Uniform approximations by Fourier sums on classes of generalized Poisson integrals

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Abstract We find asymptotic equalities for exact upper bounds of approximations by Fourier sums in uniform metric on classes of 2π–periodic functions, representable in the form of convolutions of functions φ, which belong to unit balls of spaces L_p, with generalized Poisson kernels. For obtained asymptotic equalities we introduce the estimates of remainder, which are expressed in the explicit form via the parameters of the problem.

1 Introduction

Let L_p, $1 \leq p < \infty$, be the space of 2π–periodic functions f summable to the power p on $[0, 2\pi)$, in which the norm is given by the formula $\|f\|_p = \Big(\int\limits_0^{2\pi} |f(t)|^p dt\Big)^{\frac{1}{p}}$; L_∞ be the space of measurable and essentially bounded 2π–periodic functions f with the norm $\|f\|_\infty = \operatorname*{ess\,sup}_t |f(t)|$; C be the space of continuous 2π–periodic functions f, in which the norm is specified by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Denote by $C^{\alpha,r}_{\beta,p}$, $\alpha > 0$, $r > 0$, $1 \leq p \leq \infty$, the set of all 2π–periodic functions, representable for all $x \in \mathbb{R}$ as convolutions of the form (see, e.g., [1, p. 133])

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \ \varphi \in B^0_p, \tag{1}$$

$$B^0_p = \left\{\varphi : \ ||\varphi||_p \leq 1, \ \varphi \perp 1\right\}, \ 1 \leq p \leq \infty,$$

with fixed generated kernels

$$P_{\alpha,r,\beta}(t) = \sum_{k=1}^{\infty} e^{-\alpha k^r} \cos\big(kt - \frac{\beta\pi}{2}\big), \quad \beta \in \mathbb{R}.$$

The kernels $P_{\alpha,r,\beta}(t)$ are called generalized Poisson kernels. For $r=1$ and $\beta=0$ the kernels $P_{\alpha,r,\beta}(t)$ are usual Poisson kernels of harmonic functions.

For any $r>0$ the classes $C^{\alpha,r}_{\beta,p}$ belong to set of infinitely differentiable 2π–periodic functions D^∞, i.e., $C^{\alpha,r}_{\beta,p}\subset D^\infty$ (see, e.g., [1, p. 128], [2]). For $r\geq 1$ the classes $C^{\alpha,r}_{\beta,p}$ consist of functions f, admitting a regular extension into the strip $|\mathrm{Im}\ z|\leq c,\ c>0$ in the complex plane (see, e.g., [1, p. 141]), i.e., are the classes of analytic functions. For $r>1$ the classes $C^{\alpha,r}_{\beta,p}$ consist of functions regular on the whole complex plane, i.e., of entire functions (see, e.g., [1, p. 131]). Besides, it follows from the Theorem 1 in [3] that for any $r>0$ the embedding holds $C^{\alpha,r}_{\beta,p}\subset \mathcal{J}_{1/r}$, where $\mathcal{J}_a, a>0$, are known Gevrey classes

$$\mathcal{J}_a=\Big\{f\in D^\infty:\ \sup_{k\in\mathbb{N}}\Big(\frac{\|f^{(k)}\|_C}{(k!)^a}\Big)^{1/k}<\infty\Big\}.$$

Approximation properties of classes of generalized Poisson integrals $C^{\alpha,r}_{\beta,p}$ in metrics of spaces L_s, $1\leq s\leq\infty$, were considered in [4]–[10] from the viewpoint of order or asymptotic estimates for approximations by Fourier sums, best approximations and widths.

In the present paper we obtain asymptotic equalities as $n\to\infty$ for the quantities

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C=\sup_{f\in C^{\alpha,r}_{\beta,p}}\|f(\cdot)-S_{n-1}(f;\cdot)\|_C,\ r>0,\ \alpha>0,\ 1\leq p\leq\infty, \tag{2}$$

where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of order $n-1$ for a function f.

Approximation by Fourier sums on other classes of differentiable functions in uniform metric were investigated in works [1], [11]–[15].

Nikol'skii [12, p. 221] considered the case $r=1$, $p=\infty$ and established that following asymptotic equality is true

$$\mathcal{E}_n(C^{\alpha,1}_{\beta,\infty})_C=e^{-\alpha n}\Big(\frac{8}{\pi^2}\mathbf{K}(e^{-\alpha})+O(1)n^{-1}\Big), \tag{3}$$

where

$$\mathbf{K}(q):=\int\limits_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-q^2\sin^2 t}},\ q\in(0,1),$$

is a complete elliptic integral of the first kind, and $O(1)$ is a quantity uniformly bounded in parameters n and β.

Later, the equality (3) was clarified by Stechkin [16, p. 139], who established the asymptotic formula

$$\mathcal{E}_n(C^{\alpha,1}_{\beta,\infty})_C = e^{-\alpha n}\Big(\frac{8}{\pi^2}\mathbf{K}(e^{-\alpha}) + O(1)\frac{e^{-\alpha}}{(1-e^{-\alpha})n}\Big), \ \ \alpha>0, \ \beta\in\mathbb{R}, \tag{4}$$

where $O(1)$ is a quantity uniformly bounded in all analyzed parameters.

In work [8] for $r=1$ and arbitrary values of $1\le p\le\infty$ for quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha>0$, $\beta\in\mathbb{R}$, the following equality was established

$$\mathcal{E}_n(C^{\alpha,1}_{\beta,p})_C = e^{-\alpha n}\left(\frac{2}{\pi^{1+\frac{1}{p'}}}\|\cos t\|_{p'}K(p',e^{-\alpha}) + O(1)\frac{e^{-\alpha}}{n(1-e^{-\alpha})^{s(p)}}\right), \tag{5}$$

where $p'=\frac{p}{p-1}$,

$$s(p) := \begin{cases} 1, & p=\infty, \\ 2, & p\in[1,2)\cup(2,\infty), \\ -\infty, & p=2, \end{cases}$$

$$K(p',q) := \frac{1}{2^{1+\frac{1}{p'}}}\left\|(1-2q\cos t+q^2)^{-\frac{1}{2}}\right\|_{p'}, \ \ q\in(0,1),$$

and $O(1)$ is a quantity uniformly bounded in n, p, α and β. For $p=\infty$, by virtue of the known equality $K(1,q)=\mathbf{K}(q)$ (see, e.g., formula 3.674(1) of [17, p. 401]), the estimate (5) coincides with the estimate (4).

In [23] it was proved that in the case $1\le p'<\infty$ the following equality takes place

$$K(p',q) = \frac{\pi^{\frac{1}{p'}}}{2}F^{\frac{1}{p'}}\Big(\frac{p'}{2},\frac{p'}{2};1;q^2\Big), \ \ q\in(0,1),$$

where $F(a,b;c;d)$ is Gauss hypergeometric function

$$F(a,b;c;z) = 1+\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

$$(x)_k := \frac{x}{2}\Big(\frac{x}{2}+1\Big)\Big(\frac{x}{2}+2\Big)...\Big(\frac{x}{2}+k-1\Big).$$

Note that for $p=2$ and $r=1$ formula (5) becomes the equality

$$\mathcal{E}_n(C^{\alpha,1}_{\beta,2})_C = \frac{1}{\sqrt{\pi(1-e^{-2\alpha})}}e^{-\alpha n}, \ \ \alpha>0, \ \beta\in\mathbb{R}, \ n\in\mathbb{N},$$

(see [8]). Moreover, it follows from [18] that for $p=2$ and $r>0$ for the quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$ the equalities take place

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,2})_C = \frac{1}{\sqrt{\pi}}\Big(\sum_{k=n}^{\infty}e^{-2\alpha k^r}\Big)^{\frac{1}{2}}, \ \ \alpha>0, \ \beta\in\mathbb{R}, \ n\in\mathbb{N}. \tag{6}$$

In the case of $r > 1$ and $p = \infty$ the asymptotic equalities for the quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha > 0$, $\beta \in \mathbb{R}$, were obtained by Stepanets [19, Chapter 3, Section 9], who showed that for any $n \in \mathbb{N}$

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = \Big(\frac{4}{\pi} + \gamma_n\Big)e^{-\alpha n^r}, \tag{7}$$

where

$$|\gamma_n| < 2\Big(1 + \frac{1}{\alpha r n^{r-1}}\Big)e^{-\alpha r n^{r-1}}.$$

Later Telyakovskii [4] established the asymptotic equality

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = \frac{4}{\pi}e^{-\alpha n^r} + O(1)\Big(e^{-\alpha(2(n+1)^r - n^r)} + \Big(1 + \frac{1}{\alpha r(n+2)^r}\Big)e^{-\alpha(n+2)^r}\Big), \tag{8}$$

where $O(1)$ is a quantity uniformly bounded in all analyzed parameters. Formula (8) contains more exact estimate of remainder in asymptotic decomposition of the quantity $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$ comparing with the estimate (7).

