# STIELTJES CONSTANTS OF $L$-FUNCTIONS IN THE EXTENDED SELBERG CLASS

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Abstract. Let $f$ be an arithmetic function and let $\mathcal{S}^\#$ denote the extended Selberg class. We denote by $\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ the Dirichlet series attached to $f$. The Laurent-Stieltjes constants of $\mathcal{L}(s)$ which belongs to $\mathcal{S}^\#$, are the coefficients of the Laurent expansion of $\mathcal{L}$ at its pole $s = 1$. In this paper, we give an upper bound of these constants, which is a generalization of many known results.

## 1. Introduction

Let $q$ be any positive integer $\geq 1$ and let $\chi$ be a Dirichlet character modulo $q$. Let $\gamma_n(\chi)$ denote the Laurent coefficients of the *Dirichlet $L$-function* $L(s,\chi)$ near $s = 1$. We recall that

$$\gamma_n(\chi) = \sum_{a=1}^{q} \chi(a)\gamma_n(a,q),$$

where

$$\gamma_n(a,q) = \lim_{M\to\infty} \sum_{1\leq m\equiv a \bmod q}^{M} \frac{(\log m)^n}{m} - \frac{(\log M)^{n+1}}{q(n+1)}.$$

In particular, $\gamma_0(1,1) = 0.5772156649\cdots$ is the well-known Euler constant. The constants $\gamma_n(a,q)$ are often called the *Stieltjes constants* or *generalized Euler constants*. In the particular case when $\chi = \chi_0$, where $\chi_0$ is the principal character modulo 1, the Dirichlet $L$-function $L(s,\chi_0)$ reduces to the *Riemann zeta function* $\zeta(s)$, that is $L(s,\chi_0) = \zeta(s)$. We write the corresponding Laurent coefficients simply $\gamma_n(\chi_0) = \gamma_n(1,1) = \gamma_n$. Stieltjes in 1885 showed that

$$\gamma_n = \frac{(-1)^n}{n!} \lim_{M\to\infty} \left( \sum_{m=1}^{M} \frac{(\log m)^n}{m} - \frac{(\log M)^{n+1}}{(n+1)} \right) \tag{1}$$

which pioneered the study of Laurent coefficients of zeta functions and $L$-functions. This gives rise to the widely used name "Stieltjes constants" for these coefficients.

The asymptotic behavior of $\gamma_n$ as $n \to \infty$ has been widely studied by many authors (for instance: Briggs [3], Mitrovič [14], Israilov [9], Matsuoka [13], and more recently, Coffey [5, 6], Knessl and Coffey [10], Adell [2], Adell and Lekuona [1], and Saad Eddin [18]). The studies mostly focused on the growth and sign changes of the sequence $(\gamma_n)$, explicit upper estimates for $|\gamma_n|$, and asymptotic expressions for $\gamma_n$. Stieltjes constants for other zeta functions and $L$-functions have also been studied by many authors. We introduce some of their results in the following section.

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In this paper, we are interested in investigating the Stieltjes constants of more general $L$-functions. We consider functions in a class larger than the Selberg class. We first introduce the Selberg class $\mathcal{S}$.

**Selberg class.** Let $f$ be an arithmetic function, and denote by $\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ the Dirichlet series attached to $f$. We say the Dirichlet series $\mathcal{L}(s)$ belongs to the Selberg class $\mathcal{S}$ if it is absolutely convergent when $\mathrm{Re}(s) > 1$ and satisfies the following properties:

**Condition $\boldsymbol{\mathcal{S}}$1.** Ramanujan hypothesis: For any $\varepsilon > 0$, we have $f(n) \ll_\varepsilon n^\varepsilon$.

**Condition $\boldsymbol{\mathcal{S}}$2.** Analytic continuation: There exists $k \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^k \mathcal{L}(s)$ is entire of finite order.

**Condition $\boldsymbol{\mathcal{S}}$3.** Functional equation: Define

$$\mathcal{F}(s) := Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$$

where $Q, \lambda_i$ are positive real numbers, $\Gamma$ is the gamma function, $\mu_j$ is a complex number satisfying $\mathrm{Re}(\mu_j) \geq 0$. Then the function $\Phi(s) := \mathcal{F}(s)\mathcal{L}(s)$ satisfies the functional equation

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})},$$

where $\omega$ is a complex number with $|\omega| = 1$.

