

# On the $N$ th linear complexity of $p$ -automatic sequences over $\mathbb{F}_p$

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## Abstract

We prove that any (not ultimately periodic)  $p$ -automatic sequence over  $\mathbb{F}_p$  has  $N$ th linear complexity of order of magnitude  $N$  for all  $N \geq 1$ . For some famous sequences including the Thue-Morse and Rudin-Shapiro sequence we determine the exact value of the  $N$ th linear complexities. These are non-trivial examples of predictable sequences with almost perfect linear complexity profile.

*Keywords and phrases:* automatic sequences, Thue-Morse sequence, Rudin-Shapiro sequence, pattern sequence, sum-of-digits sequence, Baum-Sweet sequence, paper folding sequence, linear complexity, expansion complexity, lattice profile, continued fractions

## 1 Introduction

Let  $k \geq 2$  be an integer. A  $k$ -automatic sequence  $(u_n)$  over an alphabet  $\mathcal{A}$  is the output sequence of a finite automaton, where the input is the  $k$ -ary digital expansion of  $n$ . Automatic sequences have gained much attention during the last decades. For monographs and surveys about automatic sequences we refer to [1, 2, 12, 13].

For a prime  $k = p$ ,  $p$ -automatic sequences  $(u_n)$  over the finite field  $\mathcal{A} = \mathbb{F}_p$  of  $p$  elements can be characterized by a result of Christol [5], see also [6]: Let

$$G(t) = \sum_{n=0}^{\infty} u_n t^n$$

be the *generating function* of the sequence  $(u_n)$  over  $\mathbb{F}_p$ . Then  $(u_n)$  is *p-automatic over  $\mathbb{F}_p$*  if and only if  $G(t)$  is algebraic over  $\mathbb{F}_p[t]$ , that is, there is a polynomial  $0 \neq h(s, t) \in \mathbb{F}_p[s, t]$  such that  $h(G(t), t) = 0$ .

Diem [7] defined the *Nth expansion complexity*  $E_N(u_n)$  of  $(u_n)$  as the least total degree of a nonzero polynomial  $h(s, t) \in \mathbb{F}_p[s, t]$  with

$$h(G(t), t) \equiv 0 \pmod{t^N}$$

if the first  $N$  sequence elements are not all 0 and  $E_N(u_n) = 0$  otherwise. Hence, the *p-automatic sequences over  $\mathbb{F}_p$*  are exactly the sequences over  $\mathbb{F}_p$  with

$$E(u_n) = \sup_{N \geq 1} E_N(u_n) < \infty.$$

Sequences  $(u_n)$  with small  $E(u_n)$  are predictable and not suitable in cryptography.

For example, the *Thue-Morse sequence* over  $\mathbb{F}_2$  is defined by

$$t_n = \begin{cases} t_{n/2} & \text{if } n \text{ is even,} \\ t_{(n-1)/2} + 1 & \text{if } n \text{ is odd,} \end{cases} \quad n = 1, 2, \dots$$

with initial value  $t_0 = 0$ . Taking

$$h(s, t) = s(t+1)^2 + s^2(t+1)^3 + t$$

its generating function  $G(t)$  satisfies  $h(G(t), t) = 0$  and thus  $E(t_n) \leq 5$ . More precisely, in the proof of Corollary 1 below we will see that  $G(t)$  is not rational and thus  $h(s, t) = (t+1)^3(s+G(t))(s+G(t) + \frac{1}{t+1})$  is irreducible over  $\mathbb{F}_2(t)[s]$  and we get

$$E(t_n) = 5.$$

The *Nth linear complexity*  $L(u_n, N)$  of a sequence  $(u_n)$  over  $\mathbb{F}_p$  is the length  $L$  of a shortest linear recurrence relation satisfied by the first  $N$  elements of  $(u_n)$ :

$$u_{n+L} = c_{L-1}u_{n+L-1} + \dots + c_1u_{n+1} + c_0u_n, \quad 0 \leq n \leq N - L - 1,$$