For $r > 1$ and for arbitrary values of $1 \le p \le \infty$ the asymptotic equalities for the quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha > 0$, $\beta \in \mathbb{R}$, are found in [8] and have the form

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r}\Big(\frac{\|\cos t\|_{p'}}{\pi} + O(1)\Big(1 + \frac{1}{\alpha r n^{r-1}}\Big)e^{-\alpha n^{r-1}}\Big), \tag{9}$$

where $O(1)$ is a quantity uniformly bounded in all analyzed parameters. For $p = \infty$ the formula (9) follows from (7) and (8).

Concerning the case $0 < r < 1$, except the presented above case $p = 2$, asymptotic equalities for quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha > 0$, $\beta \in \mathbb{R}$, were known only for $p = \infty$ due to the work of Stepanets [20], who showed that

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = \frac{4}{\pi^2}e^{-\alpha n^r}\ln n^{1-r} + O(1)e^{-\alpha n^r}, \tag{10}$$

where $O(1)$ is a quantity uniformly bounded in n and β.

In case of $0 < r < 1$ and $1 \le p < \infty$ the following order estimates for quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha > 0$, $\beta \in \mathbb{R}$, hold (see, e.g., [6], [9])

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \asymp e^{-\alpha n^r} n^{\frac{1-r}{p}}. \tag{11}$$

We remark that for $0 < r < 1$ and $1 \le p < \infty$ Fourier sums provide the order of best approximations of classes $C^{\alpha,r}_{\beta,p}$, $\alpha > 0$, $\beta \in \mathbb{R}$, in uniform metric, i.e. (see, e.g., [9], [10])

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \asymp E_n(C^{\alpha,r}_{\beta,p})_C \asymp e^{-\alpha n^r} n^{\frac{1-r}{p}},$$

where

$$E_n(C^{\alpha,r}_{\beta,p})_C = \sup_{f\in C^{\alpha,r}_{\beta,p}} \inf_{t_{n-1}\in\mathcal{T}_{2n-1}} \|f - t_{n-1}\|_C,$$

and $\mathcal{T}_{2n-1}$ is the subspace of all trigonometric polynomials t_{n-1} of degree not higher than $n-1$.

On the one hand, this fact encourages to research more deeply approximative properties of Fourier sums in given situations, and on the other hand it separates the case $1 \le p < \infty$ from considered earlier case $p = \infty$, where order equality $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C \asymp E_n(C^{\alpha,r}_{\beta,p})_C$, $0 < r < 1$, doesn't take place.

Besides, as follows from Temlyakov's work [6] for $2 \le p < \infty$ quantities of approximations by Fourier sums realize order of the linear widths λ_{2n} (definition of λ_m see, e.g., [21, Chapter 1, Section 1.2]) of the classes $C^{\alpha,r}_{0,p}$, i.e.

$$\lambda_{2n}(C^{\alpha,r}_{0,p}, C) \asymp \mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C.$$

In this paper we establish asymptotically sharp estimates of the quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$, $\alpha > 0$, $\beta \in \mathbb{R}$, for any $0 < r < 1$ and $1 \le p \le \infty$. In particular, it is proved, that for $r \in (0,1)$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $1 \le p \le \infty$ as $n \to \infty$ the following asymptotic equalities take place

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+ \frac{O(1)}{n^{\min\{\frac{1-r}{p}, r\}}} \Bigg), \quad 1 < p \le \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \tag{12}$$

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = e^{-\alpha n^r} n^{1-r} \bigg(\frac{1}{\pi\alpha r} + \frac{O(1)}{n^{\min\{1-r,\ r\}}} \bigg), \tag{13}$$

where $O(1)$ is a quantity uniformly bounded with respect to n and β.

Formulas (12) and (13) together with formulas (3)–(5), (7)–(10) give the solution of Kolmogoroff–Nikolsky problem about strong asymptotic of quantities (2) as $n \to \infty$ for all admissible values of parameters of problem.

Herewith, in this paper we found the estimates for remainders in (12) and (13), which are expressed via absolute values and the parameters of the problem α, r, p in the explicit form. These estimates can be used for practical application, since they allow effectively to estimate errors of uniform approximations of functions from the classes $C^{\alpha,r}_{\beta,p}$ by their particular Fourier sums.

The proof of aasymptotic formulas (12) and (13) is based on the one result (Lemma 1), which is some generalized modification of known Fejér Lemma (see [22]). Proof of this statement is located in the last part of the paper.

The following table contains exact values of constants (Kolmogoroff–Nikolsky constants) in main term A_n of asymptotic expansion of quantities $\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C$ of the form

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r}(A_n + o(A_n)).$$

A_n		r: $\mathbf{(0,1)}$	r: $\mathbf{1}$	r: $\mathbf{(1,\infty)}$
$\boldsymbol{p}$	$\boldsymbol{\infty}$	[20] Stepanets (1984) $\frac{4}{\pi^2}(1-r)\ln n$	[12] Nikolsky (1946) [16] Stechkin (1980) $\frac{8}{\pi^2}\mathbf{K}(e^{-\alpha})$	[19] Stepanets (1987) [4] Telyakovskii (1989) $\frac{4}{\pi}$
$\boldsymbol{p}$	$\mathbf{(1,\infty)}$	Our result $\frac{\|\cos t\|_{p'} F^{\frac{1}{p'}}(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1)}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}} n^{\frac{1-r}{p}}$	[8] Serdyuk (2005) $\frac{\|\cos t\|_{p'}}{\pi} F^{\frac{1}{p'}}(\frac{p'}{2},\frac{p'}{2};1;e^{-2\alpha})$	[8] Serdyuk (2005) $\frac{\|\cos t\|_{p'}}{\pi}$
$\boldsymbol{p}$	$\mathbf{1}$	Our result $\frac{1}{\pi\alpha r} n^{1-r}$	[8] Serdyuk (2005) $\frac{1}{\pi(1-e^{-\alpha})}$	[8] Serdyuk (2005) $\frac{1}{\pi}$

2 Formulation of main results and auxiliary statements

For arbitrary $\upsilon > 0$ and $1 \le s \le \infty$ assume

$$J_s(\upsilon) := \Big\| \frac{1}{\sqrt{t^2+1}} \Big\|_{L_s[0,\upsilon]}, \tag{14}$$

where

$$\|f\|_{L_s[a,b]} = \begin{cases} \left(\int\limits_a^b |f(t)|^s dt\right)^{\frac{1}{s}}, & 1 \le s < \infty, \\ \operatorname*{ess\,sup}\limits_{t\in[a,b]} |f(t)|, & s = \infty. \end{cases}$$

Also for $\alpha > 0$, $r \in (0,1)$ and $1 \le p \le \infty$ we denote by $n_0 = n_0(\alpha, r, p)$ the smallest integer n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \le \begin{cases} \frac{1}{14}, & p = 1, \\ \frac{1}{(3\pi)^3} \cdot \frac{p-1}{p}, & 1 < p < \infty, \\ \frac{1}{(3\pi)^3}, & p = \infty, \end{cases} \tag{15}$$

where $\chi(p) = p$ for $1 \le p < \infty$ and $\chi(p) = 1$ for $p = \infty$.

With the notations introduced above, the main result of this paper is formulated in the following statement:

Theorem 1. *Let $0 < r < 1$, $1 \le p \le \infty$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then for $n \ge n_0(\alpha, r, p)$ the following estimate is true*

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) +$$

$$+\gamma^{(1)}_{n,p}\Big(\frac{1}{(\alpha r)^{1+\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg), \tag{16}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and the quantity $\gamma^{(1)}_{n,p} = \gamma^{(1)}_{n,p}(\alpha, r, \beta)$ is such that $|\gamma^{(1)}_{n,p}| \le (14\pi)^2$.

Now we present some corollaries of Theorem 1.