**Condition $\boldsymbol{\mathcal{S}}$4.** Euler product: For $\mathrm{Re}(s) > 1$, the function $\mathcal{L}(s)$ can be written as a product over prime numbers $p$:

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s)$$

where

$$\mathcal{L}_p(s) = \exp\left( \sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \right),$$

with $b(n) \ll n^\theta$, for some $\theta < \frac{1}{2}$.

This class $\mathcal{S}$ is ***expected*** to be the largest class of zeta and $L$-functions satisfying the Riemann hypothesis, usually called the Grand Riemann Hypothesis: all nontrivial zeros of these functions lie on $\mathrm{Re}(s) = 1/2$. The extended Selberg class $\mathcal{S}^{\#}$ is defined to be the class of functions $\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ satisfying the above conditions $\boldsymbol{\mathcal{S}}$**2** and $\boldsymbol{\mathcal{S}}$**3**, but not necessarily $\boldsymbol{\mathcal{S}}$**1** and $\boldsymbol{\mathcal{S}}$**4**.

Notable examples of $\mathcal{L} \in \mathcal{S}$ are the Riemann zeta function $\zeta(s)$, Dirichlet $L$-functions $L(s, \chi)$ associated with non-principal primitive characters $\chi$, and the Dedekind zeta function $\zeta_K(s)$ of a number field $K$. The sum of the all parameters $\lambda_j$ in $\boldsymbol{\mathcal{S}}$**3** gives the *degree* of the $L$-function $\mathcal{L}(s)$ in $\mathcal{S}^{\#}$, and so in $\mathcal{S}$, as follows:

$$d_{\mathcal{L}} = 2 \sum_{j=1}^{r} \lambda_j.$$

It is not known if $d_{\mathcal{L}} \in \mathbb{Z}$ for all $\mathcal{L} \in \mathcal{S}$ but the degree $d_{\mathcal{L}}$ characterizes certain properties of the functions $\mathcal{L} \in \mathcal{S}$. For example, $d_{\mathcal{L}}$ characterizes several analytic properties of $\mathcal{L}$, even though the functional equations may contain non-unique information. We shall not discuss

$d_{\mathcal{L}}$ further since it is irrelevant to the aim of this paper. The readers may refer to [21, Chapter 6] for more details about the Selberg class.

From now we keep our focus on $\mathcal{L} \in \mathcal{S}^{\#}$. That is, we would like to extend beyond the Selberg class by eliminating conditions $\boldsymbol{\mathcal{S}1}$ and $\boldsymbol{\mathcal{S}4}$. Consider the Laurent expansion of $\mathcal{L}(s)$ at its possible pole $s=1$ written in the following form:

$$\mathcal{L}(s) = \sum_{n=-k}^{\infty} \gamma_n(\mathcal{L})(s-1)^n.$$

We call the coefficients $\gamma_n(\mathcal{L})$ the *generalized Laurent-Stieltjes constants* or the *Laurent-Stieltjes constants of the extended Selberg class*. In this paper, we study these coefficients and give an upper bound of $\gamma_n(\mathcal{L})$ for $\mathcal{L} \in \mathcal{S}^{\#}$. We remark that none of the arguments in this method we use requires $\boldsymbol{\mathcal{S}4}$, while eliminating $\boldsymbol{\mathcal{S}1}$ requires us to use a weaker condition (see $\boldsymbol{\mathcal{S}5}$ in the proof of Lemma 1).

Our main theorem is stated as follows.

**Theorem.** *Let $\mathcal{L} \in \mathcal{S}^{\#} \setminus \{0\}$ and let $d_{\mathcal{L}}$ be the degree of $\mathcal{L}$. Let $Q$ be the positive real number appearing in condition $\boldsymbol{\mathcal{S}3}$ and let*

$$\lambda_m := \min_{1\le j\le r} \lambda_j, \quad \lambda_M := \max_{1\le j\le r} \lambda_j, \quad \text{and} \quad \mu_M := \max_{1\le j\le r} |\mu_j|.$$

*For a positive integer $n$ with*

$$\frac{n}{\log n} > \left(\frac{1}{2} + \frac{\mu_M+1}{\lambda_m}\right) d_{\mathcal{L}} \log(Q+3),$$

*we have*

$$|\gamma_n(\mathcal{L})| \le C_{\mathcal{L}}(a) a^{-n} \left(2 + \frac{1}{n - \frac{d_{\mathcal{L}}(2a-1)}{2}}\right),$$

