On the unitary and bi-unitary analogues of generalized Ramanujan sums

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Abstract

A positive integer d is called an unitary divisor of any positive integer n if $d|n$ and $(n, n/d) = 1$, notation $d||n$. Let $(k, n)_*$ be the greatest divisor of the integer k which is an unitary divisor of n. Let $(k, n)_{**}$ be the greatest common unitary divisor of k and n. We introduce the following two functions

$$t_k(j) = \sum_{d||(j,k)_*} f(d) g\left(\frac{k}{d}\right), \quad v_k(j) = \sum_{d||(j,k)_{**}} f(d) g\left(\frac{k}{d}\right),$$

for any arithmetical functions f and g. Here $d||(j, k)_*$ holds if and only if $d|j$ and $d||k$. In this paper, we give some asymptotic formulas for the weighted averages of $t_k(j)$ and $v_k(j)$ with weights concerning the monomial factor, the Gamma function, and the Bernoulli polynomials. We also derive useful formulas for the unitary and the bi-unitary analogues of the gcd-sum function.

1 Introduction and statements of the results

Let (k, n) be the greatest common divisor of the integers k and n. In 1885, Cesàro published an important result on the arithmetic function, showing that for any arithmetic function f, we have

$$P(n) := \sum_{k=1}^{n} f((k, n)) = \sum_{d|n} f(d) \phi\left(\frac{n}{d}\right). \tag{1}$$

In a special case when $f = \mathrm{id}$, one can write

$$\sum_{k=1}^{n} (k, n) = \sum_{d|n} d\phi\left(\frac{n}{d}\right) = (\mathrm{id} * \phi)(n). \tag{2}$$

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This latter sum is called by the gcd-sum function (sometime by Pallai's arithmetic function) and due to Pallai (1937). Of course, the symbol $*$ denotes the Dirichlet convolution of two arithmetical functions f and g defined by $(f*g)(n) = \sum_{d|n} f(d)g(n/d)$, for every positive integer n. Various generalizations and analogues of Eqs. (1) and (2) have been widely studied by many authors. For a nice survey see [5]. It is well known that the Ramanujan sum is defined by

$$c_k(j) = \sum_{d|(k,j)} d\mu\left(\frac{k}{d}\right),$$

for any positive integers k and j. One of the most common generalization of the Ramanujan sum is due to Anderson-Apostol [1] and defined by

$$s_k(j) := \sum_{d|(k,j)} f(d)g\left(\frac{k}{d}\right), \tag{3}$$

for any arithmetical functions f and g. In this paper, we introduce the unitary and bi-unitary analogues of the Anderson-Apostol sums. The next two subsections describe precisely our functions and results.

1.1 Unitary analogues

A positive integer d is called an unitary divisor of any positive integer n if $d|n$ and $(n, n/d) = 1$, notation $d||n$. Let $(k,n)_*$ be the greatest divisor of k which is an unitary divisor of n, namely

$$(k,n)_* = \max\{d \in \mathbb{N} \ : \ d|k, d||n\}.$$

In 1989, Tóth [4] introduced the unitary analogue of Eq. (2) as follows

$$P^*(n) = \sum_{k=1}^{n} (k,n)_*. \tag{4}$$

In that paper, he showed that the function $P^*(n)$ is multiplicative and that

$$P^*(n) = \sum_{d||n} d\phi^*\left(\frac{n}{d}\right), \tag{5}$$

where ϕ^* denotes the unitary analogue of Euler's function and defined by

$$\phi^*(n) = \#\{k \in \mathbb{N} \ : \ 1 \le k \le n, (k,n)_* = 1\}. \tag{6}$$

In other words, $\phi^*(n)$ is rewritten in the form

$$\phi^*(n) = \sum_{d|n} d\mu^*\left(\frac{n}{d}\right) = (\mathrm{id} * \mu^*)(n).$$

Here $\mu^*(n)$ the unitary analogue of the Möbius function given by $\mu^*(n) = (-1)^{\omega(n)}$ where $\omega(n)$ is the he number of distinct prime factors of n.