Theorem 2. *Let $0 < r < 1$, $1 \le p < \infty$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then for $1 < p < \infty$ and $n \ge n_0(\alpha, r, p)$ the following estimate is true*

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}} (\alpha r)^{\frac{1}{p}}} F^{\frac{1}{p'}}\Big(\frac{1}{2}, \frac{3-p'}{2}; \frac{3}{2}; 1\Big) +$$

$$+\gamma^{(2)}_{n,p}\Bigg(\Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}} + \frac{(p)^{\frac{1}{p'}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r}\Bigg)\Bigg), \tag{17}$$

and for $p = 1$ and $n \ge n_0(\alpha, r, 1)$ the estimate is true

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,1})_C = e^{-\alpha n^r} n^{1-r} \Bigg(\frac{1}{\pi \alpha r} + \gamma^{(2)}_{n,1}\Big(\frac{1}{(\alpha r)^2}\frac{1}{n^r} + \frac{1}{n^{1-r}}\Big)\Bigg), \tag{18}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, *and the quantity* $\gamma_{n,p}^{(2)}=\gamma_{n,p}^{(2)}(\alpha,r,\beta)$ *is such that* $|\gamma_{n,p}^{(2)}|\le(14\pi)^2$.

The following statement follows from the Theorem 2 in the case $p=2$.

Corollary 1. *Let* $0<r<1$, $\alpha>0$ *and* $\beta\in\mathbb{R}$. *Then for* $n\ge n_0(\alpha,r,2)$ *the following estimate is true*

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C=e^{-\alpha n^r}n^{\frac{1-r}{2}}\Bigg(\frac{1}{\sqrt{2\pi\alpha r}}+\gamma_{n,2}^{(2)}\Big((1+\sqrt{\alpha r})\frac{1}{n^{\frac{1-r}{2}}}+\frac{\sqrt{2}}{(\alpha r)^{\frac{3}{2}}}\frac{1}{n^r}\Big)\Bigg),\tag{19}$$

where the quantity $\gamma_{n,2}^{(2)}=\gamma_{n,2}^{(2)}(\alpha,r,\beta)$ *is such that* $|\gamma_{n,2}^{(2)}|\le(14\pi)^2$.

Proof of the Corollary 1. Indeed, setting $p=p'=2$ in the equality (17), we obtain for $n\ge n_0(\alpha,r,2)$

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C=e^{-\alpha n^r}n^{\frac{1-r}{2}}\Bigg(\frac{\|\cos t\|_2}{\pi^{\frac{3}{2}}(\alpha r)^{\frac{1}{2}}}F^{\frac{1}{2}}\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};1\Big)+$$

$$+\gamma_{n,2}^{(2)}\Big((1+\sqrt{\alpha r})\frac{1}{n^{\frac{1-r}{2}}}+\frac{\sqrt{2}}{(\alpha r)^{\frac{3}{2}}}\frac{1}{n^r}\Big)\Bigg).\tag{20}$$

Taking into account that (see, e.g., formula 9.121(13) of [17]) $F\Big(\frac{1}{2},\frac{1}{2};\frac{3}{2};1\Big)=\frac{\pi}{2}$, from (20) we have (19). Corollary 1 is proved. □

However, it is possible to obtain more accurate estimate than (19) on the basis of equality (6). Namely, for $\alpha>0$, $r\in(0,1)$, $\beta\in\mathbb{R}$ and $n\ge n_0(\alpha,r,2)$ the following estimate is true

$$\mathcal{E}_n(C_{\beta,2}^{\alpha,r})_C=\frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}}n^{\frac{1-r}{2}}\Bigg(1+\gamma_n^{(2)}\Big(\frac{1}{2\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big)\Bigg),\tag{21}$$

where the quantity $\gamma_n^{(2)}=\gamma_n^{(2)}(\alpha,r)$ is such that $|\gamma_n^{(2)}|\le\sqrt{\frac{54\pi^3}{54\pi^3-1}}$. In order to prove (21) we use the following estimate, which will be useful in what follows.

Let $\gamma>0$, $r>0$, $m\ge1$ and $\delta\in\mathbb{R}$. Then for $m\ge\big(\frac{14|\delta+1-r|}{\gamma r}\big)^{\frac{1}{r}}$ the estimate takes place

$$\int\limits_m^\infty e^{-\gamma t^r}t^\delta dt=\frac{e^{-\gamma m^r}}{\gamma r}m^{\delta+1-r}\Big(1+\Theta_{\gamma,m}^{r,\delta}\frac{|\delta+1-r|}{\gamma r}\frac{1}{m^r}\Big),\quad|\Theta_{\gamma,m}^{r,\delta}|\le\frac{14}{13}.\tag{22}$$

Indeed, integrating by parts, we obtain

$$\int\limits_m^\infty e^{-\gamma t^r}t^\delta dt=\frac{e^{-\gamma m^r}}{\gamma r}m^{\delta+1-r}+\frac{\delta+1-r}{\gamma r}\int\limits_m^\infty e^{-\gamma t^r}t^{-r+\delta}dt.\tag{23}$$

Since

$$(24)\qquad \int_m^\infty e^{-\gamma t^r} t^{-r+\delta} dt = \frac{\overline{\Theta}^{r,\delta}_{\gamma,m}}{m^r} \int_m^\infty e^{-\gamma t^r} t^{\delta} dt, \quad 0 < \overline{\Theta}^{r,\delta}_{\gamma,m} < 1,$$

by virtue of (23) for $m \geq \left(\frac{14|\delta+1-r|}{\gamma r}\right)^{\frac{1}{r}}$ we have

$$\int_m^\infty e^{-\gamma t^r} t^{\delta} dt \leq \frac{e^{-\gamma m^r}}{\gamma r} m^{\delta+1-r} + \frac{1}{14} \int_m^\infty e^{-\gamma t^r} t^{\delta} dt,$$

whence

$$(25)\qquad \int_m^\infty e^{-\gamma t^r} t^{\delta} dt \leq \frac{14 e^{-\gamma m^r}}{13\gamma r} m^{\delta+1-r}.$$

The estimate (22) follows from (23)–(25).

From the equality (6) and relation

$$(26)\qquad \int_n^\infty \xi(u) du < \sum_{j=n}^\infty \xi(j) < \int_n^\infty \xi(u) du + \xi(n),$$

which takes place for any positive and decreasing function $\xi(u)$, $u \geq 1$, such that $\int_n^\infty \xi(u) du < \infty$, we get

$$(27)\qquad \mathcal{E}_n(C^{\alpha,r}_{\beta,2})_C = \frac{1}{\sqrt{\pi}} \Big(\int_n^\infty e^{-2\alpha t^r} dt + \Theta^{(1)}_{\alpha,r,n} e^{-2\alpha n^r} \Big)^{\frac{1}{2}}, \quad |\Theta^{(1)}_{\alpha,r,n}| < 1.$$

In order to estimate the integral $\int_n^\infty e^{-2\alpha t^r} dt$ it suffices to use the equality (22) for $\gamma = 2\alpha$, $\delta = 0$, $m = n$ and $r \in (0,1)$. Then, taking into account that $n_0(\alpha, r, 2) > \left(\frac{7(1-r)}{\alpha r}\right)^{\frac{1}{r}}$, for $n \geq n_0(\alpha, r, 2)$ from (22) and (27) we get

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,2})_C = \frac{1}{\sqrt{\pi}} \Big(\frac{e^{-2\alpha n^r}}{2\alpha r} n^{1-r} \Big(1 + \Theta^{r,0}_{2\alpha,n} \frac{(1-r)}{2\alpha r} \frac{1}{n^r} \Big) + \Theta^{(1)}_{\alpha,r,n} e^{-2\alpha n^r} \Big)^{\frac{1}{2}} =$$

$$(28)\qquad = \frac{e^{-\alpha n^r}}{\sqrt{2\pi\alpha r}} n^{\frac{1-r}{2}} \Big(1 + \Theta^{(2)}_{\alpha,r,n} \Big(\frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \Big) \Big)^{\frac{1}{2}}, \quad |\Theta^{(2)}_{\alpha,r,n}| \leq 2.$$

Since for $n > n_0(\alpha, r, 2)$

$$\left| \Big(1 + \Theta^{(2)}_{\alpha,r,n} \Big(\frac{1}{2\alpha r} \frac{1}{n^r} + \frac{\alpha r}{n^{1-r}} \Big) \Big)^{\frac{1}{2}} - 1^{\frac{1}{2}} \right| \leq$$

$$\leq \frac{1}{\sqrt{1-\left(\frac{1}{\alpha r}\frac{1}{n^r}+\frac{2\alpha r}{n^{1-r}}\right)}}\Big(\frac{1}{2\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big) \leq$$

$$\leq \sqrt{\frac{54\pi^3}{54\pi^3-1}}\Big(\frac{1}{2\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big),$$

then (21) follows from (28).

In the case of $p=\infty$ Theorem 1 allows to clarify the asymptotic equality (10).