*where $a$ satisfies $1 + \frac{\mu_M+1}{\lambda_m} < a < \frac{1}{2} + \frac{n}{d_{\mathcal{L}}}$ and*

$$C_{\mathcal{L}}(a) = \frac{2^r Q^{2a-1}}{\pi} \exp\left(\frac{1}{5}\sum_{j=1}^{r} \frac{1}{\lambda_j(a-1) - \mathrm{Re}(\mu_j)}\right) \left(\sum_{m=1}^{\infty} \frac{|f(m)|}{m^a}\right) (8\lambda_M^2 a^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}.$$

Finally we remark that the Laurent-Stieltjes constants of zeta and $L$-functions have many applications not only in analytic number theory, but also in algebraic number theory and even fields outside of number theory. They can be used to determine zero-free regions for $L(s,\chi)$ near the real axis in the critical strip $0 \le \mathrm{Re}(s) \le 1$, to compute the values of $\zeta(s)$ in the complex plane, to study the class number of a quadratic field, etc.

## 2. Some known results on the Laurent-Stieltjes constants of zeta and $L$-functions

The first explicit upper bound for $|\gamma_n|$ has been given by Briggs [3], which was later improved by Berndt and Israilov. In 1985, the theory made a huge progress via an asymptotic expansion shown by Matsuoka [13], for these constants. He gave an excellent upper bound of $|\gamma_n|$ for $n \ge 10$ and proved that

$$|\gamma_n| \le 10^{-4} e^{n \log\log n}.$$

This result had been the best upper bound of $|\gamma_n|$ for more than 20 years. Thanks to the above result, Matsuoka showed that $\zeta(s)$ has no zeros in the region $|s-1| \le \sqrt{2}$, with $0 \le \mathrm{Re}(s) \le 1$.

Many have tried to improve on the Matsuoka bound, with few successful attempts. Matsuoka's work relied on a formula that is essentially a consequence of Cauchy's integral theorem and the functional equation. More recently, the second author [18] extended this formula to Dirichlet $L$-functions. She gave the following upper bound for $|\gamma_n(\chi)|$ for primitive Dirichlet characters $\chi$ modulo $q$ and for every $1 \leq q \leq \pi e^{(n+1)/2}/(2n+2)$. We have

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-1/2} C(n,q) \min\left(1 + D(n,q), \frac{\pi^2}{6}\right)$$

where

$$C(n,q) = 2\sqrt{2} \exp\left\{-(n+1)\log\theta(n,q) + \theta(n,q)\left(\log\theta(n,q) + \log\frac{2q}{\pi e}\right)\right\}$$

and

$$\theta(n,q) = \frac{n+1}{\log\frac{2q(n+1)}{\pi}} - 1, \quad D(n,q) = 2^{-\theta(n,q)-1}\frac{\theta(n,q)+1}{\theta(n,q)-1}.$$

In the case when $\chi = \chi_0$ and $q = 1$, this leads to a sizable improvement of Matsuoka's bound and of previous results. As an application of this upper bound, the second author showed in [19] that this result enables us to approximate $L(s,\chi)$ in the neighborhood of $s = 1$ by a short Taylor polynomial. For $N = 4\log q$ and $q \geq 150$, we have

$$\left| L(s,\chi) - \sum_{n \leq N} \frac{(-1)^n \gamma_n(\chi)}{n!}(s-1)^n \right| \leq \frac{32.3}{q^{2.5}},$$

where $|s-1| \leq e^{-1}$. She also proved that the function $\zeta(s)$ has no zeros in the region $|s-1| \leq 2.2093$ with $0 \leq \mathrm{Re}(s) \leq 1$. This result is an improvement of Matsuoka's result.

Finally, let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. Define for $\mathrm{Re}(s) > 1$ the *Dedekind zeta function*

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N\mathfrak{p}^{-s}},$$

where $\mathfrak{a}$ runs over non-zero ideals in $\mathcal{O}_K$, $\mathfrak{p}$ runs over the prime ideals in $\mathcal{O}_K$ and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It is known that $\zeta_K(s)$ can be analytically continued to $\mathbb{C} \setminus \{1\}$, and that at $s = 1$ it has a simple pole, with residue $\gamma_{-1}(K)$ given by the analytic class number formula:

$$\gamma_{-1}(K) = \frac{2^{r_1}(2\pi)^{r_2} h(K) R(K)}{\omega(K)\sqrt{|D(K)|}}.$$