We introduce the function $t_k(j)$ defined by

$$t_k(j) = \sum_{d \| (j,k)_*} f(d) g\left(\frac{k}{d}\right), \tag{7}$$

such that $d \| (j,k)_*$ holds if and only if $d|j$ and $d \| k$. The function $t_k(j)$ is an unitary analogue of Eq.(3). From the above, we immediately deduce the identity

$$\sum_{j=1}^{k} t_k(j) = \sum_{d \| k} f(d) g\left(\frac{k}{d}\right) \sum_{\ell=1}^{k/d} 1 = (f \star g \cdot \mathrm{id})(k). \tag{8}$$

Here the symbol $\star$ denotes the unitary convolution of two arithmetical functions f and g defined by $(f \star g)(n) = \sum_{d \| n} f(d) g(n/d)$, for every positive integer n. Eq.(8) is a generalization of Eq. (5). Moreover, we have

$$\sum_{k \le x} \frac{1}{k} \sum_{j=1}^{k} t_k(j) = \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} g(\ell). \tag{9}$$

Since ϕ is a multiplicative function, one can also see that

$$\sum_{k \le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} t_k(j) = \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{\phi(d)} \frac{g(\ell)\ell}{\phi(\ell)}. \tag{10}$$

For any complex z we define the functions $B_n(x)$ by the generating function

$$\frac{z e^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n$$

for $|z| < 2\pi$. The functions $B_n(x)$ are known as Bernoulli polynomials, and the numbers $B_n(0)$ are called Bernoulli numbers and are denoted by B_n.

Our first goal of this paper is to derive some formulas of the weighted averages of $t_k(j)$ with weight function w concerning the monomial factor, the Gamma function $\Gamma(.)$, and the Bernoulli polynomials $B_n(.)$. This is

$$\sum_{k \le x} \frac{1}{W(k)} \sum_{j=1}^{k} w(j) t_k(j) \tag{11}$$

with certain weight function W. For any fixed positive integer r, we define the following function:

$$T_r(x; f, g) := \sum_{k \le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r t_k(j), \tag{12}$$

Then we prove that

Theorem 1. *For any positive real number $x \ge 2$ and any fixed positive integer r, we have*

$$T_r(x; f, g) = \frac{1}{2} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell} + \frac{1}{r+1} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} g(\ell)$$
$$+ \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell^{2m}} \tag{13}$$

In a special case of Theorem 1 when $f = f * \mu$ and $g = \mathbf{1}$, we get an useful formula for the unitary analogue of gcd-sum function, that is

Corollary 1. *Under the hypotheses of Theorem 1, we have*

$$T_r(x; f * \mu, \boldsymbol{1}) = \frac{1}{2} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{(f*\mu)(d)}{d} \frac{1}{\ell} + \frac{1}{r+1} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{(f*\mu)(d)}{d}$$
$$+ \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{(f*\mu)(d)}{d} \frac{1}{\ell^{2m}}. \tag{14}$$

Now, we define the following two functions

$$G_{f,g}(x) = \sum_{k \le x} \frac{1}{k} \sum_{j=1}^{k} \log \Gamma\left(\frac{j}{k}\right) t_k(j),$$

and

$$Y_{f,g}(x) = \sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) t_k(j),$$

for any fixed positive integer m. We also prove that

Theorem 2. *for any positive real number $x \ge 2$ and any fixed positive integer m, we have*

$$G_{f,g}(x) = \log\sqrt{2\pi} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} g(\ell) - \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell} \log\sqrt{2\pi\ell}, \tag{15}$$

and

$$Y_{f,g}(x) = B_m \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell^m} \tag{16}$$

By taking $f*\mu$ in place of f and $g=\mathbf{1}$ in the above, we get the following interesting formulas.

Corollary 2. *Under the hypotheses of Theorem 2, we have*

$$G_{f*\mu,\mathbf{1}}(x)=\log\sqrt{2\pi}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{(f*\mu)(d)}{d}-\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{(f*\mu)(d)}{d}\frac{\log\sqrt{2\pi\ell}}{\ell},$$

and

$$Y_{f*\mu,\mathbf{1}}(x)=B_m\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{(f*\mu)(d)}{d}\frac{1}{\ell^m}.$$