We set $n_1 = n_1(\alpha, r)$ be the smallest number n such that

$$\frac{1}{\alpha r}\frac{1}{n^r}\Big(1+\ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\Big)+\frac{\alpha r}{n^{1-r}} \leq \frac{1}{(3\pi)^3}. \tag{29}$$

The following assertion takes place.

Theorem 3. *Let $0<r<1$, $\alpha>0$ and $\beta\in\mathbb{R}$. Then for $n\geq n_1(\alpha,r)$ the following estimate is true*

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = \frac{4}{\pi^2}e^{-\alpha n^r}\ln\Big(\frac{n^{1-r}}{\alpha r}\Big)+\gamma^{(2)}_{n,\infty}e^{-\alpha n^r}, \tag{30}$$

where the quantity $\gamma^{(2)}_{n,\infty}=\gamma^{(2)}_{n,\infty}(\alpha,r,\beta)$ is such that $|\gamma^{(2)}_{n,\infty}|\leq 20\pi^4$.

The asymptotic equality (10), which was established by Stepanets (see [20]), follows from the relation (30).

The next three statements are the crucial part for the proof of Theorem 1. Section 4 of this paper contains proofs of Lemmas 1-2.

Lemma 1. *Let $1\leq s\leq\infty$, 2π–periodic functions $g(t)$ and $h(t)$ have finite derivatives and satisfy the conditions:*

$$r(t) := \sqrt{g^2(t)+h^2(t)} \neq 0, \tag{31}$$

$$M := \sup_{t\in\mathbb{R}} \frac{\sqrt{(g'(t))^2+(h'(t))^2}}{\sqrt{g^2(t)+h^2(t)}} < \infty. \tag{32}$$

Then for the function

$$\phi(t) = g(t)\cos(nt+\gamma)+h(t)\sin(nt+\gamma), \quad \gamma\in\mathbb{R}, \quad n\in\mathbb{N}, \tag{33}$$

for all numbers $n \geq \begin{cases} 4\pi s M, & 1\leq s<\infty, \\ 1, & s=\infty, \end{cases}$ *the following estimates take place*

$$\|\phi\|_s = \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}}+\delta^{(1)}_{s,n}\frac{M}{n}\Big), \tag{34}$$

$$\inf_{\lambda\in\mathbb{R}} \|\phi(t)-\lambda\|_s = \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(2)}\frac{M}{n}\Big), \tag{35}$$

$$\sup_{h\in\mathbb{R}} \frac{1}{2}\|\phi\big(t+h\big)-\phi(t)\|_s = \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} + \delta_{s,n}^{(3)}\frac{M}{n}\Big), \tag{36}$$

where

$$|\delta_{s,n}^{(i)}| < 14\pi, \quad i = 1...3. \tag{37}$$

Assertion 1 [24, Chapter 2, Section 2.8]. *Let continuous function* $\phi(x)$ *be a function of bounded variation in the interval* $(0,\infty)$, $\lim\limits_{x\to\infty}\phi(x)=0$ *and*

$$\int\limits_0^\infty \phi(t)dt < \infty.$$

Then the following equality takes place

$$\sqrt{a}\Big(\frac{\phi(0)}{2}+\sum_{k=1}^{\infty}\phi(ka)\Big) = \sqrt{\frac{2\pi}{a}}\Big(\frac{\Phi_c(0)}{2}+\sum_{k=1}^{\infty}\Phi_c(\frac{2\pi k}{a})\Big), \quad a>0, \tag{38}$$

where $\Phi_c(x)$ *is the Fourier cosine transform of the function* $\phi(x)$ *of the form*

$$\Phi_c(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^\infty \phi(u)\cos xu du.$$

Denote by $\mathfrak{M}$ the set of all convex downwards, continuous functions $\psi(t) > 0$, $t\geq 1$, such that $\lim\limits_{t\to\infty}\psi(t)=0$.

Lemma 2. *Let* $\psi\in\mathfrak{M}$. *Then*

$$0 < \int\limits_0^\infty \psi(\tau+u)\cos vu du \leq \frac{\pi}{v^2}|\psi'(\tau)|, \quad v\in\mathbb{R}\setminus\{0\}, \quad \tau\geq 1. \tag{39}$$

3 Proof of Theorems 1-3.

Proof of the Theorem 1. According to (1) and (2) we have

$$\mathcal{E}_n(C_{\beta,p}^{\alpha,r})_C = \frac{1}{\pi}\sup_{\varphi\in B_p^0}\Big\|\int\limits_{-\pi}^{\pi} P_{\alpha,r,\beta}^{(n)}(x-t)\varphi(t)dt\Big\|_C, \quad 1\leq p\leq\infty, \tag{40}$$

where

$$(41)\qquad P^{(n)}_{\alpha,r,\beta}(t):=\sum_{k=n}^{\infty}e^{-\alpha k^r}\cos\Big(kt-\frac{\beta\pi}{2}\Big),\ 0<r<1,\ \alpha>0,\ \beta\in\mathbb{R}.$$

Taking into account the invariance of the sets $B^0_p,\ 1\le p\le\infty$, under shifts of the argument, from (40) we conclude that

$$(42)\qquad \mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C=\frac{1}{\pi}\sup_{\varphi\in B^0_p}\int\limits_{-\pi}^{\pi}P^{(n)}_{\alpha,r,\beta}(t)\varphi(t)dt.$$

On the basis of the duality relation (see, e.g., [21, Chapter 1, Section 1.4])

$$(43)\qquad \sup_{\varphi\in B^0_p}\int\limits_{-\pi}^{\pi}P^{(n)}_{\alpha,r,\beta}(t)\varphi(t)dt=\inf_{\lambda\in\mathbb{R}}\|P^{(n)}_{\alpha,r,\beta}(t)-\lambda\|_{p'},\ \frac{1}{p}+\frac{1}{p'}=1.$$

In order to find the estimate for the quantity $\inf\limits_{\lambda\in\mathbb{R}}\|P^{(n)}_{\alpha,r,\beta}(t)-\lambda\|_{p'}$ we use the Lemma 1.

We represent the function $P^{(n)}_{\alpha,r,\beta}(t)$, which is defined by formula (41), in the form

$$(44)\qquad P^{(n)}_{\alpha,r,\beta}(t)=g_{\alpha,r,n}(t)\cos\Big(nt-\frac{\beta\pi}{2}\Big)+h_{\alpha,r,n}(t)\sin\Big(nt-\frac{\beta\pi}{2}\Big),$$

where

$$(45)\qquad g_{\alpha,r,n}(t):=\sum_{k=0}^{\infty}e^{-\alpha(k+n)^r}\cos kt,$$

$$(46)\qquad h_{\alpha,r,n}(t):=-\sum_{k=0}^{\infty}e^{-\alpha(k+n)^r}\sin kt.$$

Let us show, that for functions $g_{\alpha,r,n}$ and $h_{\alpha,r,n}$ the following conditions are satisfied

$$(47)\qquad \sqrt{g^2_{\alpha,r,n}(t)+h^2_{\alpha,r,n}(t)}\neq 0$$

and

$$(48)\qquad M_n=M_n(\alpha;r):=\sup_{t\in\mathbb{R}}\frac{\sqrt{(g'_{\alpha,r,n}(t))^2+(h'_{\alpha,r,n}(t))^2}}{\sqrt{g^2_{\alpha,r,n}(t)+h^2_{\alpha,r,n}(t)}}<\infty.$$

Since, for arbitrary $\alpha>0,\ 0<r<1$ the sequence $\left\{e^{-\alpha(k+n)^r}\right\}_{k=0}^{\infty}$ is convex downwards, then (see, e.g., [25, Chapter 10, Section 2])

$$\frac{1}{2}e^{-\alpha n^r}+\sum_{k=1}^{\infty}e^{-\alpha(k+n)^r}\cos kt\geq 0,$$

and

$$\sqrt{g^2_{\alpha,r,n}(t)+h^2_{\alpha,r,n}(t)}\geq\frac{1}{2}e^{-\alpha n^r}>0. \tag{49}$$

Further, since

$$g'_{\alpha,r,n}(t)=-\sum_{k=1}^{\infty}ke^{-\alpha(k+n)^r}\sin kt, \tag{50}$$

$$h'_{\alpha,r,n}(t)=-\sum_{k=1}^{\infty}ke^{-\alpha(k+n)^r}\cos kt, \tag{51}$$

it is clear that

$$\sqrt{(g'_{\alpha,r,n}(t))^2+(h'_{\alpha,r,n}(t))^2}<\sum_{k=1}^{\infty}ke^{-\alpha(k+n)^r}<\infty. \tag{52}$$