for some  $c_0, \dots, c_{L-1} \in \mathbb{F}_p$ . We use the convention that  $L(u_n, N) = 0$  if the first  $N$  elements of  $(u_n)$  are all zero and  $L(u_n, N) = N$  if  $u_0 = \dots = u_{N-2} = 0 \neq u_{N-1}$ . The sequence  $(L(u_n, N))$  is called *linear complexity profile* of  $(u_n)$  and

$$L(u_n) = \sup_{N \geq 1} L(u_n, N)$$

is the *linear complexity* of  $(u_n)$ . It is well-known ([18, Lemma 1]) that  $L(u_n) < \infty$  if and only if  $(u_n)$  is ultimately periodic, that is, its generating function is rational:  $G(t) = g(t)/f(t)$  with polynomials  $g(t), f(t) \in \mathbb{F}_p[t]$ . The *Nth linear complexity* is a measure for the unpredictability of a sequence as well. A large *Nth linear complexity* (up to sufficiently large  $N$ ) is necessary (but not sufficient) for cryptographic applications. Sequences of small linear complexity

are also weak in view of Monte-Carlo methods, see [8, 9, 10, 11]. For more background on linear complexity and related measures of pseudorandomness we refer to [16, 19, 21, 22]. In particular, for ultimately periodic sequences the relation between linear complexity and expansion complexity was studied in [17].

In this paper, we show that any  $p$ -automatic sequence over  $\mathbb{F}_p$  which is not ultimately periodic has  $N$ th linear complexity of (best possible) order of magnitude  $N$ . Hence, we provide many examples of sequences with high  $N$ th linear complexity which are still predictable (since  $E(u_n)$  is small). For example, for the Thue-Morse sequence over  $\mathbb{F}_2$  we prove

$$L(t_n, N) = 2 \left\lfloor \frac{N+2}{4} \right\rfloor, \quad N = 1, 2, \dots \quad (1)$$

In Section 2 we prove a bound on the  $N$ th linear complexity of any  $p$ -automatic sequence over  $\mathbb{F}_p$  which is not ultimately periodic. We apply this result to several famous automatic sequences including pattern sequences and sum-of-digits sequences. For example, for the Thue-Morse sequence we get the bound

$$\left\lceil \frac{N-1}{2} \right\rceil \leq L(t_n, N) \leq \left\lfloor \frac{N}{2} \right\rfloor + 1.$$

By (1) the lower bound is attained if  $N \equiv 0, 1 \pmod{4}$  and the upper bound if  $N \equiv 2, 3 \pmod{4}$ .

In Section 3 we determine the exact value of the  $N$ th linear complexity in the special case of binary pattern sequences with the all one pattern. The Thue-Morse sequence is the simplest example of such a sequence.

Besides a small expansion complexity, the Thue-Morse and Rudin-Shapiro sequences have another deficiency, a very large correlation measure of order 2, see [15]. By a result of [4], small correlation measures of order  $k$  (up to a sufficiently large  $k$ ) imply large  $N$ th linear complexities. The converse is not true. For example, the correlation measure of order 4 of the *Jacobi-sequence* ( $j_n$ ) of length  $pq$  with two distinct odd primes  $p$  and  $q$  is of order of magnitude  $pq$ , see [20], which is based on the relation  $j_n + j_{n+p} + j_{n+q} + j_{n+p+q} = 0$  for all  $1 \leq n < pq$  with  $\gcd(n, pq) = 1$ . However, its linear complexity profile is quite large, see [3]. This lower bound on the linear complexity profile can be obtained using an analog of the result of [4], see [14], for a modified correlation measure with *bounded lags*. Examples of sequences with large correlation measure of small order with bounded lags but large linear complexity profile have not been known yet. However, the results of this paper for the Thue-Morse and Rudin-Shapiro sequence and the results of [15] (using lags  $(0, 1)$ ) show that both sequences are such examples.

## 2 Arbitrary $p$ -automatic sequences

The ultimately periodic automatic sequences (that are linear recurrence sequences) correspond to rational generating functions [18, Lemma 1]. If  $(u_n)$

is an ultimately periodic automatic sequence, then it has finite linear complexity. However, we show now that if  $(u_n)$  is not ultimately periodic, then the  $N$ th linear complexity of  $(u_n)$  is of order of magnitude  $N$  for all  $N$ .