1.2 Bi-unitary analogues

Let $(k,n)_{**}$ be the greatest common unitary divisor of k and n, namely

$$(k,n)_{**}=\max\{d\in\mathbb{N}\;\; d||k,d||n\}.$$

In 2008, Haukkanen [3] defined a generalization of the gcd-sum function as follows:

$$P^{**}(n)=\sum_{k=1}^{n}(k,n)_{**}. \tag{17}$$

It is called by the bi-unitary gcd-sum function. Not surprisingly the study of $P^{**}(n)$ has a lot of similarities with that of $P^{*}(n)$ or even $P(n)$. Haukkanen also showed that

$$P^{**}(n)=\sum_{d||n}\phi^*(d)\phi\left(\frac{n}{d},d\right), \tag{18}$$

where $\phi(x,d)$ is the Legendre function. More recently, Tóth [6] gave an asymptotic formula for the partial sum of $P^{**}(n)$. He proved that

$$\sum_{n\le x}P^{**}(n)=\frac{1}{2}Bx^2\log x+O\left(x^2\right),$$

where

$$B=\prod_p\left(1-\frac{3p-1}{p^2(p+1)}\right)=\zeta(2)\prod_p\left(1-\frac{(2p-1)^2}{p^4}\right).$$

Using the above and the partial summation, he deduced that

$$\sum_{n\le x}\frac{P^{**}(n)}{n}=Bx\log x+O\left(x\right). \tag{19}$$

By similar considerations, we introduce the bi-unitary analogue of Anderson-Apostol sums $v_k(j)$ defined by

$$v_k(j) = \sum_{d\|(j,k)_{**}} f(d) g\left(\frac{k}{d}\right). \tag{20}$$

With some careful calculations, one can check that

$$\begin{aligned}
\sum_{j=1}^{k} v_k(j) &= \sum_{d\|k} f(d) g\left(\frac{k}{d}\right) \sum_{\delta|d} \mu(\delta) \sum_{\ell=1}^{k/(d\delta)} 1 && (21)\\
&= (f \cdot \phi \cdot \mathrm{id}_{-1} \star g \cdot \mathrm{id})(k). && (22)
\end{aligned}$$

Then, we get

$$\sum_{k\le x} \frac{1}{k} \sum_{j=1}^{k} v_k(j) = \sum_{\substack{d\ell\le x\\(d,\ell)=1}} \frac{f(d)}{d} \frac{\phi(d)}{d} g(\ell), \tag{23}$$

$$\sum_{k\le x} \frac{1}{\phi(k)} \sum_{j=1}^{k} v_k(j) = \sum_{\substack{d\ell\le x\\(d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)\ell}{\phi(\ell)}. \tag{24}$$

For any positive integer r, we consider the function $V_r(x; f, g)$

$$V_r(x; f, g) := \sum_{k\le x} \frac{1}{k^{r+1}} \sum_{j=1}^{k} j^r v_k(j) \tag{25}$$

to get the following result.

Theorem 3. *For any positive real number $x > 1$ and any fixed positive integer r, we have*

$$\begin{aligned}
V_r(x; f, g) = \frac{f(1)}{2} \sum_{k\le x} \frac{g(k)}{k} &+ \frac{1}{r+1} \sum_{\substack{d\ell\le x\\(d,\ell)=1}} \frac{f(d)}{d} \frac{\phi(d)}{d} g(\ell)\\
&+ \frac{1}{r+1} \sum_{m=1}^{[r/2]} \binom{r+1}{2m} B_{2m} \sum_{\substack{d\ell\le x\\(d,\ell)=1}} \frac{f(d)}{d} \frac{\phi_{1-2m}(d)}{d^{1-2m}} \frac{g(\ell)}{\ell^{2m}} \qquad (26)
\end{aligned}$$

As a consequence of Theorem 3, we deduce the following bi-unitary analogue of the gcd-sum function by replacing f by $f * \mu$ and $g = \mathbf{1}$.

Corollary 3. *Under the hypotheses of Theorem 3. For any arithmetical function f, we have*

$$V_r(x; f*\mu, \boldsymbol{1}) = \frac{(f*\mu)(1)}{2}\sum_{k\le x}\frac{1}{k} + \frac{1}{r+1}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{(f*\mu)(d)}{d}\frac{\phi(d)}{d}$$
$$+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{\substack{d\ell\le x\\ \gcd(d,\ell)=1}}\frac{(f*\mu)(d)}{d}\frac{\phi_{1-2m}(d)}{d^{1-2m}}\frac{1}{\ell^{2m}} \quad (27)$$

Our second goal of this paper is to give asymptotic formulas of the weighted averages of $v_k(j)$ with weights concerning the Gamma function and the Bernoulli polynomials. Put

$$\widetilde{G}_{f,g}(x) = \sum_{k\le x}\frac{1}{k}\sum_{j=1}^{k}\log\Gamma\left(\frac{j}{k}\right)v_k(j),$$

and

$$\widetilde{Y}_{f,g}(x) = \sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_m\left(\frac{j}{k}\right)v_k(j).$$