On the basis of (49) and (52), the functions $g_{\alpha,r,n}(t)$ and $h_{\alpha,r,n}(t)$ satisfy the conditions (47) and (48). Therefore, setting in Lemma 1 $g(t)=g_{\alpha,r,n}(t)$, $h(t)=h_{\alpha,r,n}(t)$, $s=p'$ and $\gamma=-\frac{\beta\pi}{2}$, we get that for

$$n\geq\begin{cases}4\pi p'M_n, & 1\leq p'<\infty,\\ 1, & p'=\infty,\end{cases} \tag{53}$$

the estimate takes place

$$\inf_{\lambda\in\mathbb{R}}\|P^{(n)}_{\alpha,r,\beta}(t)-\lambda\|_{p'}=\left\|\sqrt{(g_{\alpha,r,n}(t))^2+(h_{\alpha,r,n}(t))^2}\right\|_{p'}\left(\frac{\|\cos t\|_{p'}}{(2\pi)^{\frac{1}{p'}}}+\delta^{(1)}_n\frac{M_n}{n}\right), \tag{54}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, quantity M_n is defined by equality (48), and the quantity $\delta^{(1)}_n=\delta^{(1)}_n(\alpha,r,\beta,p)$ is such that $|\delta^{(1)}_n|<14\pi$.

Setting

$$\mathcal{P}_{\alpha,r,n}(t):=g_{\alpha,r,n}(t)-ih_{\alpha,r,n}(t)=\sum_{k=0}^{\infty}e^{-\alpha(k+n)^r}e^{ikt}, \tag{55}$$

we have

$$\sqrt{(g'_{\alpha,r,n}(t))^2+(h'_{\alpha,r,n}(t))^2}=\left|\mathcal{P}'_{\alpha,r,n}(t)\right|$$

and therefore

$$M_n = \sup_{t\in\mathbb{R}} \frac{\Big|\mathcal{P}'_{\alpha,r,n}(t)\Big|}{\Big|\mathcal{P}_{\alpha,r,n}(t)\Big|}. \tag{56}$$

Then, by virtue of the formulas (42), (43), (54) and (55), for all numbers n, which satisfy the condition (53), the estimate holds

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = \|\mathcal{P}_{\alpha,r,n}(t)\|_{p'}\Big(\frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}}\pi^{1+\frac{1}{p'}}} + \delta_n^{(2)}\frac{M_n}{n}\Big),\ 1\leq p\leq\infty, \tag{57}$$

where M_n is defined by equality (56), and for the quantity $\delta_n^{(2)} = \delta_n^{(2)}(\alpha,r,\beta,p)$ is such that $|\delta_n^{(2)}| < 14$.

Since

$$\Big|\mathcal{P}_{\alpha,r,n}(t)\Big|^2 = \mathcal{P}_{\alpha,r,n}(t)\widetilde{\mathcal{P}}_{\alpha,r,n}(t), \tag{58}$$

where

$$\widetilde{\mathcal{P}}_{\alpha,r,n}(t) = g_{\alpha,r,n}(t) + ih_{\alpha,r,n}(t) = \sum_{k=0}^{\infty} e^{-\alpha(k+n)^r}e^{-ikt},$$

by expanding the product $\mathcal{P}_{\alpha,r,n}\widetilde{\mathcal{P}}_{\alpha,r,n}$ in the Fourier series (see, e.g., [25, Chapter 1, Section 23]), we get

$$\mathcal{P}_{\alpha,r,n}(t)\widetilde{\mathcal{P}}_{\alpha,r,n}(t) = \Big(\sum_{k=0}^{\infty} e^{-\alpha(k+n)^r}e^{ikt}\Big)\Big(\sum_{k=-\infty}^{0} e^{-\alpha(-k+n)^r}e^{ikt}\Big) =$$

$$= \sum_{k=-\infty}^{\infty}\sum_{j=0}^{\infty} e^{-\alpha(j+n)^r}e^{-\alpha(j+|k|+n)^r}e^{ikt} =$$

$$= \sum_{j=n}^{\infty} e^{-2\alpha j^r} + 2\sum_{k=1}^{\infty}\sum_{j=n}^{\infty} e^{-\alpha j^r}e^{-\alpha(j+k)^r}\cos kt. \tag{59}$$

Let convert the sum $\sum\limits_{j=n}^{\infty} e^{-2\alpha j^r} + 2\sum\limits_{k=1}^{\infty}\sum\limits_{j=n}^{\infty} e^{-\alpha j^r}e^{-\alpha(j+k)^r}\cos kt$ with a help of Poisson summation formula from Assertion 1.

Let fix $t\in[-\pi,\pi]$, $\alpha>0$, $r\in(0,1)$ and set

$$\phi(x) = 2\sum_{j=n}^{\infty} e^{-\alpha j^r}e^{-\alpha(j+x)^r}\cos xt, \qquad x\geq 0$$

. One can easily check that all conditions of the Assertion 1 are satisfied, and therefore, setting in (38) $a = 1$, we obtain

$$\sum_{j=n}^{\infty} e^{-2\alpha j^r} + 2\sum_{k=1}^{\infty}\sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+k)^r} \cos kt =$$

$$= 2\int_0^{\infty}\sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+u)^r} \cos utdu +$$

$$+4\sum_{k=1}^{\infty}\int_0^{\infty}\sum_{j=n}^{\infty} e^{-\alpha j^r} e^{-\alpha(j+u)^r} \cos ut \cos 2\pi kudu =$$

$$= Q_n(t) + R_n(t), \tag{60}$$

where

$$Q_n(t) = Q_n(\alpha; r; t) := 2\sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^{\infty} e^{-\alpha(j+u)^r} \cos utdu, \tag{61}$$

$$R_n(t) = R_n(\alpha; r; t) :=$$

$$:= 2\sum_{k=1}^{\infty}\sum_{j=n}^{\infty} e^{-\alpha j^r} \int_0^{\infty} e^{-\alpha(j+u)^r} \big(\cos((t-2\pi k)u) + \cos((t+2\pi k)u)\big) du. \tag{62}$$

Hence, as a consequence of (58), (59) and (60)

$$\Big|\mathcal{P}_{\alpha,r,n}(t)\Big|^2 = Q_n(t) + R_n(t). \tag{63}$$

Denote by $n_2 = n_2(\alpha, r, p)$ the smallest number n such that

$$\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r \chi(p)}{n^{1-r}} \le \frac{1}{14}, \tag{64}$$

where

$$\chi(p) = \begin{cases} p, & 1 \le p < \infty, \\ 1, & p = \infty, \end{cases}$$

and let us show that for the quantity $Q_n(t)$ for $n \ge n_2(\alpha, r, p)$ and arbitrary $t \in [-\pi, \pi]$ the following estimate takes place

$$Q_n(t) = \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2}\Big(1 + \Theta^{(4)}_{\alpha,r,n}(t)\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)\Big), \quad |\Theta^{(4)}_{\alpha,r,n}(t)| < 5. \tag{65}$$

Making some elementary calculations, one can easily check that

$$\frac{d}{du}\left(e^{-\alpha(j+u)^r}\frac{-\alpha r(j+u)^{r-1}\cos ut+t\sin ut}{(\alpha r(j+u)^{r-1})^2+t^2}\right)+$$

$$+\alpha r(1-r)e^{-\alpha(j+u)^r}(j+u)^{r-2}\frac{((\alpha r(j+u)^{r-1})^2-t^2)\cos ut-2t\alpha r(j+u)^{r-1}\sin ut}{((\alpha r(j+u)^{r-1})^2+t^2)^2}=$$

$$=e^{-\alpha(j+u)^r}\cos ut.$$

So,

$$\int e^{-\alpha(j+u)^r}\cos ut du=$$

$$=e^{-\alpha(j+u)^r}\frac{-\alpha r(j+u)^{r-1}\cos ut+t\sin ut}{(\alpha r(j+u)^{r-1})^2+t^2}+\alpha r(1-r)\times$$

$$\times\int e^{-\alpha(j+u)^r}(j+u)^{r-2}\frac{((\alpha r(j+u)^{r-1})^2-t^2)\cos ut-2t\alpha r(j+u)^{r-1}\sin ut}{((\alpha r(j+u)^{r-1})^2+t^2)^2}du.$$

Hence, we obtain the equality

$$\int_0^\infty e^{-\alpha(j+u)^r}\cos ut du=\frac{\alpha r j^{r-1}}{t^2+(\alpha r j^{r-1})^2}e^{-\alpha j^r}+\alpha r(1-r)\times$$