Here we denote by $r_1$ the number of real embeddings of $K$, $r_2$ the number of complex embeddings of $K$, $h(K)$ the class number of $K$, $R(K)$ the regulator of $K$, $\omega(K)$ the number of roots of unity contained in $K$ and $D(K)$ the discriminant of the extension $K/\mathbb{Q}$. Consider the Laurent expansion

$$\zeta_K(s) = \frac{\gamma_{-1}(K)}{s-1} + \sum_{n=0}^{\infty} \gamma_n(K)(s-1)^n$$

of $\zeta_K(s)$ at $s = 1$. The constants $\gamma_n(K)$ are sometimes called the Stieltjes constants associated with the Dedekind zeta function. In [7] they are called higher Euler's constants of $K$. The second author [20] studied these constants and showed that, for $n \geq 1$, we have

$$\gamma_n(K) = \frac{(-1)^n}{n!} \lim_{x\to\infty} \left( \sum_{N\mathfrak{a}\leq x} \frac{(\log N\mathfrak{a})^n}{N\mathfrak{a}} - \gamma_{-1}(K) \frac{(\log x)^{n+1}}{n+1} \right),$$

and

$$\gamma_0(K) = \lim_{x\to\infty} \left( \sum_{N\mathfrak{a}\leq x} \frac{1}{N\mathfrak{a}} - \gamma_{-1}(K) \log x \right) + \gamma_{-1}(K).$$

To conclude this section, we remark that only the first constant $\gamma_K = \gamma_0(K)/\gamma_{-1}(K)$, called the Euler-Kronecker constant, which is closely related to values of the logarithmic derivative of $L$-functions, has been studied so far. For more details see [15]. This raises questions on the other Stieltjes constants associated with $\zeta_K(s)$. The authors were motivated to give partial answers to these questions in a much more general context, that is, for all $L$-functions in the Selberg class.

## 3. Auxiliary lemmas

In order to prove our main result, we first show a proposition and two necessary lemmas. Recall the notation used when we defined $\mathcal{S}^\#$ in Section 1.

**Lemma 1.** *Let $\mathcal{L} \in \mathcal{S}^\#$ and let $d_\mathcal{L}$ be the degree of $\mathcal{L}$. Then we have*

$$\mathcal{L}(\sigma+it) \asymp_\mathcal{L} |t|^{d_\mathcal{L}(\frac{1}{2}-\sigma)} |\mathcal{L}(1-\sigma+it)|. \tag{2}$$

*In particular,*

$$\mathcal{L}(\sigma+it) \ll_{\mathcal{L},\varepsilon} \begin{cases} |t|^\varepsilon & \textit{if} \quad \sigma \geq 1, \\ |t|^{\frac{1-\sigma}{2}d_\mathcal{L}+\varepsilon} & \textit{if} \quad 0 \leq \sigma \leq 1, \\ |t|^{(\frac{1}{2}-\sigma)d_\mathcal{L}+\varepsilon} & \textit{if} \quad \sigma \leq 0. \end{cases} \tag{3}$$

*Proof.* For the standard case when we assume $\boldsymbol{\mathcal{S}}\mathbf{1}$, see [21, Theorem 6.8]. Note here that the first-half (2) is obtained from the functional equation $\boldsymbol{\mathcal{S}}\mathbf{3}$.

Now without $\boldsymbol{\mathcal{S}}\mathbf{1}$, we note that the absolute convergence of the Dirichlet series can be rewritten as follows:

**Condition $\boldsymbol{\mathcal{S}}$5.** For any $\varepsilon > 0$, we have $\sum_{n\leq x} |f(n)| \ll_\varepsilon x^{1+\varepsilon}$.

