\tilde{B}-1) - Nb^{-2j-1} \right) - \frac{1}{b^{j+1}}(T_{1,1}+T_2),
$$

and so the triangle inequality gives

$$
|\mu_{j,m,\ell}^{N,1-\varphi}| \le \frac{1}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|} \left| \frac{1}{b^j}(\widetilde{B}-1) - Nb^{-2j-1} \right| + \frac{1}{b^{j+1}}(|T_{1,1}|+|T_2|).
$$

One can check in the same manner as done above that $\left| \frac{1}{b^j}(\widetilde{B}-1) - Nb^{-2j-1} \right| \le \frac{1}{N}\frac{3}{b^j}$ and $|T_{1,1}| \le \frac{7b}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}$. We also find

$$
|T_2| \le \left| \sum_{k=1}^{b-1} \sum_{r=0}^{k-1} \mathrm{e}^{\frac{2\pi \mathrm{i}}{b} r\ell} \right| \le \frac{1}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|} \left| \sum_{k=0}^{b-1} \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{b} k\ell} - 1 \right) \right| = \frac{b}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|} \le \frac{2b}{|\mathrm{e}^{\frac{2\pi \mathrm{i}}{b}\ell}-1|^2}.
$$

By combining all these results we finally arrive at

$$|\mu_{j,m,\ell}^{N,1-\varphi}| \leq \frac{1}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2}\left(\frac{6}{b^j}+\frac{7}{b^j}+\frac{2}{b^j}\right) = \frac{1}{b^j}\frac{15}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2}.$$

The equality $|\mu_{j,m,\ell}^{N,1-\varphi}| = \frac{Nb^{-2j-1}}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|}$ for $j \geq \lceil \log_b N \rceil$ can be verified analogously as in the case of $|\mu_{j,m,\ell}^{N,\varphi}|$ and the proof of the lemma is complete. □

Corollary 4.36. The Haar coefficients of the symmetrized van der Corput sequence in base b for $j \in \mathbb{N}_0$ satisfy

$$|\widetilde{\mu}_{j,m,\ell}^N| \begin{cases} \leq \frac{1}{b^j}\frac{26}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|^2} \lesssim_b \frac{1}{b^j} & \text{if } j < \lceil \log_b N \rceil, \\ = \frac{Nb^{-2j-1}}{|\mathrm{e}^{\frac{2\pi\mathrm{i}}{b}\ell}-1|} \lesssim_b \frac{N}{b^{2j}} & \text{if } j \geq \lceil \log_b N \rceil. \end{cases}$$

Proof. We combine Lemma 4.31, Lemma 4.33 and Lemma 4.35 to obtain the result. □

In order to prove the result on the non-symmetrized sequences we only need to consider the first Haar coefficient $\mu_{-1,0,1}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))$.

Proposition 4.37. *We have*

$$|\mu_{-1,0,1}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))| \gtrsim \log N$$

for infinitely many $N \in \mathbb{N}$.

Proof. By Parseval's equality (2.21) we have

$$\|\Delta_N(\cdot,\mathcal{V}_b^\sigma)\|_{L_2([0,1))}^2 = |\mu_{-1,0,1}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))|^2 + \sum_{j\in\mathbb{N}_0, m\in\mathbb{D}_j, \ell\in\mathbb{B}_j} b^{|j|}|\mu_{j,m,\ell}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))|^2$$

Using Lemma 4.35, we find

$$\begin{aligned} &\sum_{j\in\mathbb{N}_0, m\in\mathbb{D}_j, \ell\in\mathbb{B}_j} b^{|j|}|\mu_{j,m,\ell}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))|^2 \\ &\lesssim_b \sum_{j=0}^{\lceil \log_b N\rceil -1} b^{2j}\left(\frac{1}{b^j}\right)^2 + \sum_{j=\lceil \log_b N\rceil}^{\infty} b^{2j}\left(\frac{N}{b^{2j}}\right)^2 \leq c_1 \log N, \end{aligned}$$

for some positive constant c_1 depending only on b and for all $N \geq 2$. From the fact that $L_{2,N}(\mathcal{V}_b^\sigma) \geq c_2 \log N$ for a constant $c_2 > 0$ depending only on b and for infinitely many N, we derive

$$|\mu_{-1,0,1}(\Delta_N(\cdot,\mathcal{V}_b^\sigma))|^2 \geq c_2^2(\log N)^2 - c_1\log N \gtrsim_b (\log N)^2$$

for infinitely many N. □

Remark 4.38. A direct proof of Proposition 4.37 for $b = 2$ can be found in [25] (see also [45, Remark 1]).

Proof of Theorem 4.29 We first prove the result on the non-symmetrized van der Corput sequences. Let $\mu_{j,m,\ell}^N$ be the Haar coefficients of the discrepancy function of the first N elements of $\mathcal{V}_b^\sigma$. We employ the lower bound given in Proposition 2.12 and obtain for all $p \in (1,\infty)$

$$\begin{aligned}(L_{p,N}(\mathcal{V}_b^\sigma))^2 =& \|\Delta_N(\cdot,\mathcal{V}_b^\sigma)\|_{L_p([0,1))}^2 \gtrsim \sum_{j\in\mathbb{N}_{-1}} b^{2j(1-1/p')}\left(\sum_{m\in\mathbb{D}_j,\ell\in\mathbb{B}_j} |\mu_{j,m,\ell}^N|^{p'}\right)^{2/p'} \\ \geq & b^{-2(1-1/p')}\left(|\mu_{-1,0,1}^N|^{p'}\right)^{2/p'} \gtrsim |\mu_{-1,0,1}^N|^2 \gtrsim (\log N)^2\end{aligned}$$

for infinitely many N, where we regarded Proposition 4.37 in the last step. To verify the result on the symmetrized van der Corput sequences, we need to take into account all the Haar coefficients. We use now the upper bound on the L_p norm as given in Proposition 2.12 and insert there the bounds on the Haar coefficients according to Corollary 4.36. We have

$$\begin{aligned}(L_{p,N}(\widetilde{\mathcal{V}}_b^\sigma))^2 =& \|\Delta_N(\cdot,\widetilde{\mathcal{V}}_b^\sigma)\|_{L_p([0,1))}^2 \lesssim \sum_{j\in\mathbb{N}_{-1}} b^{2j(1-1/\bar{p})}\left(\sum_{m\in\mathbb{D}_j,\ell\in\mathbb{B}_j} |\tilde{\mu}_{j,m,\ell}^N|^{\bar{p}}\right)^{2/\bar{p}} \\ \lesssim & b^{-2(1-1/\bar{p})}\left(|\mu_{-1,0,1}^N|^{\bar{p}}\right)^{2/\bar{p}} + \sum_{j=0}^{\lceil\log_b N\rceil-1} b^{2j(1-1/\bar{p})}\left(\sum_{m\in\mathbb{D}_j,\ell\in\mathbb{B}_j}\left(\frac{1}{b^j}\right)^{\bar{p}}\right)^{2/\bar{p}} \\ &+ \sum_{j=\lceil\log_b N\rceil}^{\infty} b^{2j(1-1/\bar{p})}\left(\sum_{m\in\mathbb{D}_j,\ell\in\mathbb{B}_j}\left(\frac{N}{b^{2j}}\right)^{\bar{p}}\right)^{2/\bar{p}} \\ \lesssim & b^{-2(1-1/\bar{p})} + \sum_{j=0}^{\lceil\log_b N\rceil-1} b^{2j(1-1/\bar{p})}b^{2j/\bar{p}}\frac{1}{b^{2j}} + \sum_{j=\lceil\log_b N\rceil}^{\infty} b^{2j(1-1/\bar{p})}b^{2j/\bar{p}}\frac{N^2}{b^{4j}} \\ \lesssim & 1 + \sum_{j=0}^{\lceil\log_b N\rceil-1} 1 + N^2\sum_{j=\lceil\log_b N\rceil}^{\infty}\frac{1}{b^{2j}} \lesssim \log N.\end{aligned}$$

The results follows now by taking the square root in both cases, respectively. □

4.2.2. Optimal discrepancy rate of $\widetilde{\mathcal{V}}_b^\sigma$ in several other norms

As we have already seen in Section 2.3, the estimation of the Haar coefficients of the discrepancy function is the key to give upper bounds of its norm in various normed function spaces. In this subsection, we will give such upper bounds for the class of symmetrized van der Corput sequences. We will consider Besov, Triebel-Lizorkin, BMO and exponential Orlicz norms. We start with a theorem on the Besov norm, which demonstrates that the symmetrization of the van der Corput sequence is only necessary if the smoothness parameter r is zero, whereas for $r > 0$ also the non-symmetrized version achieves the optimal discrepancy rate.