(66)

$$\times\int_0^\infty e^{-\alpha(j+u)^r}(j+u)^{r-2}\frac{((\alpha r(j+u)^{r-1})^2-t^2)\cos ut-2t\alpha r(j+u)^{r-1}\sin ut}{(t^2+(\alpha r(j+u)^{r-1})^2)^2}du.$$

It is easy to verify that

$$\left|\int_0^\infty e^{-\alpha(j+u)^r}(j+u)^{r-2}\frac{((\alpha r(j+u)^{r-1})^2-t^2)\cos ut-2t\alpha r(j+u)^{r-1}\sin ut}{((\alpha r(j+u)^{r-1})^2+t^2)^2}du\right|\le$$

$$\le\int_0^\infty e^{-\alpha(j+u)^r}(j+u)^{r-2}\left(\frac{1}{t^2+(\alpha r(j+u)^{r-1})^2}+\frac{2t\alpha r(j+u)^{r-1}}{(t^2+(\alpha r(j+u)^{r-1})^2)^2}\right)du\le$$

$$\le 2\int_0^\infty e^{-\alpha(j+u)^r}\frac{(j+u)^{r-2}}{t^2+(\alpha r(j+u)^{r-1})^2}du. \tag{67}$$

For fixed $\alpha>0$, $r\in(0,1)$ and $t\in[-\pi,\pi]$ the function $\frac{v^{r-2}}{t^2+(\alpha r v^{r-1})^2}$, $v\ge 1$ decreases. Besides, according to (25), for $\delta=0$, $\gamma=\alpha$, $m=j$, $j\ge n_2(\alpha,r,p)$ the estimate takes place

$$\int_0^\infty e^{-\alpha(j+u)^r}\frac{(j+u)^{r-2}}{t^2+(\alpha r(j+u)^{r-1})^2}du \le \frac{j^{r-2}}{t^2+(\alpha rj^{r-1})^2}\int_0^\infty e^{-\alpha(j+u)^r}du =$$

$$(68)\qquad = \frac{j^{r-2}}{t^2+(\alpha rj^{r-1})^2}\int_j^\infty e^{-\alpha u^r}du \le \frac{14}{13}\frac{e^{-\alpha j^r}}{\alpha rj(t^2+(\alpha rj^{r-1})^2)}.$$

It follows from relations (66)–(68) that for $j \ge n_2(\alpha, r, p)$

$$\int_0^\infty e^{-\alpha(j+u)^r}\cos ut du =$$

$$(69)\qquad = \frac{\alpha rj^{r-1}}{t^2+(\alpha rj^{r-1})^2}e^{-\alpha j^r}\Big(1+\Theta^{(5)}_{\alpha,r,j}(t)\frac{1-r}{\alpha r}\frac{1}{j^r}\Big),\quad |\Theta^{(5)}_{\alpha,r,j}(t)| \le \frac{28}{13}.$$

Therefore, taking into account (61), for $n \ge n_2(\alpha, r, p)$ we have

$$(70)\quad Q_n(t) = 2\alpha r\sum_{j=n}^\infty \frac{e^{-2\alpha j^r}j^{r-1}}{t^2+(\alpha rj^{r-1})^2}\Big(1+\Theta^{(6)}_{\alpha,r,n}(t)\frac{1-r}{\alpha r}\frac{1}{n^r}\Big),\quad |\Theta^{(6)}_{\alpha,r,n}(t)| \le \frac{28}{13}.$$

Further, let us find bilateral estimates for the quantities $\sum\limits_{j=n}^\infty \frac{e^{-2\alpha j^r}j^{r-1}}{t^2+(\alpha rj^{r-1})^2}$ for $n \ge n_2(\alpha, r, p)$. It can be shown that for fixed $\alpha > 0$, $r \in (0, 1)$ and $t \in [-\pi, \pi]$ the function $\xi(u) = \frac{e^{-2\alpha u^r}u^{r-1}}{t^2+(\alpha ru^{r-1})^2}$ decreases for $u \ge n_2(\alpha, r, p)$. Therefore, on basis of (26)

$$2\alpha r\sum_{j=n}^\infty e^{-2\alpha j^r}\frac{j^{r-1}}{t^2+(\alpha rj^{r-1})^2} =$$

(71)

$$= 2\alpha r\int_n^\infty e^{-2\alpha u^r}\frac{u^{r-1}}{t^2+(\alpha ru^{r-1})^2}du + \Theta^{(7)}_{\alpha,r,n}(t)\frac{\alpha r\ e^{-2\alpha n^r}n^{r-1}}{t^2+(\alpha rn^{r-1})^2},\quad 0 \le \Theta^{(7)}_{\alpha,r,n}(t) \le 2.$$

Integrating by parts, we have

$$2\alpha r\int_n^\infty \frac{e^{-2\alpha u^r}u^{r-1}}{t^2+(\alpha ru^{r-1})^2}du =$$

$$= \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} + 2(\alpha r)^2(1-r) \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{2r-3}}{(t^2 + (\alpha r u^{r-1})^2)^2} du. \tag{72}$$

Since

$$(\alpha r)^2 \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{2r-3}}{(t^2 + (\alpha r u^{r-1})^2)^2} du \le \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{-1}}{t^2 + (\alpha r u^{r-1})^2} du \le$$

$$\le \frac{1}{n^r} \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du, \tag{73}$$

it follows from (72) that for $n \ge n_2(\alpha, r, p)$ the following inequalities are true

$$\int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \le$$

$$\le \frac{1}{2\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} + \frac{1-r}{\alpha r} \frac{1}{n^r} \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \le$$

$$\le \frac{1}{2\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} + \frac{1}{14} \int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du.$$

Hence, for $n \ge n_2(\alpha, r, p)$

$$\int\limits_n^\infty \frac{e^{-2\alpha u^r} u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du \le \frac{7}{13\alpha r} \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2}. \tag{74}$$

From (72)–(74) for $n \ge n_2(\alpha, r, p)$ we arrive at the following estimate

$$2\alpha r \int\limits_n^\infty e^{-2\alpha u^r} \frac{u^{r-1}}{t^2 + (\alpha r u^{r-1})^2} du =$$

$$= \frac{e^{-2\alpha n^r}}{t^2 + (\alpha r n^{r-1})^2} \Big(1 + \Theta^{(8)}_{\alpha,r,n}(t) \frac{1-r}{\alpha r} \frac{1}{n^r}\Big), \quad 0 < \Theta^{(8)}_{\alpha,r,n}(t) \le \frac{14}{13}. \tag{75}$$

It follows from formulas (71) and (75) that

$$2\alpha r \sum_{j=n}^\infty \frac{e^{-2\alpha j^r} j^{r-1}}{t^2 + (\alpha r j^{r-1})^2} =$$

$$= \frac{e^{-2\alpha n^r}}{t^2+(\alpha r n^{r-1})^2}\Big(1+\Theta^{(9)}_{\alpha,r,n}(t)\Big(\frac{1-r}{\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big)\Big), \tag{76}$$

where $n \geq n_2(\alpha,r,p)$ and $0<\Theta^{(9)}_{\alpha,r,n}(t)\leq 2$.

In view of (70) and (76) for all $n \geq n_2(\alpha,r,p)$ we obtain (65). In particular, it follows from formulas (64) and (65) that

$$Q_n(t)>0,\ \ t\in[-\pi,\pi],\ \ n\geq n_2(\alpha,r,p). \tag{77}$$

Let us find upper estimate for the quantity $R_n(t)$ of the form (62).