Note that this is a weaker condition than $\boldsymbol{\mathcal{S}}\mathbf{1}$. This and again the functional equation $\boldsymbol{\mathcal{S}}\mathbf{3}$ easily give us the bounds (3) for the case $\sigma \geq 1+\varepsilon$ and $\sigma \leq -\varepsilon$. That is

$$\mathcal{L}(\sigma+it) \ll_{\mathcal{L},\varepsilon} \begin{cases} 1 & \text{if} \quad \sigma \geq 1+\varepsilon, \\ |t|^{(\frac{1}{2}-\sigma)d_\mathcal{L}+\varepsilon} & \text{if} \quad \sigma \leq -\varepsilon. \end{cases}$$

Since the function $\mathcal{L}(s)$ is entire of finite order from condition $\boldsymbol{\mathcal{S}}\mathbf{2}$, for any $\delta > 0$,

$$\mathcal{L}(\sigma+it) \ll_{\mathcal{L},\varepsilon} \exp\exp(\delta|t|)$$

holds in the strip $-1 \leq \sigma \leq 2$. Substituting this into (2), we can show that this also holds for $0 \leq \sigma \leq 1/2$. Applying the theorem of Phragmén-Lindelöf [16, Proposition 8.15], we have

$$\mathcal{L}(\sigma+it) \ll_{\mathcal{L},\varepsilon} |t|^{\frac{1-\sigma}{2}d_\mathcal{L}+\varepsilon}$$

for $0 \leq \sigma \leq 1$. $\square$

**Remark.** We remark that this is the only statement for which we need condition $\boldsymbol{\mathcal{S}}\mathbf{1}$ or at least $\boldsymbol{\mathcal{S}}\mathbf{5}$.

**Proposition 1.** *Let $\mathcal{L} \in \mathcal{S}^{\#}$ with degree $d_{\mathcal{L}} > 0$. Let $n$ be an integer with $n > \max\left\{0, \frac{d_{\mathcal{L}}}{2} - 1\right\}$. For $1 < a < \frac{n+1}{d_{\mathcal{L}}} + \frac{1}{2}$ such that $\lambda_j(1-a) + \mathrm{Re}(\mu_j) \notin \mathbb{Z}$ for each $j = 1, 2, \ldots, r$, we have*

$$\gamma_n(\mathcal{L}) = \frac{(-1)^n}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{G_{\mathcal{L}}(s)}{s^{n+1}} \overline{\mathcal{L}(\overline{s})} ds,$$

*where the function $G_{\mathcal{L}}$ is defined by*

$$G_{\mathcal{L}}(s) := \frac{\omega Q^{2s-1}}{\pi^r} \prod_{j=1}^{r} \Gamma(\lambda_j s + \overline{\mu}_j) \sin(\pi(\lambda_j(1-s) + \mu_j)) \Gamma(\lambda_j(s-1) + 1 - \mu_j). \tag{4}$$

*Here $Q, \lambda_i$ are positive real numbers, $\mu_j$ and $\omega$ are complex numbers with $\mathrm{Re}(\mu_j) \geq 0$ and $|\omega| = 1$.*

*Proof.* By Cauchy's formula, we can write

$$\gamma_n(\mathcal{L}) = \frac{1}{2\pi i} \int_D \frac{\mathcal{L}(s)}{(s-1)^{n+1}} ds,$$

where $D$ is the positively oriented rectangular path passing through the vertices $-a+1+iT$, $-a+1-iT$, $A-iT$ and $A+iT$, where $A$ and $T$ are sufficiently large numbers. Let us now divide $D$ into the line segments $D_1, D_2, D_3$ and $D_4$ joining $-a+1+iT$, $-a+1-iT$, $A-iT$, $A+iT$ and $-a+1+iT$, as in Figure 1. Then, we have

$$\gamma_n(\mathcal{L}) = \frac{1}{2\pi i} \left( \int_{D_1} + \int_{D_2} + \int_{D_3} + \int_{D_4} \right) \frac{\mathcal{L}(s)}{(s-1)^{n+1}} ds.$$

By Lemma 1, the integral over $D_2$ is bounded by

$$\begin{aligned} \left| \int_{D_2} \frac{\mathcal{L}(s)}{(s-1)^{n+1}} ds \right| &= \left| \left( \int_{-a+1-iT}^{-iT} + \int_{-iT}^{1-iT} + \int_{1-iT}^{A-iT} \right) \frac{\mathcal{L}(s)}{(s-1)^{n+1}} ds \right| \\ &\ll_{\mathcal{L},\varepsilon} T^{-n-1+\varepsilon} \times \left( \int_{-a+1}^{0} T^{d_{\mathcal{L}}(\frac{1}{2}-\sigma)} d\sigma + \int_0^1 T^{\frac{1}{2} d_{\mathcal{L}}(1-\sigma)} d\sigma + \int_1^A d\sigma \right) \\ &\ll_{\mathcal{L},\varepsilon} T^{-n-1+d_{\mathcal{L}}(a-1/2)+\varepsilon}. \end{aligned}$$

Since $a < \frac{n+1}{d_{\mathcal{L}}} + \frac{1}{2}$, the last term vanishes as $T \to +\infty$. Therefore, the integral over $D_2$ tends to 0 as $T \to +\infty$. A similar argument shows that the integral over $D_4$ tends to 0 as $T \to +\infty$.