**Theorem 1.** *Let  $p$  be a prime number and  $(u_n)$  be a  $p$ -automatic sequence over  $\mathbb{F}_p$  which is not ultimately periodic. Let  $h(s, t) = h_0(t) + h_1(t)s + \cdots + h_d(t)s^d \in \mathbb{F}_p[s, t]$  be a non-zero polynomial with  $h(G(t), t) = 0$  with no rational zero. Put*

$$M = \max_{0 \leq i \leq d} \{\deg h_i - i\}.$$

Then we have

$$\frac{N - M}{d} \leq L(u_n, N) \leq \frac{(d - 1)N + M + 1}{d}.$$

*Proof.* Since  $(u_n)$  is not ultimately periodic,  $G(t) \notin \mathbb{F}_p(t)$  is not rational by [18, Lemma 1].

Let  $g(t)/f(t) \in \mathbb{F}_p(t)$  be a rational zero of  $h(s, t)$  modulo  $t^N$  with  $\deg(f) \leq L(u_n, N)$  and  $\deg(g) < L(u_n, N)$ . More precisely, put  $L = L(u_n, N)$ . Then we have

$$\sum_{\ell=0}^L c_\ell u_{n+\ell} = 0 \quad \text{for } 0 \leq n \leq N - L - 1$$

for some  $c_0, \dots, c_L \in \mathbb{F}_p$  with  $c_L \neq 0$ . Take

$$f(t) = \sum_{\ell=0}^L c_\ell t^{L-\ell}$$

and

$$g(t) = \sum_{m=0}^{L-1} \left( \sum_{\ell=L-m}^L c_\ell u_{m+\ell-L} \right) t^m$$

and verify

$$f(t)G(t) \equiv g(t) \pmod{t^N}.$$

Then

$$h_0(t)f^d(t) + h_1(t)g(t)f^{d-1}(t) + \cdots + h_d(t)g(t)^d = K(t)t^N.$$

Here  $K(t) \neq 0$  since  $h(s, t)$  has no rational zero. Comparing the degrees of both sides we get

$$dL + M \geq N$$

which gives the lower bound.

The upper bound for  $N = 1$  is trivial. For  $N \geq 2$  the result follows from the well-known bound (see for example [11, Lemma 3])

$$L(u_n, N) \leq \max \{L(u_n, N - 1), N - L(u_n, N - 1)\}$$

by induction. □

## Examples

Now we state bounds on the  $N$ th linear complexity of some famous automatic sequences as corollaries of Theorem 1.

### Pattern sequences

Let  $P \in \mathbb{F}_p^k \setminus \{00\dots 0\}$  be a *pattern of length  $k$* . Let  $e_P(n)$  be the number of occurrences of  $P$  in the  $p$ -ary representation of  $n$ . For example if  $p = 2$ , then  $e_{11}(7) = 2$ ,  $e_1(9) = 2$ ,  $e_{11}(9) = 0$  and  $e_{101}(21) = 2$ .

For a pattern  $P$  of length  $k$  define the sequence  $(r_n)$  by

$$r_n \equiv e_P(n) \pmod{p}, \quad r_n \in \mathbb{F}_p, \quad n = 0, 1, \dots$$

The sequence  $(r_n)$  over  $\mathbb{F}_p$  satisfies the following recurrence relation

$$r_n = \begin{cases} r_{\lfloor n/p \rfloor} + 1 & \text{if } n \equiv a \pmod{p^k}, \\ r_{\lfloor n/p \rfloor} & \text{otherwise,} \end{cases} \quad n = 1, 2, \dots \quad (2)$$

with initial value  $r_0 = 0$ , where  $a$  is the  $0 < a < p^k$  such that its  $p$ -ary expansion corresponds to the pattern  $P$ .

Classical examples for binary pattern sequences are the Thue-Morse sequence ( $p = 2$ ,  $k = 1$  and  $P = 1$  ( $a = 1$ )) and the *Rudin-Shapiro sequence* ( $p = 2$ ,  $k = 2$  and  $P = 11$  ( $a = 3$ )).