Setting in inequality (39) $v=t\pm 2\pi k,\ \ k\in\mathbb{N}$, and $\tau=j$, we obtain that for arbitrary $\psi\in\mathfrak{M}$ and $t\in[-\pi,\pi]$

$$0<\sum_{k=1}^{\infty}\sum_{j=n}^{\infty}\psi(j)\int\limits_0^{\infty}\psi(j+u)\big(\cos((t-2\pi k)u)+\cos((t+2\pi k)u)\big)du\leq$$

$$\leq \pi\sum_{k=1}^{\infty}\Big(\frac{1}{(t-2k\pi)^2}+\frac{1}{(t+2k\pi)^2}\Big)\sum_{j=n}^{\infty}\psi(j)|\psi'(j)|\leq$$

$$\leq \pi\sum_{k=1}^{\infty}\Big(\frac{1}{(\pi-2k\pi)^2}+\frac{1}{(\pi+2k\pi)^2}\Big)\psi(n)\bigg(|\psi'(n)|+\int\limits_n^{\infty}|\psi'(u)|du\bigg)=$$

$$=\frac{1}{\pi}\sum_{k=1}^{\infty}\Big(\frac{1}{(2k-1)^2}+\frac{1}{(2k+1)^2}\Big)\psi(n)\bigg(|\psi'(n)|+\psi(n)\bigg)=$$

$$=\Big(\frac{\pi}{4}-\frac{1}{\pi}\Big)\psi(n)\bigg(|\psi'(n)|+\psi(n)\bigg),\ \ n\in\mathbb{N}. \tag{78}$$

Setting in (78) $\psi(t)=e^{-\alpha t^r}$, $0<r<1$, $\alpha>0$, we get that for the function $R_n(t)$ of the form (62) the following estimate takes place

$$0<R_n(t)\leq\Big(\frac{\pi}{2}-\frac{2}{\pi}\Big)e^{-2\alpha n^r}\big(\frac{\alpha r}{n^{1-r}}+1\big)\leq\Big(\frac{\pi}{2}-\frac{2}{\pi}\Big)\frac{15}{14}e^{-2\alpha n^r}<\frac{\pi}{3}e^{-2\alpha n^r}, \tag{79}$$

where $n\geq n_2(\alpha,r,p)$.

By virtue of (63)

$$\big|\mathcal{P}_{\alpha,r,n}(t)\big|=\sqrt{Q_n(t)+R_n(t)}, \tag{80}$$

and therefore, taking into account (77) and (79), we have

$$\big\|\mathcal{P}_{\alpha,r,n}\big\|_{p'}=\big\|\sqrt{Q_n}\big\|_{L_{p'}[-\pi,\pi]}+\Theta^{(2)}_{\alpha,r,p,n}e^{-\alpha n^r},\quad 1\leq p'\leq\infty, \tag{81}$$

where $|\Theta^{(2)}_{\alpha,r,p,n}| < \frac{2\pi^2}{3}$ and $n \geq n_2(\alpha, r, p)$.

Let us show, that for $1 \leq p' \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $n \geq n_2(\alpha, r, p)$ the estimate is true

$$\left\|\mathcal{P}_{\alpha,r,n}\right\|_{p'} = e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) +$$

$$+\Theta^{(3)}_{\alpha,r,p,n}\Big(\frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg), \tag{82}$$

where

$$|\Theta^{(3)}_{\alpha,r,p,n}| \leq \begin{cases} \pi^2, & 1 \leq p' < \infty, \\ \frac{14}{13}, & p' = \infty. \end{cases} \tag{83}$$

Since, on the basis of estimate (65) for $n \geq n_2(\alpha, r, p)$ and $1 \leq p' \leq \infty$

$$\Bigg|\Big(1 + \Theta^{(4)}_{\alpha,r,n}(t)\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)^{\frac{1}{2}} - 1\Bigg| \leq$$

$$\leq \frac{5}{2}\frac{1}{\sqrt{1 - 5\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)}}\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big) \leq \frac{5\sqrt{7}}{3\sqrt{2}}\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)$$

we get

$$\sqrt{Q_n(t)} =$$

$$= \frac{e^{-\alpha n^r}}{\sqrt{t^2 + (\alpha r n^{r-1})^2}}\Big(1 + \Theta^{(10)}_{\alpha,r,n}(t)\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)\Big), \quad |\Theta^{(10)}_{\alpha,r,n}(t)| \leq \frac{5\sqrt{7}}{3\sqrt{2}}. \tag{84}$$

For $1 \leq p' < \infty$ from (84) we have

$$\left\|\sqrt{Q_n}\right\|_{L_{p'}[-\pi,\pi]} =$$

$$= e^{-\alpha n^r}\Bigg(\int\limits_{-\pi}^{\pi} \frac{dt}{(t^2 + (\alpha r n^{r-1})^2)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}}\Big(1 + \Theta^{(4)}_{\alpha,r,p,n}\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)\Big) =$$

$$= 2^{\frac{1}{p'}} e^{-\alpha n^r}\Big(\frac{n^{1-r}}{\alpha r}\Big)^{\frac{1}{p}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\Big(1 + \Theta^{(4)}_{\alpha,r,p,n}\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)\Big), \tag{85}$$

where $|\Theta^{(4)}_{\alpha,r,p,n}| \leq \frac{5\sqrt{7}}{3\sqrt{2}}$ and $J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)$ is defined by equality (14).

Combining (81) and (85), we obtain that for $1 \le p' < \infty$ the following relation takes place

$$\big\|\mathcal{P}_{\alpha,r,n}\big\|_{p'} = e^{-\alpha n^r} n^{\frac{1-r}{p}} \Bigg(\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) +$$

$$+\Theta^{(4)}_{\alpha,r,p,n} \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\Big(\frac{1-r}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big) + \frac{\Theta^{(2)}_{\alpha,r,p,n}}{n^{\frac{1-r}{p}}}\Bigg). \tag{86}$$

However, for all $n > n_2(\alpha, r, p)$

$$\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{\alpha r}{n^{1-r}} < \frac{1}{n^{\frac{1-r}{p}}},\ \ 1 \le p' < \infty. \tag{87}$$

Indeed, taking into account (14) and (64), for all $1 < p' < \infty$ and $n \ge n_2(\alpha, r, p)$ we find

$$\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{\alpha r}{n^{1-r}} n^{\frac{1-r}{p}} = \Big(\frac{2\alpha r}{n^{1-r}}\Big)^{\frac{1}{p'}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) <$$

$$< \Big(\frac{2\alpha r}{n^{1-r}}\Big)^{\frac{1}{p'}} \Big(\int\limits_0^\infty \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} < \Big(\frac{2\alpha r}{n^{1-r}}\Big)^{\frac{1}{p'}} \Big(1 + \int\limits_1^\infty \frac{dt}{t^{p'}}\Big)^{\frac{1}{p'}} =$$

$$= \Big(\frac{2\alpha r p}{n^{1-r}}\Big)^{\frac{1}{p'}} < \Big(\frac{1}{7}\Big)^{\frac{1}{p'}} < 1, \tag{88}$$

and for $p' = 1$ and $n \ge n_2(\alpha, r, p)$, taking into account decreasing on the interval $[e, \infty)$ of the function $\frac{\ln v}{v}$, we have

$$\frac{2\alpha r}{n^{1-r}} J_1\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) = \frac{2\alpha r}{n^{1-r}} \int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}} < \frac{2\alpha r}{n^{1-r}} \Big(1 + \int\limits_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}}\Big) <$$

$$< \frac{2\alpha r}{n^{1-r}} + \frac{2\alpha r}{n^{1-r}} \ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) \le \frac{2}{14} + \frac{2\pi \ln 14\pi}{14\pi} < 1. \tag{89}$$

Formulas (88) and (89) prove (87). For $1 \le p' < \infty$ estimate (82) follows from (86) and (87).

Let us verify validity of the estimate (82) for $p' = \infty$. It follows from (55) and (26) that

$$\big\|\mathcal{P}_{\alpha,r,n}\big\|_\infty = \sum_{k=0}^\infty e^{-\alpha(k+n)^r} = \int\limits_n^\infty e^{-\alpha t^r} dt + \Theta^{(11)}_{\alpha,r,n} e^{-\alpha n^r},\ \ |\Theta^{(11)}_{\alpha,r,n}| \le 1. \tag{90}$$

Setting in formula (22) $\gamma=\alpha$, $\delta=0$ and $m=n$, from (90) we obtain that for arbitrary $n\ge n_2(\alpha,r,p)$

$$\big\|\mathcal{P}_{\alpha,r,n}\big\|_\infty=\frac{e^{-\alpha n^r}}{\alpha r}n^{1-r}\Big(1+\Theta^{(12)}_{\alpha,r,n}\Big(\frac{1-r}{\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big)\Big), \tag{91}$$

where $|\Theta^{(12)}_{\alpha,r,n}|\le\frac{14}{13}$.

For $p'=\infty$ the validity of (82) follows from (91) and the equality $J_\infty\big(\frac{\pi n^{1-r}}{\alpha r}\big)=1$.