Theorem 4.39. *Let $1 \leq p,q \leq \infty$ and $0 \leq r < \frac{1}{p}$. Then for any integer $b \geq 2$ we have*

$$\left\|\Delta_N(\cdot,\widetilde{\mathcal{V}}_b^\sigma)\right\|_{S_{p,q}^rB([0,1))} \lesssim (\log N)^{\frac{1}{q}}$$

for all $N \geq 2$ and

$$\|\Delta_N(\cdot,\mathcal{V}_b^\sigma)\|_{S_{p,q}^rB([0,1))} \gtrsim \log N$$

for infinitely many $N \in \mathbb{N}$ *if* $r = 0$ *and*

$$\left\| \Delta_N(\cdot, \widetilde{\mathcal{V}}_b^\sigma) \right\|_{S_{p,q}^r B([0,1))} \lesssim N^r$$

as well as

$$\| \Delta_N(\cdot, \mathcal{V}_b^\sigma) \|_{S_{p,q}^r B([0,1))} \lesssim N^r$$

if $0 < r < \frac{1}{p}$ *for all* $N \geq 2$.

Proof. Let us first consider the symmetrized van der Corput sequences. From Proposition 2.11 it follows that it suffices to show

$$\left(\sum_{j \in \mathbb{N}_{-1}} b^{j\left(r - \frac{1}{p} + 1\right)q} \left(\sum_{m \in \mathbb{D}_j, \ell \in \mathbb{B}_j} \left| \widetilde{\mu}_{j,m,\ell}^N \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \begin{cases} (\log N)^{\frac{1}{q}} & \text{if } r = 0, \\ N^r & \text{if } 0 < r < \frac{1}{p}. \end{cases}$$

Since $q \geq 1$, we have

$$\begin{aligned}
&\left(\sum_{j \in \mathbb{N}_{-1}} b^{j\left(r - \frac{1}{p} + 1\right)q} \left(\sum_{m \in \mathbb{D}_j, \ell \in \mathbb{B}_j} \left| \widetilde{\mu}_{j,m,\ell}^N \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\lesssim |\mu_{-1,0,1}| + \left(\sum_{j=0}^{\lceil \log_b N \rceil - 1} b^{j\left(r - \frac{1}{p} + 1\right)q} \left(\sum_{m \in \mathbb{D}_j, \ell \in \mathbb{B}_j} \left| \widetilde{\mu}_{j,m,\ell}^N \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{j=\lceil \log_b N \rceil}^{\infty} b^{j\left(r - \frac{1}{p} + 1\right)q} \left(\sum_{m \in \mathbb{D}_j, \ell \in \mathbb{B}_j} \left| \widetilde{\mu}_{j,m,\ell}^N \right|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} =: S_1 + S_2 + S_3.
\end{aligned}$$

We apply Lemma 4.32 and Corollary 4.36. We have $S_1 \lesssim 1 \lesssim N^r$ for all $0 \leq r < \frac{1}{p}$. We also find

$$S_2 \lesssim \left(\sum_{j=0}^{\lceil \log_b N \rceil - 1} b^{j\left(r - \frac{1}{p} + 1\right)q} \left(b^j \left(\frac{1}{b^j} \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} = \left(\sum_{j=0}^{\lceil \log_b N \rceil - 1} b^{jqr} \right)^{\frac{1}{q}}.$$

The assumption $r = 0$ leads to

$$S_2 \lesssim \left(\sum_{j=0}^{\lceil \log_b N \rceil - 1} 1 \right)^{\frac{1}{q}} \lesssim (\log N)^{\frac{1}{q}},$$

whereas for $0 < r < \frac{1}{p}$ we obtain

$$S_2 \lesssim \left(\sum_{j=0}^{\lceil \log_b N \rceil - 1} b^{jqr} \right)^{\frac{1}{q}} \lesssim (b^{\log_b N})^r = N^r.$$

It remains to estimate S_3. We have

$$S_3 \lesssim \left(\sum_{j=\lceil \log_b N \rceil}^{\infty} b^{j\left(r-\frac{1}{p}+1\right)q} \left(b^j \left(\frac{N}{b^{2j}} \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} = N \left(\sum_{j=\lceil \log_b N \rceil}^{\infty} b^{jq(r-1)} \right)^{\frac{1}{q}}$$
$$\lesssim N b^{\log_b N (r-1)} = N^r,$$

which concludes the proof of the claims on the symmetrized van der Corput sequences. Now we investigate the non-symmetrized van der Corput sequences $\mathcal{V}_b^\sigma$. Note that for $j \neq -1$ the corresponding Haar coefficients of $\Delta_N(\cdot, \mathcal{V}_b^\sigma)$ are of the same order as those of $\Delta_N(\cdot, \widetilde{\mathcal{V}}_b^\sigma)$. It is therefore enough to consider only the Haar coefficient $\mu_{-1,0,1}(\Delta_N(\cdot, \mathcal{V}_b^\sigma))$. If $r > 0$, we have $\mu_{-1,0,1}(\Delta_N(\cdot, \mathcal{V}_b^\sigma)) \lesssim \log N \lesssim N^r$ and the proof is complete. If, however, r is zero, then we have $\mu_{-1,0,1}(\Delta_N(\cdot, \mathcal{V}_b^\sigma)) \gtrsim \log N$ and hence $\|\Delta_N(\cdot, \mathcal{V}_b^\sigma)\|_{S^r_{p,q}B([0,1))} \gtrsim \log N$ by Proposition 2.11. □

From the embedding Theorem 2.24 we immediately obtain the following results on the Triebel-Lizorkin spaces:

Theorem 4.40. *Let $1 \leq p, q \leq \infty$ and $0 \leq r < \frac{1}{\max\{p,q\}}$. Then for any integer $b \geq 2$ we have*

$$\left\| \Delta_N(\widetilde{\mathcal{V}}_b^\sigma) \right\|_{S^r_{p,q}F([0,1))} \lesssim (\log N)^{\frac{1}{q}}$$

for all $N \geq 2$ and

$$\|\Delta_N(\mathcal{V}_b^\sigma)\|_{S^r_{p,q}F([0,1))} \gtrsim \log N$$

for infinitely many $N \in \mathbb{N}$ if $r = 0$ and

$$\left\| \Delta_N(\widetilde{\mathcal{V}}_b^\sigma) \right\|_{S^r_{p,q}F([0,1))} \lesssim N^r$$

as well as

$$\|\Delta_N(\mathcal{V}_b^\sigma)\|_{S^r_{p,q}F([0,1))} \lesssim N^r$$

if $0 < r < \frac{1}{\max\{p,q\}}$ for all $N \geq 2$.

By putting $q = 2$ in Theorem 4.40 we obtain results for Sobolov spaces. We recover Theorem 4.29 by additionally setting $r = 0$.

For the rest of this section, we restrict ourselves to the dyadic case, where we can use the apparatus we have for the BMO and exponential Orlicz norms. We first prove the following result on the BMO norm of the discrepancy function.

Theorem 4.41. *We have*

$$\|\Delta_N(\cdot, \mathcal{V}_2)\|_{\mathrm{BMO}([0,1))} \leq \sqrt{\log N}$$

and

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\mathrm{BMO}([0,1))} \leq \sqrt{\log N}$$

for all $N \geq 2$.

Proof. We start with the classical van der Corput sequence. We fix an arbitrary measurable set $U \subset [0,1)$. Since for a fixed $j \in \mathbb{N}_0$ we can estimate

$$|U| \geq \sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} |I_{j,m}| = 2^{-j} \sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} 1,$$

we have $\sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} 1 \leq 2^j|U|$. By the definition of the BMO norm we need to prove

$$|U|^{-1} \sum_{j\in\mathbb{N}_0} 2^j \sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} |\mu_{j,m}^N|^2 \lesssim \log N,$$

because then the supremum extended over all $U \subset [0,1)$ satisfies this upper bound as well. We distinguish between two cases and use Lemma 4.35. We first assume that $0 \leq j < \lceil \log_2 N \rceil$ and obtain

$$|U|^{-1} \sum_{j=0}^{\lceil \log_2 N\rceil -1} 2^j \sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} |\mu_{j,m}^N|^2 \lesssim |U|^{-1} \sum_{j=0}^{\lceil \log_2 N\rceil -1} 2^j \frac{1}{2^{2j}} 2^j |U| \lesssim \log N.$$