**Corollary 1.** *Let  $p$  be a prime number and  $a, k$  integers with  $1 \leq a < p^k$ . If  $(r_n)$  is the pattern sequence defined by (2), then*

$$\frac{N - p^k + 1}{p} \leq L(r_n, N) \leq \frac{(p-1)N + p^k}{p}.$$

*Proof.* By the recurrence relation (2) we have

$$G(t) = (1 + t + \dots + t^{p-1})G(t)^p + \sum_{i=0}^{\infty} t^{p^k i + a}.$$

Multiplying with  $t^{p^k} - 1 = (t-1)^{p^k}$  we have

$$(t-1)^{p^k} G(t) = (t-1)^{p^k + p-1} G(t)^p - t^a. \quad (3)$$

Put  $h(s, t) = (t-1)^{p^k + p-1} s^p - (t-1)^{p^k} s - t^a$ . Then  $h(s, t)$  has no rational zero in  $\mathbb{F}_p(t)$ . Indeed, if  $g(t)/f(t)$  is a zero (with  $\gcd(g, f) = 1$ ), that is

$$(t-1)^{p^k + p-1} g(t)^p - (t-1)^{p^k} g(t) f(t)^{p-1} - t^a f(t)^p = 0,$$

then  $(t-1) \mid f(t)$  and so  $(t-1)^{p^k + p-1} \mid f(t)^p$  thus  $(t-1)^{p^{k-1} + 1} \mid f(t)$  which means that  $t-1 \mid g(t)$ , a contradiction.

Finally, the result follows from Theorem 1.  $\square$

### The sum-of-digits sequence

Let  $k > 1$  be an integer and  $\sigma_m(n)$  be the sum of digits of  $n$  in the  $k$ -ary representation. Then define  $s_n = \sigma_m(n) \bmod k$ . Clearly  $(s_n)$  satisfies the following recurrence relation

$$s_n \equiv s_{\lfloor \frac{n}{k} \rfloor} + n \pmod{k}, \quad n = 1, 2, \dots \quad (4)$$

with initial value  $s_0 = 0$ .

**Corollary 2.** *Let  $p$  be a prime number and  $(s_n)$  be the sum-of-digit sequence modulo  $p$  defined by (4) with  $k = p$ . Then*

$$\frac{N-1}{p} \leq L(s_n, N) \leq \frac{(p-1)N+2}{p}.$$

For  $p = 2$   $(s_n)$  is the Thue-Morse sequence again and in this case Corollary 2 coincides with Corollary 1.

*Proof.* As before, by the recurrence relation (4) we have

$$\begin{aligned} G(t) &= (1 + t + \dots + t^{p-1})G(t)^p + \sum_{i=0}^{\infty} t^{ip} \cdot \sum_{a=1}^{p-1} at^a \\ &= \frac{1-t^p}{1-t} G(t)^p + \frac{1}{1-t^p} \cdot t(1-t)^{p-2}. \end{aligned}$$

Multiplying by  $(1-t)^2$  we get that  $G(t)$  is a root of

$$h(s, t) = (1-t)^{p+1} s^p - (1-t)^2 s + t.$$

As in the previous proof, it can be shown that  $h(s, t)$  has no rational zero in  $\mathbb{F}_p(t)$ , thus the result follows from Theorem 1.  $\square$

### Baum-Sweet sequence

The Baum-Sweet sequence  $(b_n)$  is a 2-automatic sequence defined by the rule  $b_0 = 1$  and for  $n \geq 1$

$$b_n = \begin{cases} 1 & \text{if the binary representation of } n \text{ contains no block of consecutive} \\ & \text{0's of odd length;} \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently we have for  $n \geq 1$  of the form  $n = 4^k m$  with  $m$  not divisible by 4

$$b_n = \begin{cases} 0 & \text{if } m \text{ is even,} \\ b_{(m-1)/2} & \text{if } m \text{ is odd.} \end{cases}$$

**Corollary 3.** *Let  $(b_n)$  be the Baum-Sweet sequence. Then*

$$\frac{N}{3} \leq L(b_n, N) \leq \frac{2N+1}{3}.$$

*Proof.* It can be easily checked that the generating function  $G(t)$  of  $(b_n)$  is the unique root of  $h(s, t) = s^3 + t \cdot s + 1$ . Then the bounds follow from Theorem 1.  $\square$

### Regular paperfolding sequence

The value of any given term  $v_n \in \mathbb{F}_2$  in the regular paperfolding sequence can be defined as follows. If  $n = m \cdot 2^k$  where  $m$  is odd, then

$$v_n = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{4}, \\ 0 & \text{if } m \equiv 3 \pmod{4}, \end{cases} \quad n = 1, 2, \dots$$

and any  $v_0 \in \mathbb{F}_2$ .