We prove that:

Theorem 4. *Let the notation be as above, we have*

$$\widetilde{G}_{f,g}(x) = \log\sqrt{2\pi}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)\phi(d)}{d^2}g(\ell)$$
$$- f(1)\sum_{k\le x}\frac{g(k)}{k}\log\sqrt{2\pi k} - \frac{1}{2}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)\Lambda(d)}{d}\frac{g(\ell)}{\ell} \quad (28)$$

where Λ is the von Mangoldt function. Furthermore, we have

$$\widetilde{Y}_{f,g}(x) = B_m\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{\phi_{1-m}(d)}{d^{1-m}}\frac{g(\ell)}{\ell^m} \quad (29)$$

As an application of Theorem 4, we take $f*\mu$ in place of f and $g = \mathbf{1}$ into the above to get the interesting formulas

Corollary 4. *Let the notation be as above, we have*

$$\begin{aligned}
\widetilde{G}_{f*\mu,\mathbf{1}}(x) &:= \sum_{k\le x}\frac{1}{k}\sum_{j=1}^{k}\log\Gamma\left(\frac{j}{k}\right)\sum_{d\|(j,k)_{**}}(f*\mu)(d)\\
&= \log\sqrt{2\pi}\sum_{\substack{d\ell\le x\\ \gcd(d,\ell)=1}}\frac{(f*\mu)(d)\phi(d)}{d^2} - f(1)\sum_{k\le x}\frac{\log\sqrt{2\pi k}}{k}\\
&\quad -\frac{1}{2}\sum_{\substack{d\ell\le x\\ \gcd(d,\ell)=1}}\frac{(f*\mu)(d)\Lambda(d)}{d}\frac{1}{\ell},
\end{aligned}\tag{30}$$

and

$$\begin{aligned}
\widetilde{Y}_{f*\mu,\mathbf{1}}(x) &:= \sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1}B_m\left(\frac{j}{k}\right)\sum_{d\|(j,k)_{**}}(f*\mu)(d)\\
&= B_m\sum_{\substack{d\ell\le x\\ \gcd(d,\ell)=1}}\frac{(f*\mu)(d)}{d}\frac{\phi_{1-m}(d)}{d^{1-m}}\frac{1}{\ell^m}.
\end{aligned}\tag{31}$$

2 Proof of Theorems 1 and 3

2.1 Proof of Theorem 1

By the definition of $t_k(j)$, we have

$$\begin{aligned}
\sum_{j=1}^{k}\left(\frac{j}{k}\right)^r t_k(j) &= \sum_{j=1}^{k}\left(\frac{j}{k}\right)^r\sum_{\substack{d||k\\ d|j}}f(d)g\left(\frac{k}{d}\right)\\
&= \frac{1}{k^r}\sum_{d||k}f(d)g\left(\frac{k}{d}\right)\sum_{\substack{j=1\\ d|j}}^{k}j^r\\
&= \frac{1}{k^r}\sum_{d||k}d^r f(d)g\left(\frac{k}{d}\right)\sum_{\ell=1}^{k/d}\ell^r
\end{aligned}$$

We use the well known identity, see [2, Proposition 9.2.12],

$$\sum_{\ell=1}^{N}\ell^r = \frac{N^r}{2}+\frac{1}{r+1}\sum_{m=0}^{[r/2]}\binom{r+1}{2m}B_{2m}N^{r+1-2m}\tag{32}$$

for any positive integer $N>1$, to obtain

$$\sum_{j=1}^{k}\left(\frac{j}{k}\right)^r t_k(j) = \frac{1}{2}\sum_{d||k}f(d)g\left(\frac{k}{d}\right)+\frac{1}{r+1}\sum_{m=0}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d||k}f(d)g\left(\frac{k}{d}\right)\left(\frac{k}{d}\right)^{1-2m}$$

Using the fact that $d||k$ is $d|k$ and $(d, k/d) = 1$, we conclude that

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r t_k(j) = \frac{1}{2}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{g(\ell)}{\ell} + \frac{1}{r+1}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}g(\ell)$$
$$+\frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{g(\ell)}{\ell^{2m}}.$$

This completes the proof of Theorem 1.