For $j \geq \lceil \log_2 N \rceil$ we can estimate

$$|U|^{-1} \sum_{j=\lceil \log_2 N\rceil}^{\infty} 2^j \sum_{\substack{m\in\mathbb{D}_j \\ I_{j,m}\subset U}} |\mu_{j,m}^N|^2 \lesssim |U|^{-1} \sum_{j=\lceil \log_2 N\rceil}^{\infty} 2^j \frac{N^2}{2^{4j}} 2^j |U| = N^2 \sum_{j=\lceil \log_2 N\rceil}^{\infty} \frac{1}{2^{2j}} \lesssim 1.$$

This yields the BMO norm estimate for the local discrepancy of $\mathcal{V}_2$. We can proceed completely analogously for the symmetrized version of the van der Corput sequence by using Corollary 4.36 (note that we do not need the delicate first Haar coefficient $\mu_{-1,0}^N$ in the estimation of the BMO norm). □

Theorem 4.42. *Let $\beta > 0$. Then we have*

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\exp(L^\beta)} \lesssim (\log N)^{1-\frac{1}{\max\{2,\beta\}}}.$$

Proof. Let us first consider the case $\beta = 2$. The Chang-Wilson-Wolff inequality (Proposition 2.14) and Corollary 4.36 give

$$\begin{aligned}
\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\exp(L^2)} &\lesssim \left\| \left(\sum_{j\in\mathbb{N}_{-1}} 2^{2|j|} \sum_{m\in\mathbb{D}_j} \left|\left\langle \Delta_N(\cdot, \widetilde{\mathcal{V}}_2), h_{j,m} \right\rangle\right|^2 \mathbf{1}_{I_{j,m}} \right)^{\frac{1}{2}} \right\|_{L_\infty([0,1))} \\
&\leq \left(\sum_{j\in\mathbb{N}_{-1}} 2^{2|j|} \left\| \sum_{m\in\mathbb{D}_j} |\tilde{\mu}_{j,m}^N|^2 \mathbf{1}_{I_{j,m}} \right\|_{L_\infty([0,1))} \right)^{\frac{1}{2}} \\
&\lesssim \left(\left\| \mathbf{1}_{[0,1)} \right\|_{L_\infty([0,1))} \right)^{\frac{1}{2}} + \left(\sum_{j=0}^{\lfloor \log_2 N \rfloor -1} \left\| \sum_{m\in\mathbb{D}_j} \mathbf{1}_{I_{j,m}} \right\|_{L_\infty([0,1))} \right)^{\frac{1}{2}} \\
&\quad + N \left(\sum_{j=\lfloor \log_2 N\rfloor}^{\infty} 2^{-2j} \left\| \sum_{m\in\mathbb{D}_j} \mathbf{1}_{I_{j,m}} \right\|_{L_\infty([0,1))} \right)^{\frac{1}{2}} \lesssim \sqrt{\log N},
\end{aligned}$$

where we regarded $\left\|\sum_{m\in\mathbb{D}_j} \mathbf{1}_{I_{j,m}}\right\|_{L_\infty([0,1))} = 1$.
The result for $0 < \beta < 2$ follows from the fact that $\|f\|_{\exp(L^\alpha)} < \|f\|_{\exp(L^\beta)}$ for $0 < \alpha < \beta < \infty$. Hence we have $\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\exp(L^\beta)} \lesssim \sqrt{\log N}$ in this case.
For $\beta > 2$ we can obtain our result by interpolation between Orlicz spaces and the L_∞ space. For $0 < \alpha < \beta < \infty$ and $f \in L_\infty([0,1))$ we have

$$\|f\|_{\exp(L^\beta)} \lesssim \left(\|f\|_{\exp(L^\alpha)}\right)^{\frac{\alpha}{\beta}} \left(\|f\|_{L_\infty([0,1))}\right)^{1-\frac{\alpha}{\beta}}$$

(see [7, Proposition 2.4]). We show $L_{\infty,N}(\widetilde{\mathcal{V}}_2) \lesssim \log N$. Let $N = 2M$ and $n \in \mathbb{N}$ such that $2^{n-1} \leq M < 2^n$ for an arbitrary $n \in \mathbb{N}$. Then we have

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{L_\infty([0,1))} \leq \|\Delta_M(\cdot, \mathcal{V}_2)\|_{L_\infty([0,1))} + \|\Delta_M(\cdot, 1-\varphi_2)\|_{L_\infty([0,1))},$$

where $1-\varphi_2$ denotes the sequence $(1-\varphi_2(n))_{n\geq 0}$. With similar arguments as in Lemma 2.3 and in the proof of Theorem 3.29 we find

$$\left|\|\Delta_M(\cdot, 1-\varphi_2)\|_{L_\infty([0,1))} - \|\Delta_M(\cdot, \varphi_2^{\tau_2})\|_{L_\infty([0,1))}\right| \leq 1,$$

where $\varphi_2^{\tau_2}$ is as defined in the proof of Theorem 3.29. This fact yields

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{L_\infty([0,1))} \lesssim \|\Delta_M(\cdot, \mathcal{V}_2)\|_{L_\infty([0,1))} + \|\Delta_M(\cdot, \varphi_2^{\tau_2})\|_{L_\infty([0,1))}.$$

We already know from [26] that $\|\Delta_M(\cdot, \mathcal{V}_2)\|_{L_\infty([0,1))} \lesssim \log M$ and $\|\Delta_M(\cdot, \mathcal{V}_2^{\tau_2})\|_{L_\infty([0,1))} \lesssim \log M$. Therefore we have

$$\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{L_\infty([0,1))} \leq 2\log M \lesssim \log N.$$

This yields the claim on the star discrepancy of the symmetrized van der Corput sequence since n was arbitrary (the proof for odd N is basically the same). Hence we obtain

$$\begin{aligned}\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\exp(L^\beta)} &\lesssim \left(\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{\exp(L^2)}\right)^{\frac{2}{\beta}} \left(\|\Delta_N(\cdot, \widetilde{\mathcal{V}}_2)\|_{L_\infty([0,1))}\right)^{1-\frac{2}{\beta}} \\ &\lesssim \left(\sqrt{\log N}\right)^{\frac{2}{\beta}} (\log N)^{1-\frac{2}{\beta}} \lesssim (\log N)^{1-\frac{1}{\beta}}\end{aligned}$$

and the proof is complete. □

4.3. Conclusions

We would like to summarize several general phenomena which occur with the analysis of the L_p discrepancy of point sets in $[0,1)^2$ and sequences in $[0,1)$.

- We observe that the reason for the large L_p discrepancy of the Hammersley point set and the van der Corput sequence is the large Haar coefficient of the corresponding local discrepancy for $\boldsymbol{j} = (-1,-1)$ and $j = -1$, respectively, whereas all the other Haar coefficients are small enough to achieve the optimal order of L_p discrepancy. Modifying the Hammersley point set and the van der Corput sequence (e.g. by digit scrambling with permutations or symmetrization) leads to a significant reduction of the aforementioned first Haar coefficient. While for Hammersly point sets this reduction can occur for suitable permutations, this is not the case for the van der Corput sequence. However, the method of symmetrization leads to the optimal L_p discrepancy rate in both cases. It has already been observed by Davenport [16] that the symmetrization of point sets has the effect that the zeroth Fourier coefficient of the discrepancy function is reduced and therefore the symmetrized point sets $\widetilde{\mathcal{L}}_N(\alpha)$ as considered in Example 1.9 achieve the optimal order of L_2 discrepancy.

- We have seen the following: If our modified point sets and sequences achieve the optimal order of L_2 discrepancy, they always have the optimal order of L_p discrepancy for all $p \in (1, \infty)$ simultaneously. We do not know if there is a general law behind this observation.

- Finally, we note that constructions of point sets and sequences with the optimal order of L_p discrepancy seem to achieve the optimal discrepancy rate in other function spaces such as Besov spaces with dominating mixed smoothness too.

A. Appendix - Some arithmetics concerning c_b^σ

In this paragraph we give all the missing proofs of Section 3.2.2. Our aim is to prove the following three propositions.

Proposition A.1 provides the formula for the constant c_b^σ appearing in Theorem 3.17, which has already been mentioned in Lemma 3.27 and was essential to obtain the numerical results for c_b^σ in Section 3.2.2.

Proposition A.1. *Let $\sigma \in \mathcal{A}_b(\tau)$. Then we have*

$$\begin{aligned} c_b^\sigma =& \frac{16 - 12b - 111b^2 + 228b^3 - 112b^4}{72b^2} - \frac{1-(-1)^b}{16b^3} \\ &+ \frac{4}{b^3} \sum_{k_1,k_2=0}^{b-1} \max\{\sigma(k_1), \sigma(k_2)\} \left(\frac{b}{2} \Big(\max\{k_1, k_2\} + \max\{k_1 + k_2, b-1\} \Big) - k_1^2 - k_1 \right). \end{aligned}$$

Proposition A.2 contains the relations we used in Remark 3.26 to simplify our formula for $L_2(\widetilde{\mathcal{H}}_{b,n}^\Sigma)$ as stated in Theorem 3.17.