**Corollary 4.** *Let  $(v_n)$  be the regular paperfolding sequence. Then*

$$\frac{N-3}{2} \leq L(v_n, N) \leq \frac{N}{2} + 2.$$

*Proof.* It can be checked that the generating function  $G(t)$  of  $(v_n)$  is a zero of  $h(s, t) = (t^4 + 1) \cdot s^2 + (t^4 + 1) \cdot s + t = (t^4 + 1)(s + G)(s + G + 1)$  which has no rational zero. Then the bounds follow from Theorem 1.  $\square$

### A sequence with perfect lattice profile and perfect linear complexity profile

The generating function  $G(t)$  of the sequence  $(w_n)$  over  $\mathbb{F}_2$  defined by

$$w_{2n} = 1 \quad \text{and} \quad w_{2n+1} = w_n + 1, \quad n = 0, 1, \dots$$

satisfies the functional equation

$$t(t+1)G(t)^2 + (t+1)G(t) + 1 = 0.$$

Hence,  $L(w_n, N) = \lfloor \frac{N+1}{2} \rfloor$ . Such a linear complexity profile is called perfect. This is the only sequence with both a perfect linear complexity profile and a perfect 'lattice profile', see [9] for more details. Sequences with the first are characterized by  $w_0 = 1$  and  $w_{2n+2} = w_{2n+1} + w_n$  but the choice of  $w_{2n+1}$  is free for  $n \geq 1$ , see [18]. Sequences with the latter are characterized by  $w_{2n+1} = w_n + 1$  but the choice of any  $w_{2n}$  is free, see [9].

## 3 Pattern sequences with the all one pattern

In this section we slightly improve Theorem 1 for the binary pattern sequences when the pattern  $P$  is the all one string  $11 \dots 1$  of length  $k$  (that is  $a = 2^k - 1$ ).

**Theorem 2.** *The  $N$ th linear complexity  $L(r_n, N)$  of the sequence  $(r_n)$  defined by (2) with  $a = 2^k - 1$  satisfies*

$$L(r_n, N) = \begin{cases} 2(2^k - 1) \left\lfloor \frac{N}{4(2^k - 1)} \right\rfloor + 2^k & \text{if } 2^k \leq N \pmod{4(2^k - 1)} \leq 3 \cdot (2^k - 1), \\ 2(2^k - 1) \left\lfloor \frac{N+2^k-2}{4(2^k-1)} \right\rfloor & \text{otherwise.} \end{cases}$$

The proof of the theorem is based on the theory of continued fractions. Thus first we summarize some basic facts about them (Section 3.1), then we prove the result (Section 3.2).

### 3.1 Linear complexity and continued fractions

Now we describe the connection between linear complexity profile of a binary sequence and the continued fraction expansion of its the generating function (see for example [18]).

Let  $\mathbb{F}_2((x^{-1}))$  be the ring of formal Laurent series

$$R = \sum_{i=-m}^{\infty} r_i x^{-i}, \quad r_{-m}, r_{-m+1}, \dots \in \mathbb{F}_2$$

over  $\mathbb{F}_2$  in  $x^{-1}$ . In fact  $\mathbb{F}_2((x^{-1}))$  is a field and the coefficients of the inverse of a non-zero  $R$  can be computed recursively, namely, if  $r_{-m} = 1$ , then

$$R^{-1} = \sum_{i=m}^{\infty} z_i x^{-i}$$

with coefficients

$$z_m = 1 \quad \text{and} \quad z_{i+m} = r_{i-m} + \sum_{j=-m+1}^{i-m-1} r_j z_{i-j}, \quad i = 1, 2, \dots \quad (5)$$

Every formal non-rational Laurent series  $R \in \mathbb{F}_2((x^{-1}))$  has a unique continued fraction expansion

$$R = A_0 + \frac{1}{A_1 + \frac{1}{A_2 + \frac{1}{\ddots}}},$$

where  $A_j \in \mathbb{F}_2[x]$  are polynomials for  $j \geq 0$  and  $\deg A_j \geq 1$  for  $j \geq 1$ .