Next we consider the integral over $D_3$. For $n > 0$, we find that

$$\left| \int_{D_3} \right| \ll_{\mathcal{L},n} \int_0^\infty \frac{dt}{((A-1)^2 + t^2)^{(n+1)/2}} < +\infty$$

and for any $t \geq 0$,

$$\lim_{A \to +\infty} \frac{1}{((A-1)^2 + t^2)^{(n+1)/2}} = 0.$$

Hence by Lebesgue's convergence theorem, we have $\lim\limits_{A \to +\infty} \lim\limits_{T \to +\infty} \int_{D_3} = 0$.

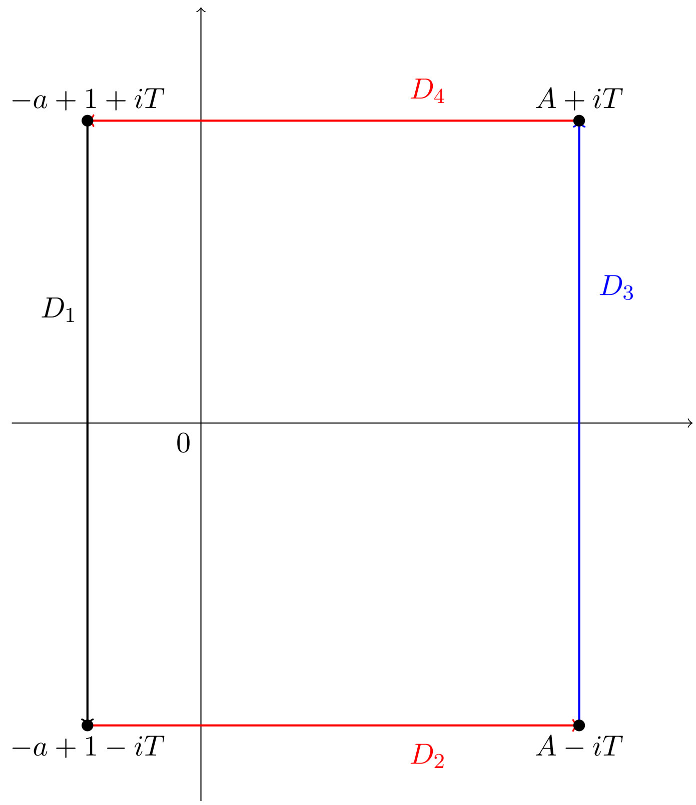


FIGURE 1. The rectangle $D$ in the complex $s$ plane

Thus, for $n > \max\left\{0, \frac{d_{\mathcal{L}}}{2} - 1\right\}$, we have

$$\gamma_n(\mathcal{L}) = \frac{1}{2\pi i} \int_{-a+1+i\infty}^{-a+1-i\infty} \frac{\mathcal{L}(s)}{(s-1)^{n+1}} ds = \frac{(-1)^n}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\mathcal{L}(1-s)}{s^{n+1}} ds.$$

Here, by using the functional equation $\boldsymbol{\mathcal{S}3}$ for $\mathcal{L}(s)$ and the formula $\Gamma(s)\Gamma(1-s)\sin(\pi s) = \pi$, we have

$$\begin{aligned}\mathcal{L}(1-s) &= \overline{\mathcal{L}(\overline{s})} \left(\omega \frac{\overline{\mathcal{F}(\overline{s})}}{\mathcal{F}(1-s)}\right) = \overline{\mathcal{L}(\overline{s})} \left(\omega Q^{2s-1} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j s + \overline{\mu_j})}{\Gamma(\lambda_j(1-s) + \mu_j)}\right) \\ &= \overline{\mathcal{L}(\overline{s})} G_{\mathcal{L}}(s).\end{aligned}$$

Hence

$$\gamma_n(\mathcal{L}) = \frac{(-1)^n}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{G_{\mathcal{L}}(s)}{s^{n+1}} \overline{\mathcal{L}(\overline{s})} ds,$$

where the function $G_{\mathcal{L}}(s)$ is as defined in (4). This completes the proof of Proposition 1. $\square$