Proposition A.2. *For $\sigma \in \mathcal{A}_b(\tau)$ we have the relation*

$$\widetilde{\Phi}_b^\sigma - \frac{1}{2}\widetilde{\Phi}_{b,1}^\sigma - \frac{1}{2}\widetilde{\Phi}_{b,2}^\sigma = \begin{cases} -\frac{1}{24} & \text{if } b \text{ is even },\\ -\frac{1}{24}\frac{b^2-1}{b^2} & \text{if } b \text{ is odd }. \end{cases}$$

Finally, Proposition A.3 confirms the fact that the value of the constant c_b^σ is invariant with respect to certain transformations on the permutation σ.

Proposition A.3. *Let $\sigma \in \mathcal{A}_b(\tau)$ and $d \in \{0, \ldots, b-1\}$. Then we define the permutation $\widehat{\sigma} \in \mathcal{A}_b(\tau)$ in the following way: For $k \in \{0, \ldots, b-1\} \setminus \{d, b-1-d\}$ we set $\widehat{\sigma}(k) = \sigma(k)$ and additionally we set $\widehat{\sigma}(d) = \sigma(b-1-d)$ and $\widehat{\sigma}(b-1-d) = \sigma(d)$. Then we have $c_b^\sigma = c_b^{\widehat{\sigma}}$.*

The proofs are similar and of elementary, but technical nature. The fundamental properties of the function $\psi_{b,h}^\sigma$, which are required for the proofs, are the following: Let $l \in \{0, 1, \ldots, b-1\}$ and $\sigma \in \mathfrak{S}_b$. Then we have

$$\textbf{(P1)} \qquad (\psi_{b,h}^\sigma)'\left(\frac{l}{b} + 0\right) = (\psi_{b,h}^{id})'\left(\frac{\sigma(l)}{b} + 0\right),$$

$$\textbf{(P2)} \qquad \psi_{b,h}^\sigma\left(\frac{l}{b}\right) = \frac{1}{b}\sum_{k=0}^{l-1}(\psi_{b,h}^\sigma)'\left(\frac{k}{b} + 0\right),$$

$$\textbf{(P2)}' \qquad \sum_{k=0}^{b-1}(\psi_{b,h}^\sigma)'\left(\frac{k}{b} + 0\right) = 0.$$

Note that **(P2)'** follows directly from **(P2)** by setting $l = b$. We will often use the same tricks. It is evident that

$$\sum_{k=0}^{b-1} f(k) = \sum_{k=0}^{b-1} f(b-1-k) \quad \text{or} \quad \sum_{k=0}^{b-1} f(\sigma(k)) = \sum_{k=0}^{b-1} f(k),$$

where f is some expression depending on the index k. We will also need simple relations concerning the maximum of two real numbers; e.g.

$$\max\{a+b, a+c\} = a + \max\{b, c\} \quad \text{or} \quad \max\{-a, -b\} = -\min\{a, b\}$$

for $a, b, c \in \mathbb{R}$.

In order to prove the central propositions in this Appendix, we need several auxiliary results. First, we proof the following lemma, which holds for arbitrary permutations in $\mathfrak{S}_b$.

Lemma A.4. *For all $\sigma \in \mathfrak{S}_b$ we have*

$$\begin{aligned}\Phi_b^{\sigma,(2)} =& \frac{1-6b^2+9b^3-4b^4}{18b^2} \\ &- \frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}\left(\frac{k_1^2+k_1+k_2^2+k_2}{2} - b\max\{k_1,k_2\}\right)\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Phi}_b^{\sigma} =& \frac{c-6b-33b^2+78b^3-40b^4}{72b^2} \\ &- \frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}\left(\frac{k_1^2+k_1+k_2^2+k_2}{2} - b\max\{k_1+k_2, b-1\}\right),\end{aligned}$$

where $c = 4$ for even bases b and $c = 1$ for odd bases b. We also have

$$\begin{aligned}\widetilde{\Phi}_{b,1}^{\sigma} =& -\frac{c_1+18b-9b^2-54b^3+40b^4}{72b^2} \\ &- \frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}\left(\frac{k_1^2-k_1+k_2^2-k_2}{2} - (b-1)\max\{k_1+k_2, b\}\right) \\ &+ \frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b}}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}(k_1+k_2-b),\end{aligned}$$

as well as

$$\begin{aligned}\widetilde{\Phi}_{b,2}^{\sigma} =& -\frac{c_1+30b+27b^2-102b^3+40b^4}{72b^2} \\ &- \frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}\left(\frac{k_1^2+3k_1+k_2^2+3k_2}{2}\right. \\ &\qquad\qquad\left. - (b+1)\max\{k_1+k_2, b-2\}\right) \\ &- \frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b-2}}^{b-1} \max\{\sigma^{-1}(k_1), \sigma^{-1}(k_2)\}(k_1+k_2-b+2)\end{aligned}$$

where $c_1 = 8$ for even bases b and $c_1 = 5$ for odd bases b, respectively.

Proof. The first formula is [31, Lemma 8]. We show the second formula. By Simpson's rule we have

$$\begin{aligned}\widetilde{\Phi}_b^\sigma =&\frac{1}{b}\int_0^1 \widetilde{\psi}_b^\sigma(x)\,\mathrm{d}x = \frac{1}{3b^2}\sum_{l=0}^{b-1}\widetilde{\psi}_b^\sigma\left(\frac{l}{b}\right) + \frac{2}{3b^2}\sum_{l=0}^{b-1}\widetilde{\psi}_b^\sigma\left(\frac{l}{b}+\frac{1}{2b}\right)\\ =&\frac{1}{3b^2}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\psi_{b,h}^\sigma\left(\frac{l}{b}\right)\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)\\ &+\frac{2}{3b^2}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\left(\psi_{b,h}^\sigma\left(\frac{l}{b}\right)+\frac{1}{2b}(\psi_{b,h}^\sigma)'\left(\frac{l}{b}+0\right)\right)\left(\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)+\frac{1}{2b}(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right)\right),\end{aligned}$$

where in the last step we regarded the fact that $\psi_{b,h}^\sigma$ and $\psi_{b,h}^{\overline{\sigma}}$ are piecewise linear on intervals $\left[\frac{k}{b},\frac{k+1}{b}\right)$ for $k\in\mathbb{N}$. Now we have

$$\begin{aligned}\widetilde{\Phi}_b^\sigma =&\frac{1}{b^2}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\psi_{b,h}^\sigma\left(\frac{l}{b}\right)\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)+\frac{1}{3b^3}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\psi_{b,h}^\sigma\left(\frac{l}{b}\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right)\\ &+\frac{1}{3b^3}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)(\psi_{b,h}^\sigma)'\left(\frac{l}{b}+0\right)\\ &+\frac{1}{6b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^\sigma)'\left(\frac{l}{b}+0\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right)=: S_1+S_2+S_3+S_4.\end{aligned}$$

With property (**P1**) we can write

$$S_4=\frac{1}{6b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{l}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{l}{b}+0\right)=:\frac{1}{6b^4}\sum_{l=0}^{b-1}J(l).$$

To calculate this expression, we distinguish two cases:

1. Let $l\le\frac{b-1}{2}$, i.e. $l\le b-l-1$. Then we have

$$J(l)=\sum_{h=0}^{l}h^2+\sum_{h=l+1}^{b-l-1}(b-h)(-h)+\sum_{h=b-l}^{b-1}(b-h)^2=\frac{b}{6}(1-b^2+6l+6l^2).$$

2. Let $l>\frac{b-1}{2}$, i.e. $l>b-l-1$. Then we have

$$\begin{aligned}J(l)=&\sum_{h=0}^{b-l-1}h^2+\sum_{h=b-l}^{l}(b-h)(-h)+\sum_{h=l+1}^{b-1}(b-h)^2\\ =&\frac{b}{6}(1+5b^2-6b+6l^2+6l-12bl).\end{aligned}$$

This leads to

$$S_4=\begin{cases}-\frac{1}{72}\frac{b^2+2}{b^2} & \text{if } b \text{ is even}\\ -\frac{1}{72}\frac{b^2-1}{b^2} & \text{if } b \text{ is odd.}\end{cases}$$

We turn to S_2 and S_3. Using (**P2**) we can write it as

$$S_2=\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\sum_{k=0}^{l-1}(\psi_{b,h}^\sigma)'\left(\frac{k}{b}+0\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right)$$

$$=\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{k=0}^{l-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(l)}{b}+0\right).$$