For

$$R = \sum_{i=-m}^{\infty} r_i x^{-i} \in \mathbb{F}_2((x^{-1}))$$

we define its polynomial part by

$$\text{Pol}(R) = \sum_{i=-m}^0 r_i x^{-i}.$$

The polynomials  $A_j$  ( $j \geq 0$ ) are obtained recursively by

$$\begin{aligned} A_0 &= \text{Pol}(R), & B_0 &= R - \text{Pol}(R), \\ A_{j+1} &= \text{Pol}(B_j^{-1}), & B_{j+1} &= B_j^{-1} - \text{Pol}(B_j^{-1}), \quad j \geq 0. \end{aligned}$$

If the continued fraction expansion is broken off after the term  $A_j$  ( $j \geq 0$ ), we get the rational convergent  $P_j/Q_j$ . The polynomials  $P_j, Q_j$  can be calculated recursively by



$$\begin{aligned}
P_{-1} &= 1, & P_0 &= A_0, & P_j &= A_j P_{j-1} + P_{j-2}, & j &\geq 1, \\
Q_{-1} &= 0, & Q_0 &= 1, & Q_j &= A_j Q_{j-1} + Q_{j-2}, & j &\geq 1.
\end{aligned} \tag{6}$$

The following formulas are shown by straightforward induction on  $j$ :

$$\deg(Q_j) = \sum_{h=1}^j \deg A_h \quad \text{for } j \geq 1, \tag{7}$$

$$P_{j-1}Q_j + P_jQ_{j-1} = 1 \quad \text{for } j \geq 1, \tag{8}$$

$$R = \frac{P_j + B_j P_{j-1}}{Q_j + B_j Q_{j-1}} \quad \text{for } j \geq 0. \tag{9}$$

With  $x = t^{-1} \in \mathbb{F}_2(t)$  we obtain that the generating function of the sequence  $(r_n)$  over  $\mathbb{F}_2$  is

$$G = \sum_{n=0}^{\infty} r_n x^{-n}.$$

The following lemma [18, Theorem 1] gives an explicit description of the linear complexity profile of the sequence  $(r_n)$  in terms of the polynomials  $Q_j$  that are obtained from the continued fraction expansion of

$$R = x^{-1}G = \sum_{i=1}^{\infty} r_{i-1}x^{-i}. \tag{10}$$

**Lemma 5.** *For any  $N \geq 1$  the  $N$ th linear complexity  $L(r_n, N)$  is given by*

$$L(r_n, N) = \deg Q_j$$

where  $j \geq 0$  is uniquely determined by the condition

$$\deg Q_{j-1} + \deg Q_j \leq N < \deg Q_j + \deg Q_{j+1}.$$

We define the (exponential) valuation  $v$  on  $\mathbb{F}_2((x^{-1}))$  as

$$v(R) = m, \quad \text{if } R = \sum_{i=-m}^{\infty} r_i x^{-i} \in \mathbb{F}_2((x^{-1})) \text{ and } r_{-m} \neq 0.$$

For  $R = 0$  we put  $v(R) = -\infty$ . We have the following properties of  $v$ . For  $R, S \in \mathbb{F}_2((x^{-1}))$  we have

- (i)  $v(RS) = v(R) + v(S)$ ,
- (ii)  $v(R + S) \leq \max\{v(R), v(S)\}$ ,
- (iii)  $v(R + S) = \max\{v(R), v(S)\}$  if  $v(R) \neq v(S)$ .

The valuation  $v$  extends the degree function on  $\mathbb{F}_2[x]$ : for  $f \in \mathbb{F}_2[x]$  we have  $v(f) = \deg f$  and

$$v\left(\frac{f}{g}\right) = \deg f - \deg g \quad \text{for } f, g \in \mathbb{F}_2[x], g \neq 0.$$

### 3.2 Proof of Theorem 2

It follows from the functional equation (3) for the generating function of the sequence  $(r_n)$  that the function  $R$  defined in (10) satisfies

$$(1+x)R^2 + R + U^{2^k}x^{-2^k} = 0, \quad (11)$$

where  $U = \sum_{i=0}^{\infty} x^{-i}$ .