**Lemma 2.** *For $\mathcal{L} \in \mathcal{S}^{\#} \setminus \{0\}$, consider $G_{\mathcal{L}}$ as defined in Proposition 1. Let $\lambda_m := \min_{1 \le j \le r} \lambda_j$ and $\mu_M := \max_{1 \le j \le r} |\mu_j|$. For $a > 1 + \frac{\mu_M}{\lambda_m}$, we have*

$$|G_{\mathcal{L}}(a+it)| \le c_{\mathcal{L}}(a) Q^{2a-1} \left((a\lambda_M + \mu_M + 1)^2 + (\lambda_M |t| + \mu_M)^2\right)^{\frac{d_{\mathcal{L}}}{4}(2a-1)},$$

*where the constant $c_{\mathcal{L}}(a)$ is defined by*

$$c_{\mathcal{L}}(a) = 2^r \exp\left( \frac{1}{5} \sum_{j=1}^{r} \frac{1}{\lambda_j (a-1) - \mathrm{Re}(\mu_j)} \right).$$

*Proof.* Put $\lambda_m := \min_{1\le j\le r} \lambda_j$, $\lambda_M := \max_{1\le j\le r} \lambda_j$, $\mu_M := \max_{1\le j\le r} |\mu_j|$, and let $a > 1+\frac{\mu_M}{\lambda_m}$. From (4), we have

$$\begin{aligned} |G_{\mathcal{L}}(a+it)| \le & \frac{Q^{2a-1}}{\pi^r} \\ & \times \prod_{j=1}^{r} \left| \Gamma(\lambda_j(a+it)+\overline{\mu_j}) \Gamma(\lambda_j(a-1+it)+1-\mu_j) \sin(\pi(\lambda_j(1-a-it)+\mu_j)) \right|. \end{aligned} \tag{5}$$

We can easily show that

$$|\sin(\pi(\lambda_j(1-a-it)+\mu_j))| \le \exp(\pi|\lambda_j t - \mathrm{Im}(\mu_j)|).$$

On the other hand, using Stirling's formula we can show that, for $x>0$,

$$\begin{aligned} \log|\Gamma(x+iy)| &= \frac{1}{2}\left(x-\frac{1}{2}\right)\log(x^2+y^2) - y\arctan(y/x) - x + \frac{1}{2}\log 2\pi + \varphi(x,y) \\ &\le \frac{1}{2}\left(x-\frac{1}{2}\right)\log(x^2+y^2) - \frac{\pi}{2}|y| + \frac{1}{2}\log 2\pi + \varphi(x,y), \end{aligned}$$

where the function $\varphi(x,y)$ satisfies the inequality (cf. Binet's first formula)

$$|\varphi(x,y)| \le \left| \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1} \right) \frac{e^{-t(x+iy)}}{t} dt \right| \le \frac{1}{10x}.$$

From these inequalities, we find that (note that $\mathrm{Re}(\lambda_j(a-1+it)+1-\mu_j) \ge 0$ since $a > 1+\mu_M/\lambda_m$)

$$\begin{aligned} & |\Gamma(\lambda_j(a+it)+\overline{\mu_j})\Gamma(\lambda_j(a-1+it)+1-\mu_j)\sin(\pi(\lambda_j(1-a-it)+\mu_j))| \\ & \quad \le 2\pi \exp\left( \frac{1}{5(\lambda_j(a-1)-\mathrm{Re}(\mu_j))} \right) \left( (a\lambda_M+\mu_M+1)^2 + (\lambda_M|t|+\mu_M)^2 \right)^{\frac{\lambda_j}{2}(2a-1)}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \frac{Q^{2a-1}}{\pi^r} \prod_{j=1}^{r} |\Gamma(\lambda_j(a+it)+\overline{\mu_j})\Gamma(\lambda_j(a-1+it)+1-\mu_j)\sin(\pi(\lambda_j(1-a-it)+\mu_j))| \\ & \quad \le c_{\mathcal{L}}(a) Q^{2a-1} \left( (a\lambda_M+\mu_M+1)^2 + (\lambda_M|t|+\mu_M)^2 \right)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}, \end{aligned}$$

which completes the proof. □

## 4. Proof of Theorem

Now we are ready to prove our main theorem. We again put

$$\lambda_m := \min_{1\le j\le r} \lambda_j, \quad \lambda_M := \max_{1\le j\le r} \lambda_j, \quad \mu_M := \max_{1\le j\le r} |\mu_j|,$$

and let $a$ be a real number satisfying $1+\frac{\mu_M+1}{\lambda_m} < a < \frac{1}{2} + \frac{n}{d_{\mathcal{L}}}$.