2.2 Proof of Theorem 3

By the definition of $v_k(j)$, we have

$$\begin{aligned}
\sum_{j=1}^{k}\left(\frac{j}{k}\right)^r v_k(j) &= \sum_{j=1}^{k}\left(\frac{j}{k}\right)^r \sum_{\substack{d||k\\d||j}} f(d)g\left(\frac{k}{d}\right)\\
&= \frac{1}{k^r}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\substack{j=1\\d||j}}^{k} j^r\\
&= \frac{1}{k^r}\sum_{d||k} d^r f(d)g\left(\frac{k}{d}\right)\sum_{\substack{\ell=1\\(d,\ell)=1}}^{k/d}\ell^r\\
&= \frac{1}{k^r}\sum_{d||k} d^r f(d)g\left(\frac{k}{d}\right)\sum_{\ell=1}^{k/d}\ell^r\sum_{\substack{q|d\\q|\ell}}\mu(q).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{j=1}^{k}\left(\frac{j}{k}\right)^r v_k(j) &= \frac{1}{k^r}\sum_{d||k} d^r f(d)g\left(\frac{k}{d}\right)\sum_{q|d}\mu(q)\sum_{\substack{\ell=1\\q|\ell}}^{k/d}\ell^r\\
&= \frac{1}{k^r}\sum_{d||k} d^r f(d)g\left(\frac{k}{d}\right)\sum_{q|d}\mu(q)q^r\sum_{q_1=1}^{k/(dq)} q_1^r
\end{aligned}$$

Applying Eq. (32) to the inner sum on the right-hand side above, we find that

$$\sum_{j=1}^{k}\left(\frac{j}{k}\right)^{r} v_k(j) = \frac{1}{2}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{q|d}\mu(q) + \frac{1}{r+1}\sum_{m=0}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\left(\frac{k}{d}\right)^{1-2m}\sum_{q|d}\frac{\mu(q)}{q^{1-2m}}$$

Now, we notice that $\sum_{q|d}\mu(q) = 1$ where $d=1$. Otherwise $\sum_{q|d}\mu(q) = 0$. Moreover, we have

$$\sum_{q|d}\frac{\mu(q)}{q^{1-2m}} = \frac{\phi_{1-2m}(d)}{d^{1-2m}}.$$

Thus, we get

$$\sum_{j=1}^{k}\left(\frac{j}{k}\right)^{r} v_k(j) = \frac{1}{2}f(1)g(k) + \frac{1}{r+1}\sum_{m=0}^{[r/2]}\binom{r+1}{2m}B_{2m}\left(f\cdot\phi_{1-2m}\cdot \mathrm{id}_{2m-1}\star g\cdot \mathrm{id}_{1-2m}\right)(k)$$

Therefore, we obtain that

$$\sum_{k\le x}\frac{1}{k^{r+1}}\sum_{j=1}^{k} j^r v_k(j) = \frac{f(1)}{2}\sum_{k\le x}\frac{g(k)}{k} + \frac{1}{r+1}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{\phi(d)}{d}g(\ell) + \frac{1}{r+1}\sum_{m=1}^{[r/2]}\binom{r+1}{2m}B_{2m}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{\phi_{1-2m}(d)}{d^{1-2m}}\frac{g(\ell)}{\ell^{2m}}.$$

This completes the proof.

3 Proof of Theorems 2 and 4

3.1 Proof of Theorem 2

Notice that

$$\sum_{j=1}^{k} t_k(j)\log\Gamma\left(\frac{j}{k}\right) = \sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\ell=1}^{k/d}\log\Gamma\left(\frac{d\ell}{k}\right)$$

Using the multiplication formula of Gauss–Legendre for the gamma function, see [2, Proposition 9.6.33]

$$\prod_{j=1}^{n}\Gamma\left(\frac{j}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}, \tag{33}$$

we get

$$\sum_{j=1}^{k} t_k(j) \log \Gamma\left(\frac{j}{k}\right) = \frac{\log(2\pi)}{2} \sum_{d||k} f(d) g\left(\frac{k}{d}\right)\left(\frac{k}{d}\right) - \frac{\log(2\pi)}{2} \sum_{d||k} f(d) g\left(\frac{k}{d}\right) - \frac{1}{2} \sum_{d||k} f(d) g\left(\frac{k}{d}\right) \log\left(\frac{k}{d}\right).$$