Similarly, we have

$$\begin{aligned}S_3=&\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{k=0}^{l-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(k)}{b}+0\right)(\psi_{b,h}^{id})'\left(\frac{\sigma(l)}{b}+0\right)\\=&\frac{1}{3b^4}\sum_{k=0}^{b-1}\sum_{l=0}^{k-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(l)}{b}+0\right)(\psi_{b,h}^{id})'\left(\frac{\sigma(k)}{b}+0\right)\\=&\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{k=l+1}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(l)}{b}+0\right).\end{aligned}$$

Summing S_2 and S_3 yields

$$\begin{aligned}S_2+S_3=&\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{k=0}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(l)}{b}+0\right)\\&-\frac{1}{3b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(l)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(l)}{b}+0\right)=-2S_4,\end{aligned}$$

since the first sum in the last expression vanishes due to property (**P2**)′. So far we have

$$\widetilde{\Phi}_b^\sigma=\frac{1}{b^2}\sum_{l=0}^{b-1}\sum_{h=0}^{b-1}\psi_{b,h}^{\sigma}\left(\frac{l}{b}\right)\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)-S_4.$$

We have

$$\begin{aligned}S_1=&\frac{1}{b^4}\sum_{l=0}^{b-1}\sum_{k_1,k_2=0}^{l-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{\sigma})'\left(\frac{k_1}{b}+0\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{k_2}{b}+0\right)\\=&\frac{1}{b^4}\sum_{k_1,k_2=0}^{b-1}\sum_{l=\max\{k_1,k_2\}+1}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k_1)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(k_2)}{b}+0\right)\\=&\frac{b-1}{b^4}\sum_{k_1,k_2=0}^{b-1}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k_1)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(k_2)}{b}+0\right)\\&-\frac{1}{b^4}\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{\sigma(k_1)}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{\sigma(k_2)}{b}+0\right)\\=&-\frac{1}{b^4}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\underbrace{\sum_{h=0}^{b-1}(\psi_{b,h}^{id})'\left(\frac{k_1}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{k_2}{b}+0\right)}_{\mathcal{X}}.\end{aligned}$$

We have

$$\begin{aligned}\mathcal{X}=&\sum_{h=0}^{\min\{k_1,b-1-k_2\}}h^2+\sum_{h=\min\{k_1,b-1-k_2\}+1}^{\max\{k_1,b-1-k_2\}}(-h)(b-h)+\sum_{h=\max\{k_1,b-1-k_2\}+1}^{b-1}(b-h)^2\\=&\frac{b}{6}\left(5b^2-6b+1+3(k_1^2+k_1+k_2^2+k_2)-6bk_2-6b\max\{k_1,b-1-k_2\}\right).\end{aligned}$$

This leads to

$$\begin{aligned}
S_1 =& -\frac{1}{6b^3}\sum_{k_1,k_2=0}^{b-1}(5b^2-6b+1)\max\{k_1,k_2\} \\
&-\frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Bigg(\frac{k_1^2+k_1+k_2^2+k_2}{2} \\
&\qquad -b(k_2+\max\{k_1,b-1-k_2\})\Bigg) \\
=& -\frac{(b-1)^2(4b+1)(5b-1)}{36b^2} \\
&-\frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\left(\frac{k_1^2+k_1+k_2^2+k_2}{2}-b\max\{k_1+k_2,b-1\}\right).
\end{aligned}$$

The result for $\widetilde{\Phi}_b^{\sigma}$ follows. We consider $\widetilde{\Phi}_{b,1}^{\sigma}$ and derive analogously as before

$$\begin{aligned}
\widetilde{\Phi}_{b,1}^{\sigma} =& \frac{1}{b^2}\sum_{l=0}^{b-1}\sum_{h=0}^{b-2}\psi_{b,h+1}^{\sigma}\left(\frac{l}{b}\right)\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)+\frac{1}{3b^3}\sum_{l=0}^{b-1}\sum_{h=0}^{b-2}\psi_{b,h+1}^{\sigma}\left(\frac{l}{b}\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right) \\
&+\frac{1}{3b^3}\sum_{l=0}^{b-1}\sum_{h=0}^{b-2}\psi_{b,h}^{\overline{\sigma}}\left(\frac{l}{b}\right)(\psi_{b,h+1}^{\sigma})'\left(\frac{l}{b}+0\right) \\
&+\frac{1}{6b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-2}(\psi_{b,h+1}^{\sigma})'\left(\frac{l}{b}+0\right)(\psi_{b,h}^{\overline{\sigma}})'\left(\frac{l}{b}+0\right) =: S_1'+S_2'+S_3'+S_4'.
\end{aligned}$$

For S_4' we proceed as above and write

$$S_4' = \frac{1}{6b^4}\sum_{l=0}^{b-1}\sum_{h=0}^{b-2}(\psi_{b,h+1}^{id})'\left(\frac{l}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{l}{b}+0\right) =: \frac{1}{6b^4}\sum_{l=0}^{b-1}J'(l).$$

To calculate this expression, we distinguish two cases:

1. Let $l \leq \frac{b}{2}$, i.e. $l-1 \leq b-l-1$. Then we have

$$\begin{aligned}
J'(l) =& \sum_{h=0}^{l-1}(h+1)h+\sum_{h=l+1}^{b-l-1}(b-h-1)(-h)+\sum_{h=b-l}^{b-2}(b-h)(b-h-1) \\
=& \frac{b}{6}(-2+3b-b^2-6l+6l^2).
\end{aligned}$$

2. Let $l > \frac{b}{2}$, i.e. $l-1 > b-l-1$. Then we have

$$\begin{aligned}
J'(l) =& \sum_{h=0}^{b-l-1}(h+1)h+\sum_{h=b-l}^{l-1}(-h-1)(b-h)(-h)+\sum_{h=l}^{b-2}(b-h)^2 \\
=& \frac{b}{6}(-2+3b-6l+5b^2-12bl+6l^2).
\end{aligned}$$

This leads to

$$S_4' = \begin{cases} -\frac{1}{72}\frac{b^2-4}{b^2} & \text{if } b \text{ is even,} \\ -\frac{1}{72}\frac{b^2-1}{b^2} & \text{if } b \text{ is odd.} \end{cases}$$

As seen above, we have $S_2' + S_3' = -2S_4'$, and further

$$S_1' = -\frac{1}{b^4}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\underbrace{\sum_{h=0}^{b-1}(\psi_{b,h+1}^{id})'\left(\frac{k_1}{b}+0\right)(\psi_{b,h}^{\tau_b})'\left(\frac{k_2}{b}+0\right)}_{\mathcal{X}'}.$$

In order to evaluate $\mathcal{X}'$, we have to distinguish two cases:

1. Assume that $k_1 - 1 \leq b - 1 - k_2$. Then we have

$$\begin{aligned}\mathcal{X}' = &\sum_{h=0}^{\min\{k_1-1,b-1-k_2\}} h(h+1) + \sum_{h=\min\{k_1-1,b-1-k_2\}+1}^{\max\{k_1-1,b-1-k_2\}}(-h)(b-h-1)\\ &+\sum_{h=\max\{k_1-1,b-1-k_2\}+1}^{b-1}(b-h)(b-h-1) =: \mathcal{X}_1'.\end{aligned}$$

2. If $k_1 - 1 > b - 1 - k_2$, then we have

$$\begin{aligned}\mathcal{X}' = &\sum_{h=0}^{\min\{k_1-1,b-1-k_2\}} h(h+1) + \sum_{h=\min\{k_1-1,b-1-k_2\}+1}^{\max\{k_1-1,b-1-k_2\}}(-h-1)(b-h-1)\\ &+\sum_{h=\max\{k_1-1,b-1-k_2\}+1}^{b-1}(b-h)(b-h-1)\\ =&: \mathcal{X}_1' - b(\max\{k_1-1,b-1-k_2\} - \min\{k_1-1,b-1-k_2\})\\ =&\mathcal{X}_1' - b(k_1+k_2-b).\end{aligned}$$

A straightforward calculation yields

$$\mathcal{X}_1' = \frac{b}{6}\left(-2-3b+5b^2+3k_1^2-3k_1+3k_2^2-3k_2+6(b-1)\max\{k_1+k_2,b\}\right),$$

and therefore

$$\begin{aligned}S_1' = &-\frac{(b-1)^2(4b+1)(5b+2)}{36b^2}\\ &-\frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\left(\frac{k_1^2-k_1+k_2^2-k_2}{2}-(b-1)\max\{k_1+k_2,b\}\right)\\ &+\frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b).\end{aligned}$$

This completes the proof for $\widetilde{\Phi}_{b,1}^{\sigma}$. Since the formula for $\widetilde{\Phi}_{b,2}^{\sigma}$ can be shown in the very same way, we omit an explicit proof. □

We will make excessive use of the property $\sigma \in \mathcal{A}_b(\tau)$ in all subsequent proofs.