**Lemma 6.** *If for a polynomial  $Q \in \mathbb{F}_2[x]$  we have*

$$QU^{2^k} = \sum_{i=-\deg Q}^{\infty} b_i x^{-i},$$

then  $b_i = b_j$  for  $i, j \geq 0$  with  $i \equiv j \pmod{2^k}$  and

$$Q \equiv b_1 x^{2^k-1} + b_2 x^{2^k-2} + \cdots + b_{2^k} \pmod{x^{2^k} + 1}.$$

*Proof.* Write  $Q = x^d + q_{d-1}x^{d-1} + \cdots + q_1x + q_0$ . Then for  $i \geq 0$  the coefficient of  $QU^{2^k}$  of  $x^{-i}$  is

$$b_i = \sum_{j: i+j \equiv 0 \pmod{2^k}} q_j,$$

and the result follows.  $\square$

*Proof of Theorem 2.* We prove the following assertions by induction.

- (i)  $A_1 = x^2 + x + 1$  if  $k = 1$  and  $A_1 = x^{2^k} + x$  if  $k \geq 2$ .
- (ii) For  $j \geq 2$ ,  $A_j = x^2 + 1$  if  $k = 1$  and for  $j \geq 1$ ,  $A_{2j} = x^{2^k-2} + x^{2^k-4} + \cdots + x^2 + 1$ ,  $A_{2j+1} = x^{2^k} + 1$  if  $k \geq 2$ .
- (iii) For  $j \geq 0$ ,  $Q_j \equiv 1 \pmod{x+1}$  if  $k = 1$  and  $Q_{2j} \equiv 1 \pmod{x^{2^{k-1}} + 1}$ ,  $Q_{2j+1} \equiv x + 1 \pmod{x^{2^{k-1}} + 1}$  for  $k \geq 2$ .

Then the result follows from Lemma 5, (i), (ii) and (7).

The first part follows from straightforward computation. Namely, observe that the first  $2^{k+1}$  elements of the sequence  $(r_n)$  are zeros, except the elements  $r_n$  with  $n = 2^k - 1$  and  $n = 2^{k+1} - 2$ . Thus

$$R = \frac{1}{x^{2^k}} + \frac{1}{x^{2^{k+1}-1}} + \sum_{i=2^{k+1}+1}^{\infty} r_{i-1} x^{-i},$$

whence by (5) we have

$$R^{-1} = x^{2^k} + x + x^{2-2^k} + \dots$$

so

$$A_1 = \text{Pol}(R^{-1}) = \begin{cases} x^2 + x + 1 & \text{if } k = 1, \\ x^{2^k} + x & \text{if } k \geq 2 \end{cases}$$

which proves (i).

By Corollary 1,  $R$  is irrational and thus  $\deg A_j \geq 1$  for all  $j \geq 1$ . Now by (7), (8), (9) and the properties of  $v$  it follows for  $l \geq 1$  that

$$v(Q_{l-1}R - P_{l-1}) = -v(Q_l) \quad (12)$$

whence

$$v(Q_{l-1}^2 R^2 - P_{l-1}^2) = -2v(Q_l). \quad (13)$$

Consider

$$T = (x+1)P_{l-1}^2 + P_{l-1}Q_{l-1} + U^{2^k} x^{-2^k} Q_{l-1}^2 \quad (14)$$

$$= (x+1)(Q_{l-1}^2 R^2 - P_{l-1}^2) + Q_{l-1}(P_{l-1} + Q_{l-1}R), \quad (15)$$

where the second equality follows from (11). Now we have

$$v(T) = -v(A_l), \quad l \geq 1, \quad (16)$$

by (7), (12), (13), (15) and the properties of  $v$ . It follows from (6), (8) and (9), that

$$Q_{l-1}(P_{l-1} + Q_{l-1}R) = Q_{l-1} \frac{1}{A_l Q_{l-1} + Q_{l-2} + B_l Q_{l-1}} = \frac{1}{A_l + \frac{Q_{l-2}}{Q_{l-1}} + B_l} \quad (17)$$