This leads to

$$\sum_{k \le x} \frac{1}{k} \sum_{j=1}^{k} t_k(j) \log \Gamma\left(\frac{j}{k}\right) = \frac{\log(2\pi)}{2} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} g(\ell) - \frac{\log(2\pi)}{2} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell} - \frac{1}{2} \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell) \log \ell}{\ell}.$$

This completes the proof of Eq. (15). Now, we have

$$\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) t_k(j) = \sum_{d||k} f(d) g\left(\frac{k}{d}\right) \sum_{\ell=0}^{\frac{k}{d}-1} B_m\left(\frac{\ell}{k/d}\right).$$

Using the following property of Bernoulli polynomial, see [2, Proposition 9.1.3]

$$\sum_{j=0}^{n-1} B_m\left(\frac{j}{n}\right) = \frac{B_m}{n^{m-1}} \tag{34}$$

for any fixed positive integer m, we get

$$\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) t_k(j) = \frac{B_m}{k^{m-1}} \sum_{d||k} d^{m-1} f(d) g\left(\frac{k}{d}\right).$$

Therefore, we deduce that

$$\sum_{k \le x} \frac{1}{k} \sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right) t_k(j) = B_m \sum_{\substack{d\ell \le x \\ (d,\ell)=1}} \frac{f(d)}{d} \frac{g(\ell)}{\ell^m}.$$

The formula (16) is proved.

3.2 Proof of Theorem 4

Notice that

$$\begin{aligned}\sum_{j=1}^{k} v_k(j)\log\Gamma\left(\frac{j}{k}\right) &= \sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\substack{\ell=1\\(\ell,d)=1}}^{k/d}\log\Gamma\left(\frac{d\ell}{k}\right)\\ &= \sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\sum_{q=1}^{k/(d\delta)}\log\Gamma\left(\frac{qd\delta}{k}\right)\end{aligned}$$

Using Eq. (33) and $-\Lambda = \mu\cdot\log * \mathbf{1}$, we get

$$\begin{aligned}&\sum_{j=1}^{k} v_k(j)\log\Gamma\left(\frac{j}{k}\right)\\ =\;& \log\sqrt{2\pi}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\left(\frac{k}{d\delta}-1\right) - \frac{1}{2}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\log\left(\frac{k}{d\delta}\right)\\ =\;& \log\sqrt{2\pi}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\frac{k}{d}\sum_{\delta|d}\frac{\mu(\delta)}{\delta} - \log\sqrt{2\pi}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\\ &-\frac{1}{2}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\log\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta) + \frac{1}{2}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\log\delta\\ =\;& \log\sqrt{2\pi}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\frac{k}{d}\frac{\phi(d)}{d} - \log\sqrt{2\pi k}f(1)g(k) - \frac{1}{2}\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\Lambda(d).\end{aligned}$$

Thus, we find that

$$\begin{aligned}\sum_{k\le x}\frac{1}{k}\sum_{j=1}^{k} v_k(j)\log\Gamma\left(\frac{j}{k}\right) &= \log\sqrt{2\pi}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)\phi(d)}{d^2}g(\ell)\\ &\quad - f(1)\sum_{k\le x}\frac{g(k)}{k}\log\sqrt{2\pi k} - \frac{1}{2}\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)\Lambda(d)}{d}\frac{g(\ell)}{\ell}.\end{aligned}$$

This completes the proof of Eq. (28). For Eq. (29), we note that

$$\begin{aligned}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)v_k(j) &= \sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\substack{\ell=0\\(\ell,d)=1}}^{\frac{k}{d}-1} B_m\left(\frac{d\ell}{k}\right)\\ &= \sum_{d||k} f(d)g\left(\frac{k}{d}\right)\sum_{\delta|d}\mu(\delta)\sum_{j=0}^{\frac{k}{d\delta}-1} B_m\left(\frac{d\delta j}{k}\right)\\ &= B_m\sum_{d||k} f(d)g\left(\frac{k}{d}\right)\left(\frac{k}{d}\right)^{1-m}\frac{\phi_{1-m}(d)}{d^{1-m}}.\end{aligned}$$

We therefore conclude that

$$\sum_{k\le x}\frac{1}{k}\sum_{j=0}^{k-1} B_m\left(\frac{j}{k}\right)v_k(j) = B_m\sum_{\substack{d\ell\le x\\(d,\ell)=1}}\frac{f(d)}{d}\frac{\phi_{1-m}(d)}{d^{1-m}}\frac{g(\ell)}{\ell},$$

which gives the desired result.

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