Lemma A.5. *Let $\sigma \in \mathcal{A}_b(\tau)$. With $S_1(\sigma) = \sum_{k=0}^{b-1} k\sigma(k)$ we have*

$$\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}\sigma(k_1) = \frac{b}{2}S_1(\sigma) + \frac{1}{24}b(b-1)(5b^2-3b-2).$$

Proof. We have

$$\begin{aligned}
&\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1) \\
&=\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1) + \frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}(b-1-\sigma(b-1-k_1)) \\
&=\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \Big(\max\{k_1,k_2\} - \max\{b-1-k_1,k_2\}\Big)\sigma(k_1) + \frac{b-1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}.
\end{aligned}$$

Now we have

$$\begin{aligned}
-\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{b-1-k_1,k_2\}\sigma(k_1) &= -\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{b-1-k_1,b-1-k_2\}\sigma(k_1) \\
&=\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} (\min\{k_1,k_2\}-(b-1))\sigma(k_1),
\end{aligned}$$

where we regarded the fact that

$$\max\{b-1-k_1,b-1-k_2\} = b-1+\max\{-k_1,-k_2\} = b-1-\min\{k_1,k_2\}.$$

Since $\max\{k_1,k_2\}+\min\{k_1,k_2\} = k_1+k_2$, we get

$$\begin{aligned}
&\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1) \\
&=\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \Big(k_1+k_2-(b-1)\Big)\sigma(k_1) + \frac{b-1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\} \\
&=\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} k_1\sigma(k_1) + \frac{1}{2}\sum_{k_1,k_2=0}^{b-1} k_2\sigma(k_1) - \frac{b-1}{2}\sum_{k_1,k_2=0}^{b-1} \sigma(k_1) + \frac{b-1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\} \\
&=\frac{b}{2}S_1(\sigma) + \frac{1}{2}\left(\frac{b(b-1)}{2}\right)^2 - \left(\frac{b(b-1)}{2}\right)^2 + \frac{b-1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}.
\end{aligned}$$

The rest of the proof is straightforward. □

Lemma A.6. *Let $\sigma \in \mathcal{A}_b(\tau)$. With $S_1(\sigma) = \sum_{k=0}^{b-1} k\sigma(k)$ and $S_2(\sigma) = \sum_{k=0}^{b-1} k^2\sigma(k)^2$ we have*

$$\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1)^2 = \frac{1}{2}S_2(\sigma) + \frac{b-1}{2}S_1(\sigma) + \frac{1}{24}b(b-1)^2(4b^2-3b+2).$$

Proof. We have

$$\begin{aligned}
\sum_{k_1,k_2=0}^{b-1} & \max\{k_1,k_2\}\sigma(k_1)^2 \\
=&\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1)^2 \\
&+\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}((b-1)^2-2(b-1)\sigma(b-1-k_1)+\sigma(b-1-k_1)^2) \\
=&\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \Big(\max\{k_1,k_2\}+\max\{b-1-k_1,b-1-k_2\}\Big)\sigma(k_1)^2 \\
&+\frac{(b-1)^2}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}-(b-1)\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}(b-1-\sigma(k_1)) \\
=&\frac{1}{2}\sum_{k_1,k_2=0}^{b-1} \Big(|k_1-k_2|+(b-1)\Big)\sigma(k_1)^2-\frac{(b-1)^2}{2}\sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\} \\
&+(b-1)\left(\frac{b}{2}S_1(\sigma)+\frac{1}{24}b(b-1)(5b^2-3b-2)\right).
\end{aligned}$$

We compute

$$\begin{aligned}
\frac{1}{2}\sum_{k_1,k_2=0}^{b-1}|k_1-k_2|\sigma(k_1)^2 =&\frac{1}{2}\sum_{k_1=0}^{b-1}\sum_{k_2=0}^{k_1}(k_1-k_2)\sigma(k_1)^2+\frac{1}{2}\sum_{k_1=0}^{b-1}\sum_{k_2=k_1+1}^{b-1}(k_2-k_1)\sigma(k_1)^2 \\
=&\frac{1}{4}\sum_{k_1=0}^{b-1}k_1(k_1+1)\sigma(k_1)^2+\frac{1}{4}\sum_{k_1=0}^{b-1}(b-1-k_1)(b-k_1)\sigma(k_1)^2 \\
=&\frac{1}{2}S_2(\sigma)-\frac{b-1}{2}\sum_{k=0}^{b-1}k\sigma(k)^2+\frac{1}{4}b(b-1)\sum_{k=0}^{b-1}k^2.
\end{aligned}$$

Further we find

$$\begin{aligned}
\sum_{k=0}^{b-1}k\sigma(k)^2 =&\sum_{k=0}^{b-1}\tau_b(k)\sigma(\tau_b(k))^2=\sum_{k=0}^{b-1}\tau_b(k)\tau_b(\sigma(k))^2=\sum_{k=0}^{b-1}(b-1-k)(b-1-\sigma(k))^2 \\
=&(b-1)^2\sum_{k=0}^{b-1}(b-1-k)-2(b-1)\sum_{k=0}^{b-1}(b-1-k)\sigma(k) \\
&+(b-1)\sum_{k=0}^{b-1}\sigma(k)^2-\sum_{k=0}^{b-1}k\sigma(k)^2
\end{aligned}$$

which yields after rearranging this formula

$$\sum_{k=0}^{b-1}k\sigma(k)^2=(b-1)S_1(\sigma)-\frac{1}{12}b(b-1)(b^2-3b+2).$$

Putting all these results together yields the claim of this lemma. □

Lemma A.7. *For all $\sigma\in\mathcal{A}_b$ we have*

$$\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}\max\{\sigma(k_1)+\sigma(k_2),b-1\}=\sum_{k_1,k_2=0}^{b-1}\max\{\sigma(k_1),\sigma(k_2)\}\max\{k_1+k_2,b-1\}.$$

Proof. We have

$$\begin{aligned}
&\sum_{k_1,k_2=0}^{b-1} \max\{\sigma(k_1),\sigma(k_2)\}\max\{k_1+k_2,b-1\}\\
&\quad=\sum_{k_1,k_2=0}^{b-1} \max\{\sigma(k_1),\sigma(k_2)\}\left(k_1+\max\{k_2,b-1-k_1\}\right)\\
&\quad=\sum_{k_1,k_2=0}^{b-1} k_1\max\{\sigma(k_1),\sigma(k_2)\}+\sum_{k_1,k_2=0}\max\{b-1-\sigma(k_1),\sigma(k_2)\}\max\{k_1,k_2\}\\
&\quad=\sum_{k_1,k_2=0}^{b-1} k_1\max\{\sigma(k_1),\sigma(k_2)\}+\sum_{k_1,k_2=0}\max\{\sigma(k_1)+\sigma(k_2),b-1\}\max\{k_1,k_2\}\\
&\qquad-\sum_{k_1,k_2=0}^{b-1}\sigma(k_1)\max\{k_1,k_2\}.
\end{aligned}$$

Further we find by Lemma A.5

$$\begin{aligned}
\sum_{k_1,k_2=0}^{b-1} k_1\max\{\sigma(k_1),\sigma(k_2)\}=&\sum_{k_1,k_2=0}^{b-1}\sigma^{-1}(k_1)\max\{k_1,k_2\}\\
=&\frac{b}{2}S_1(\sigma^{-1})+\frac{1}{24}b(b-1)^2(5b+2)\\
=&\frac{b}{2}S_1(\sigma)+\frac{1}{24}b(b-1)^2(5b+2)=\sum_{k_1,k_2=0}^{b-1}\sigma(k_1)\max\{k_1,k_2\},
\end{aligned}$$

which completes the proof. □

Corollary A.8. We have

$$\begin{aligned}
c_b^\sigma=&\frac{32-57b-90b^2+228b^3-112b^4}{72b^2}+\frac{1-(-1)^b}{16b^3}\\
&-\frac{2}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1^2+k_1+k_2^2+k_2)\\
&+\frac{2}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Bigg(2b\max\{k_1,k_2\}+b\max\{k_1+k_2,b-1\}\\
&+\frac{b+1}{2}\max\{k_1+k_2,b-2\}+\frac{b-1}{2}\max\{k_1+k_2,b\}\Bigg)\\
&-\frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}.
\end{aligned}$$

Proof. At first we compute

$$\begin{aligned}
&\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b)\\
&-\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b+2)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b)\\
&-\left(\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b+2)\right.\\
&\quad\left.+\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(b-1-b+2)\right)\\
=&-\frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}-\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}.
\end{aligned}
$$

Now it holds that

$$
\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}=\frac{1}{2b^3}\sum_{k=0}^{b-1}\max\{k,b-1-k\}=\frac{3b-2}{8b^2}-\frac{1-(-1)^b}{16b^3}.
$$

Summing the results in Lemma A.4 and applying the relation above yields the claim. □

Now we are ready to give the proofs of Propositions A.1, A.2 and A.3.