By Proposition 1 and Lemma 2, we have

$$\begin{aligned}|\gamma_n(\mathcal{L})| &\le \frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{|f(m)|}{m^a}\int_{-\infty}^{\infty}\frac{|G_{\mathcal{L}}(a+it)|}{(a^2+t^2)^{(n+1)/2}}dt\\ &\le \frac{c_{\mathcal{L}}(a)}{\pi}Q^{2a-1}\sum_{m=1}^{\infty}\frac{|f(m)|}{m^a}\int_0^{\infty}\frac{\left((a\lambda_M+\mu_M+1)^2+(\lambda_M t+\mu_M)^2\right)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}}{(a^2+t^2)^{(n+1)/2}}dt,\end{aligned}$$

where

$$c_{\mathcal{L}}(a) = 2^r \exp\left(\frac{1}{5}\sum_{j=1}^{r}\frac{1}{\lambda_j(a-1)-\mathrm{Re}(\mu_j)}\right).$$

We divide the region of integration into two as follows:

$$\left(\int_0^A + \int_A^{\infty}\right)\frac{\left((a\lambda_M+\mu_M+1)^2+(\lambda_M t+\mu_M)^2\right)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}}{(a^2+t^2)^{(n+1)/2}}dt =: J_1 + J_2 \tag{6}$$

with $A = a + \frac{1}{\lambda_M}$. We estimate $J_1$ and $J_2$ in the following manner:

$$\begin{aligned}J_1 &\le 2^{\frac{3}{4}d_{\mathcal{L}}(2a-1)}\int_0^A \frac{(a\lambda_M)^{\frac{d_{\mathcal{L}}}{2}(2a-1)}}{a^{n+1}}dt \le (8\lambda_M^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}\frac{A}{a}a^{-n+\frac{d_{\mathcal{L}}}{2}(2a-1)}\\ &\le 2(8\lambda_M^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}a^{-n+\frac{d_{\mathcal{L}}}{2}(2a-1)},\end{aligned}$$

and

$$\begin{aligned}J_2 &\le 2^{\frac{3}{4}d_{\mathcal{L}}(2a-1)}\int_A^{\infty}\frac{(\lambda_M t)^{\frac{d_{\mathcal{L}}}{2}(2a-1)}}{t^{n+1}}dt \le (8\lambda_M^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}\int_A^{\infty}\frac{dt}{t^{n+1-d_{\mathcal{L}}(2a-1)/2}}\\ &\le \frac{(8\lambda_M^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}}{n-d_{\mathcal{L}}(2a-1)/2}a^{-n+\frac{d_{\mathcal{L}}}{2}(2a-1)}.\end{aligned}$$

Substituting the above into (6), we obtain

$$|\gamma_n(\mathcal{L})| \le \frac{c_{\mathcal{L}}(a)}{\pi}Q^{2a-1}\left(\sum_{m=1}^{\infty}\frac{|f(m)|}{m^a}\right)(8\lambda_M^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}a^{-n+\frac{d_{\mathcal{L}}}{2}(2a-1)}\left(2+\frac{1}{n-\frac{d_{\mathcal{L}}(2a-1)}{2}}\right).$$

Therefore putting

$$C_{\mathcal{L}}(a) = \frac{2^r Q^{2a-1}}{\pi}\exp\left(\frac{1}{5}\sum_{j=1}^{r}\frac{1}{\lambda_j(a-1)-\mathrm{Re}(\mu_j)}\right)\left(\sum_{m=1}^{\infty}\frac{|f(m)|}{m^a}\right)(8\lambda_M^2 a^2)^{\frac{d_{\mathcal{L}}}{4}(2a-1)}$$

for $1+\frac{\mu_M+1}{\lambda_m} < a < \frac{1}{2}+\frac{n}{d_{\mathcal{L}}}$, we obtain

$$|\gamma_n(\mathcal{L})| \le C_{\mathcal{L}}(a)a^{-n}\left(2+\frac{1}{n-\frac{d_{\mathcal{L}}(2a-1)}{2}}\right),$$

which completes the proof of our Theorem.

$\square$

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