where we have

$$v(B_l), v\left(\frac{Q_{l-2}}{Q_{l-1}}\right) < 0. \quad (18)$$

On the other hand

$$\begin{aligned} v((x+1)(Q_{l-1}^2 R^2 - P_{l-1}^2)) &= 1 - 2v(Q_l) = -2v(A_l) + 1 - 2v(Q_{l-1}) \\ &\leq -2v(A_l) + 1 - 2v(A_1) < -2v(A_l) \end{aligned} \quad (19)$$

for  $l \geq 2$ .

We now prove (ii) and (iii) by induction. We remark that the assertion (iii) for  $Q_0$  and  $Q_1$  follows from (6) and (i).

Take  $l \geq 2$ . If  $k = 1$  or  $k \geq 2$  and  $l = 2j + 1$  is odd, then  $Q_{l-1} \equiv 1 \pmod{x^{2^{k-1}} + 1}$ , thus  $Q_{l-1}^2 \equiv 1 \pmod{x^{2^k} + 1}$ . By Lemma 6, the coefficients of  $x^{-i}$  ( $i = 1, \dots, 2^k - 1$ ) in  $Q_{l-1}^2 U^{2^k}$  are all zero but the coefficient of  $x^{-2^k}$  is one. Since  $R$  is irrational, the degree  $\deg A_l$  and so  $v(T)$  cannot be zero. Thus by (14)  $x^{-2^k}$  is the leading term of  $T$  so  $\deg(A_l) = 2^k$  by (16).

By Lemma 6, the coefficient of  $x^{-i}$  ( $i = 2^k + 1, \dots, 2^{k+1} - 1$ ) in  $Q_{l-1}^2 U^{2^k}$  is zero. Thus by (19)

$$Q_{l-1}(P_{l-1} + Q_{l-1}R) = x^{-2^k} + x^{2^{k+1}} + \sum_{i=2^{k+1}}^{\infty} a_i x^{-i}.$$

It follows from (18) that

$$A_l^{-1} = x^{-2^k} + x^{2^{k+1}} + \sum_{i=2^{k+1}}^{\infty} a'_i x^{-i},$$

whence  $A_l = x^{2^k} + 1$  by (5).

If  $k \geq 2$  and  $l = 2j$  is even, the proof of (ii) is similar. Since  $Q_{l-1} \equiv x + 1 \pmod{x^{2^{k-1}} + 1}$ , thus  $Q_{l-1}^2 \equiv x^2 + 1 \pmod{x^{2^k} + 1}$ . By Lemma 6, the coefficients of  $x^{-i}$  ( $i = 1, \dots, 2^k - 3$ ) in  $Q_{l-1}^2 U^{2^k}$  are all zero but the coefficient of  $x^{-2^k+2}$  is one. Thus by (14)  $x^{-2^k+2}$  is the leading term of  $T$  so  $\deg(A_l) = 2^k - 2$  by (16). By Lemma 6, the coefficient of  $x^{-i}$  ( $i = 2^k - 1, \dots, 2^{k+1} - 1$ ) in  $Q_{l-1}^2 U^{2^k}$  are all zeros except for  $i = 2^{k+1} - 2$ . Thus By (19)

$$Q_{l-1}(P_{l-1} + Q_{l-1}R) = x^{-2^k+2} + x^{2^k} + x^{-2^{k+1}-2} + \sum_{i=2^{k+1}-1}^{\infty} a_i x^{-i}.$$

It follows from (18) that

$$A_l^{-1} = x^{-2^k+2} + x^{2^k} + x^{-2^{k+1}-2} + \sum_{i=2^{k+1}-1}^{\infty} a'_i x^{-i},$$

whence  $A_l = x^{2^k-2} + x^{2^k-4} + \dots + x^2 + 1$  by (5), which proves (ii).

Since  $x^{2^k} + 1 \mid A_l$  for  $l \geq 2$ , we have by (6) that

$$Q_l = A_l Q_{l-1} + Q_{l-2} \equiv Q_{l-2} \pmod{x + 1},$$

which proves (iii). □

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