Proof of Proposition A.1. We set $M=\max\{k_1+k_2,b-1\}$. Then we have

$$
\begin{aligned}
&\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Bigg(b\max\{k_1+k_2,b-1\}\\
&\quad+\frac{b+1}{2}\max\{k_1+k_2,b-2\}+\frac{b-1}{2}\max\{k_1+k_2,b\}\Bigg)\\
&=\sum_{\substack{k_1,k_2=0\\k_1+k_2\leq b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\left(bM+\frac{b+1}{2}(M-1)+\frac{b-1}{2}(M+1)\right)\\
&\quad+\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\left(bM+\frac{b+1}{2}M+\frac{b-1}{2}(M+1)\right)\\
&\quad+\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\left(bM+\frac{b+1}{2}M+\frac{b-1}{2}M\right)\\
&=2b\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\max\{k_1+k_2,b-1\}\\
&\quad-\frac{b+1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2\leq b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}+\frac{b-1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2\leq b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\\
&=2b\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\max\{k_1+k_2,b-1\}\\
&\quad-\sum_{\substack{k_1,k_2=0\\k_1+k_2\leq b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}+\frac{b+1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}.
\end{aligned}
$$

The last sum is computed easily, since

$$\begin{aligned}\frac{b+1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}=&\frac{b+1}{2}\sum_{k=0}^{b-1}\max\{k,b-1-k\}\\=&\frac{1}{8}b(b+1)(3b-2)-\frac{b+1}{16}(1-(-1)^b).\end{aligned}$$

Now we can combine this result with Corollary A.8 to obtain the expression in Proposition A.1 with σ^{-1} instead of σ. However, Lemmas A.5, A.6 and A.7 allow us to interchange σ^{-1} and σ in this formula, and the proof is complete. □

Proof of Proposition A.2. From Lemma A.4 we observe that

$$\begin{aligned}\widetilde{\Phi}_b^\sigma-\frac{1}{2}\widetilde{\Phi}_{b,1}^\sigma-\frac{1}{2}\widetilde{\Phi}_{b,2}^\sigma=&\frac{\tilde{c}+3b-4b^2}{12b^2}\\&+\frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Big(b\max\{k_1+k_2,b-1\}\\&-\frac{b+1}{2}\max\{k_1+k_2,b-2\}-\frac{b-1}{2}\max\{k_1+k_2,b\}\Big)\\&-\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b)\\&+\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b+2),\end{aligned}$$

where $\tilde{c}=2$ for even bases and $\tilde{c}=1$ for odd bases. As in the proof of Proposition A.1 we set $M=\max\{k_1+k_2,b-1\}$. Then we have

$$\begin{aligned}&\sum_{k_1,k_2=0}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Big(b\max\{k_1+k_2,b-1\}\\&\quad-\frac{b+1}{2}\max\{k_1+k_2,b-2\}-\frac{b-1}{2}\max\{k_1+k_2,b\}\Big)\\&=\sum_{\substack{k_1,k_2=0\\k_1+k_2\le b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Big(bM-\frac{b+1}{2}(M-1)-\frac{b-1}{2}(M+1)\Big)\\&\quad+\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Big(bM-\frac{b+1}{2}M-\frac{b-1}{2}(M+1)\Big)\\&\quad+\sum_{\substack{k_1,k_2=0\\k_1+k_2\ge b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\Big(bM-\frac{b+1}{2}M-\frac{b-1}{2}M\Big)\\&=\sum_{\substack{k_1,k_2=0\\k_1+k_2\le b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}-\frac{b-1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}\\&=\sum_{\substack{k_1,k_2=0\\k_1+k_2\le b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}-\frac{b+1}{2}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}.\end{aligned}$$

From the proof of Corollary A.8 we already know that

$$
\begin{aligned}
&-\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b)\\
&+\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2>b-2}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}(k_1+k_2-b+2)\\
&=\frac{1}{2b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2=b-1}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\}+\frac{1}{b^3}\sum_{\substack{k_1,k_2=0\\k_1+k_2\geq b}}^{b-1}\max\{\sigma^{-1}(k_1),\sigma^{-1}(k_2)\},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\widetilde{\Phi}_b^{\sigma}-\frac{1}{2}\widetilde{\Phi}_{b,1}^{\sigma}-\frac{1}{2}\widetilde{\Phi}_{b,2}^{\sigma}=&\frac{\tilde{c}+3b-4b^2}{12b^2}-\frac{1}{2b^2}\sum_{k=0}^{b-1}\max\{k,b-1-k\}\\
&+\frac{1}{b^3}\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}.
\end{aligned}
$$

The rest of the proof is straightforward. □

Proof of Proposition A.3. We define

$$
\begin{aligned}
A(\sigma)&=\sum_{k_1,k_2=0}^{b-1}\max\{\sigma(k_1),\sigma(k_2)\}k_1,\\
B(\sigma)&=\sum_{k_1,k_2=0}^{b-1}\max\{\sigma(k_1),\sigma(k_2)\}k_1^2,\\
C(\sigma)&=\sum_{k_1,k_2=0}^{b-1}\max\{\sigma(k_1),\sigma(k_2)\}(\max\{k_1,k_2\}+\max\{k_1+k_2,b-1\}).
\end{aligned}
$$

It follows from Proposition A.1 that in order to prove Proposition A.3, we have to show

$$
-(A(\sigma)-A(\widehat{\sigma}))-(B(\sigma)-B(\widehat{\sigma}))+\frac{b}{2}(C(\sigma)-C(\widehat{\sigma}))=0. \tag{A.1}
$$

We employ Lemma A.5 to obtain

$$
\begin{aligned}
A(\sigma)-A(\widehat{\sigma})=&\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}\sigma(k_1)-\sum_{k_1,k_2=0}^{b-1}\max\{k_1,k_2\}\widehat{\sigma}(k_1)=\frac{b}{2}(S_1(\sigma)-S_1(\widehat{\sigma}))\\
=&\frac{b}{2}\Big(d\sigma(d)+(b-1-d)(b-1-\sigma(d))\\
&-d(b-1-\sigma(d))-(b-1-d)\sigma(d)\Big)\\
=&\frac{b}{2}(b-1-2d)(b-1-2\sigma(d)).
\end{aligned} \tag{A.2}
$$

With Lemma A.6 we find similarly

$$
\begin{aligned}
B(\sigma) - B(\widehat{\sigma}) = & \sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\sigma(k_1)^2 - \sum_{k_1,k_2=0}^{b-1} \max\{k_1,k_2\}\widehat{\sigma}(k_1)^2 \\
= & \frac{1}{2}(S_2(\sigma) - S_2(\widehat{\sigma})) + \frac{b-1}{2}(S_1(\sigma) - S_1(\widehat{\sigma})) \\
= & \frac{1}{2}\Big(d^2\sigma(d)^2 + (b-1-d)^2(b-1-\sigma(d))^2 \\
& - d^2(b-1-\sigma(d))^2 - (b-1-d)^2\sigma(d)^2\Big) \\
& + \frac{b-1}{2}\Big(d\sigma(d) + (b-1-d)(b-1-\sigma(d)) \\
& - d(b-1-\sigma(d)) - (b-1-d)\sigma(d)\Big) \\
= & \frac{b(b-1)}{2}(b-1-2d)(b-1-2\sigma(d)). \qquad \text{(A.3)}
\end{aligned}
$$

It remains to compute $C(\sigma) - C(\widehat{\sigma})$. Since all summand with indices $k_1, k_2 \notin \{d, b-1-d\}$ in $C(\widehat{\sigma})$ are the same as the corresponding summands in $C(\sigma)$, we can write

$$
\begin{aligned}
& C(\sigma) - C(\widehat{\sigma}) \\
& = \sum_{\substack{k_2=0 \\ k_2 \notin \{d,b-1-d\}}}^{b-1} \Big(\max\{\sigma(d),\sigma(k_2)\} - \max\{\sigma(b-1-d),\sigma(k_2)\}\Big) \\
& \qquad \times \Big(\max\{d,k_2\} + \max\{d+k_2,b-1\}\Big) \\
& + \sum_{\substack{k_2=0 \\ k_2 \notin \{d,b-1-d\}}}^{b-1} \Big(\max\{\sigma(b-1-d),\sigma(k_2)\} - \max\{\sigma(d),\sigma(k_2)\}\Big) \\
& \qquad \times \Big(\max\{b-1-d,k_2\} + \max\{b-1-d+k_2,b-1\}\Big) \\
& + \sum_{\substack{k_1=0 \\ k_1 \notin \{d,b-1-d\}}}^{b-1} \Big(\max\{\sigma(k_1),\sigma(d)\} - \max\{\sigma(k_1),\sigma(b-1-d)\}\Big) \\
& \qquad \times \Big(\max\{k_1,d\} + \max\{k_1+d,b-1\}\Big) \\
& + \sum_{\substack{k_1=0 \\ k_1 \notin \{d,b-1-d\}}}^{b-1} \Big(\max\{\sigma(k_1),\sigma(b-1-d)\} - \max\{\sigma(k_1),\sigma(d)\}\Big) \\
& \qquad \times \Big(\max\{k_1,b-1-d\} + \max\{k_1+b-1-d,b-1\}\Big) \\
& + \Big(\max\{\sigma(d),\sigma(d)\} - \max\{\widehat{\sigma}(d),\widehat{\sigma}(d)\}\Big)\Big(\max\{d,d\} + \max\{2d,b-1\}\Big) \\
& + \Big(\max\{\sigma(d),\sigma(b-1-d)\} - \max\{\widehat{\sigma}(d),\widehat{\sigma}(b-1-d)\}\Big) \\
& \qquad \times \Big(\max\{d,b-1-d\} + \max\{b-1,b-1\}\Big) \\
& + \Big(\max\{\sigma(b-1-d),\sigma(d)\} - \max\{\widehat{\sigma}(b-1-d),\widehat{\sigma}(d)\}\Big) \\
& \qquad \times \Big(\max\{b-1-d,d\} + \max\{b-1,b-1\}\Big) \\
& + \Big(\max\{\sigma(b-1-d),\sigma(b-1-d)\} - \max\{\widehat{\sigma}(b-1-d),\widehat{\sigma}(b-1-d)\}\Big) \\
& \qquad \times \Big(\max\{b-1-d,b-1-d\} + \max\{2b-2-2d,b-1\}\Big).
\end{aligned}
$$

Let Z_1 denote the first four sums in the above expressions and Z_2 the last four summands, respectively. Obviously Z_1 may be written as

$$\begin{aligned} Z_1 =& 2 \sum_{\substack{k=0 \\ k\notin\{d,b-1-d\}}}^{b-1} (\max\{\sigma(d),\sigma(k)\} - \max\{b-1-\sigma(d),\sigma(k)\}) \\ &\qquad \times (\max\{d,k\} + \max\{d+k,b-1\}) \\ &+ 2 \sum_{\substack{k=0 \\ k\notin\{d,b-1-d\}}}^{b-1} (\max\{b-1-\sigma(d),\sigma(k)\} - \max\{\sigma(d),\sigma(k)\}) \\ &\qquad \times (\max\{b-1-d,k\} + \max\{b-1-d+k,b-1\}). \end{aligned}$$

Since

$$\begin{aligned} &\max\{b-1-d,k\} + \max\{b-1-d+k,b-1\} \\ &\quad = \max\{b-1,k+d\} - d + \max\{d,k\} + b - 1 - d \\ &\quad = \max\{b-1,k+d\} + \max\{d,k\} + b - 1 - 2d, \end{aligned}$$

we can simplify Z_1 to

$$Z_1 = 2(b-1-2d) \sum_{\substack{k=0 \\ k\notin\{d,b-1-d\}}}^{b-1} (\max\{b-1-\sigma(d),\sigma(k)\} - \max\{\sigma(d),\sigma(k)\}).$$

By including the summands for $k = d$ and $k = b-1-d$, we find

$$\begin{aligned} Z_1 =& -2(b-1-2d)(b-1-2\sigma(d)) \\ &+ 2(b-1-2d) \sum_{k=0}^{b-1} (\max\{b-1-\sigma(d),\sigma(k)\} - \max\{\sigma(d),\sigma(k)\}). \end{aligned}$$

We calculate the last sum. Assuming that $\sigma(d) \leq b-1-\sigma(d)$ yields

$$\begin{aligned} &\sum_{k=0}^{b-1} (\max\{b-1-\sigma(d),\sigma(k)\} - \max\{\sigma(d),\sigma(k)\}) \\ &\quad = \sum_{k=0}^{b-1} (\max\{b-1-\sigma(d),k\} - \max\{\sigma(d),k\}) \\ &\quad = \sum_{k=0}^{\sigma(d)-1} (b-1-\sigma(d)-\sigma(d)) + \sum_{k=\sigma(d)}^{b-1-\sigma(d)-1} (b-1-\sigma(d)-k) \\ &\quad = \sigma(d)(b-1-2\sigma(d)) + \frac{1}{2}(b-1-2\sigma(d))(b-2\sigma(d)) \\ &\quad = \frac{b}{2}(b-1-2\sigma(d)). \end{aligned}$$

If $\sigma(d) > b-1-\sigma(d)$, we obtain the same result. Inserting it into our expression for Z_1, we find

$$Z_1 = (b-2)(b-1-2d)(b-1-2\sigma(d)).$$

It remains to evaluate Z_2. Since $\max\{\sigma(d),\sigma(b-1-d)\} - \max\{\widehat{\sigma}(d),\widehat{\sigma}(b-1-d)\} = 0$, it is clear that

$$
\begin{aligned}
Z_2 =& (2\sigma(d) - b + 1)(d + \max\{2d, b-1\}) \\
&+ (b - 1 - 2\sigma(d))(b - 1 - d + \max\{2b - 2 - 2d, b - 1\}).
\end{aligned}
$$

By the elementary relation

$$
b - 1 - d + \max\{2b - 2 - 2d, b - 1) = 2b - 2 - 3d + \max\{2d, b - 1\}
$$

we find

$$
Z_2 = 2(b - 1 - 2d)(b - 1 - 2\sigma(d))
$$

and hence

$$
C(\sigma) - C(\widehat{\sigma}) = b(b - 1 - 2d)(b - 1 - 2\sigma(d)). \tag{A.4}
$$

Now the identity (A.1) can be easily checked by inserting (A.2), (A.3) and (A.4). □

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Teaching experience

09/2013–07/2014	**Internship for teaching at school**, *Georg von Peuerbach-Gymnasium*, Linz.
02/2015	**Project week „Applied mathematics“ for students (as project leader)**, *Kefermarkt.*
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Publications

A.Hinrichs, R. Kritzinger, F. Pillichshammer	**Optimal order of L_p-discrepancy of digit shifted Hammersley point sets in dimension 2**, *Unif. Distrib. Theory (10), Nr. 1*, pp. 115–133, 2015.
R. Kritzinger, F. Pillichshammer	**L_p-discrepancy of the symmetrized van der Corput sequence**, *Arch. Math. (104)*, pp. 407–418, 2015.
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R. Kritzinger	**L_p- and $S_{p,q}^r B$-discrepancy of the symmetrized van der Corput sequence and modified Hammersley point sets in arbitrary bases**, *J. Complexity (33)*, pp. 145–168, 2016.
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R. Kritzinger **An exact formula for the L_2 discrepancy of the symmetrized Hammersley point set**, *Math. Comput. Simulat., to appear*, 2017.

R. Kritzinger, H. Laimer, M. Neumüller **A reduced fast construction of polynomial lattice point sets with low weighted star discrepancy**, *submitted*, 2017.

R. Kritzinger **Optimal discrepancy rate of point sets in Besov spaces with negative smoothness**, *submitted*, 2017.

Talks

04/2015 **L_p discrepancy of modified van der Corput sequences and Hammersley point sets**, *Workshop on Information-Based Complexity*, Bedlewo, Poland.

07/2015 **On symmetrized van der Corput sequences and generalized Hammersley point sets**, *10th IMACS Seminar on Monte Carlo Methods 2015*, Linz, Austria.

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07/2016 **Symmetrized point sets in the unit square with small L_p discrepancy**, *5th International Conference on Uniform Distribution Theory*, Sopron, Hungary.

08/2016 **A reduced fast component-by-component construction of lattice point sets with small weighted star discrepancy** , *12th International Conference on Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing 2015*, Stanford, California, USA.