To complete the proof of Theorem 1 it suffices to find the upper estimate of the quantity M_n in formula (57). It is clear that

$$M_n=\sup_{t\in\mathbb{R}}\frac{\big|\mathcal{P'}_{\alpha,r,n}(t)\big|\big|\mathcal{P}_{\alpha,r,n}(t)\big|}{\big|\mathcal{P}_{\alpha,r,n}(t)\big|^2}=$$

$$=\max\Bigg\{\sup_{|t|\le\frac{\alpha r}{n^{1-r}}}\frac{\big|\mathcal{P'}_{\alpha,r,n}(t)\big|\big|\mathcal{P}_{\alpha,r,n}(t)\big|}{\big|\mathcal{P}_{\alpha,r,n}(t)\big|^2},\sup_{\frac{\alpha r}{n^{1-r}}\le|t|\le\pi}\frac{\big|\mathcal{P'}_{\alpha,r,n}(t)\big|\big|\mathcal{P}_{\alpha,r,n}(t)\big|}{\big|\mathcal{P}_{\alpha,r,n}(t)\big|^2}\Bigg\}. \tag{92}$$

In view of formulas (64) and (65) and the fact that $R_n(t)>0$ for $n\ge n_2(\alpha,r,p)$ we obtain

$$\big|\mathcal{P}_{\alpha,r,n}(t)\big|^2>Q_n(t)>\frac{9}{14}\frac{e^{-2\alpha n^r}}{t^2+(\alpha r n^{r-1})^2}. \tag{93}$$

It directly follows from (55) that

$$\big|\mathcal{P}_{\alpha,r,n}(t)\big|\le\sum_{k=0}^{\infty}e^{-\alpha(k+n)^r},\quad \big|\mathcal{P'}_{\alpha,r,n}(t)\big|\le\sum_{k=1}^{\infty}ke^{-\alpha(k+n)^r}. \tag{94}$$

By virtue of (91) for $n\ge n_2(\alpha,r,p)$ we have

$$\big|\mathcal{P}_{\alpha,r,n}(t)\big|\le\sum_{k=0}^{\infty}e^{-\alpha(k+n)^r}<\frac{14}{13}e^{-\alpha n^r}\frac{n^{1-r}}{\alpha r}. \tag{95}$$

The function $te^{-\alpha t^r}$ is monotone decreasing for $t>(\alpha r)^{-\frac{1}{r}}$. Therefore, according to (26), for $n\ge n_2(\alpha,r,p)$ the following estimate takes place

$$\sum_{k=1}^{\infty}e^{-\alpha(k+n)^r}k=\sum_{k=n}^{\infty}e^{-\alpha k^r}k-n\sum_{k=n}^{\infty}e^{-\alpha k^r}<$$

$$<e^{-\alpha n^r}n+\int\limits_n^\infty e^{-\alpha t^r}tdt-n\int\limits_n^\infty e^{-\alpha t^r}dt. \tag{96}$$

Setting in (22) $\gamma=\alpha$, $\delta=1$, $m=n$, and also $\gamma=\alpha$, $\delta=0$, $m=n$, from (94) and (96) we have

$$\big|\mathcal{P}'_{\alpha,r,n}(t)\big| \le e^{-\alpha n^r}\Big(\frac{42}{13}\Big(\frac{n^{1-r}}{\alpha r}\Big)^2+n\Big), \quad n\ge n_2(\alpha,r,p), \tag{97}$$

where $\mathcal{P}_{\alpha,r,n}(t)$ is defined by formula (55).

In view of (93), (95) and (97) for $n\ge n_2(\alpha,r,p)$ we arrive at the estimate

$$\sup_{|t|\le\frac{\alpha r}{n^{1-r}}}\frac{\big|\mathcal{P}'_{\alpha,r,n}(t)\big|\big|\mathcal{P}_{\alpha,r,n}(t)\big|}{\big|\mathcal{P}_{\alpha,r,n}(t)\big|^2}\le$$

$$\le\frac{14}{9}e^{2\alpha n^r}\sup_{|t|<\frac{\alpha r}{n^{1-r}}}\big|\mathcal{P}'_{\alpha,r,n}(t)\big|\big|\mathcal{P}_{\alpha,r,n}(t)\big|(t^2+\Big(\frac{\alpha r}{n^{1-r}}\Big)^2)\le$$

$$\le\frac{5488}{507}\Big(\Big(\frac{n^{1-r}}{\alpha r}\Big)^2+n\Big)\frac{n^{1-r}}{\alpha r}\Big(\frac{\alpha r}{n^{1-r}}\Big)^2=$$

$$=\frac{5488}{507}\Big(\frac{n^{1-r}}{\alpha r}+\alpha r n^r\Big). \tag{98}$$

Applying the Abel transformation to the function $\mathcal{P}_{\alpha,r,n}(t)$ for $0<|t|\le\pi$, and taking into account the inequallity

$$\Big|\sum_{j=0}^{k}e^{ijt}\Big|\le\frac{\pi}{|t|},\ 0<|t|\le\pi,$$

we get

$$\big|\mathcal{P}_{\alpha,r,n}(t)\big|=\Big|\sum_{k=0}^{\infty}(e^{-\alpha(k+n)^r}-e^{-\alpha(k+n+1)^r})\sum_{j=0}^{k}e^{ijt}\Big|\le\frac{\pi}{|t|}e^{-\alpha n^r}. \tag{99}$$

By analogy, for $0<|t|\le\pi$

$$\big|\mathcal{P}'_{\alpha,r,n}(t)\big|=\Big|\sum_{k=0}^{\infty}(e^{-\alpha(k+n)^r}k-e^{-\alpha(k+n+1)^r}(k+1))\sum_{j=0}^{k}e^{ijt}\Big|\le$$

$$\le\frac{\pi}{|t|}\sum_{k=0}^{\infty}|e^{-\alpha(k+n)^r}k-e^{-\alpha(k+n+1)^r}(k+1)|\le$$

$$\le\frac{\pi}{|t|}\Big(\sum_{k=0}^{\infty}k(e^{-\alpha(k+n)^r}-e^{-\alpha(k+n+1)^r})+\sum_{k=0}^{\infty}e^{-\alpha(k+n+1)^r}\Big)= \tag{100}$$

According to (95) and (100)

$$(101)\qquad \left|\mathcal{P'}_{\alpha,r,n}(t)\right| \le \frac{2\pi}{|t|}\sum_{k=0}^{\infty} e^{-\alpha(k+n+1)^r} \le \frac{28\pi}{13|t|}e^{-\alpha n^r}\frac{n^{1-r}}{\alpha r}.$$

In view of (93), (99) and (101) we obtain the estimate

$$\sup_{\frac{\alpha r}{n^{1-r}}\le|t|\le\pi} \frac{\left|\mathcal{P'}_{\alpha,r,n}(t)\right|\left|\mathcal{P}_{\alpha,r,n}(t)\right|}{\left|\mathcal{P}_{\alpha,r,n}(t)\right|^2} \le$$

$$\le \frac{14}{9}e^{2\alpha n^r} \sup_{\frac{\alpha r}{n^{1-r}}\le|t|\le\pi} \left|\mathcal{P'}_{\alpha,r,n}(t)\right|\left|\mathcal{P}_{\alpha,r,n}(t)\right|\Big(t^2+\Big(\frac{\alpha r}{n^{1-r}}\Big)^2\Big) \le$$

$$(102)\qquad \le \frac{392\pi^2}{117}\frac{n^{1-r}}{\alpha r} \sup_{\frac{\alpha r}{n^{1-r}}\le|t|\le\pi} \frac{t^2+\big(\frac{\alpha r}{n^{1-r}}\big)^2}{t^2} \le \frac{784\pi^2}{117}\frac{n^{1-r}}{\alpha r}.$$

Combining (92), (98) and (102), we arrive at the estimate

$$(103)\qquad M_n \le \frac{784\pi^2}{117}\Big(\frac{n^{1-r}}{\alpha r}+\alpha r n^r\Big), \quad n\ge n_2(\alpha,r,p).$$

It follows from conditions (15) and (64) that $n_0(\alpha,r,p) \ge n_2(\alpha,r,p)$ for arbitrary $1\le p\le\infty$. It means that estimates (82) and (103) are true also for $n\ge n_0(\alpha,r,p)$. Let us show that for $n\ge n_0(\alpha,r,p)$ the condition (53) is satisfied. This is obvious for $p'=\infty$. For $1\le p'<\infty$ by virtue of (103), we have

$$(104)\qquad 4\pi M_n p' \le \frac{3136\pi^3}{117}\Big(\frac{n^{1-r}}{\alpha r}+\alpha r n^r\Big)p' < 27\pi^3\Big(\frac{n^{1-r}}{\alpha r}+\alpha r\chi(p)n^r\Big)p'.$$

According to (15) and (104) for any $n\ge n_0(\alpha,r,p)$ the following inequality is true

$$4\pi p' M_n \le n,$$

which is equivalent to (53) for $1\le p'<\infty$.

By using formulas (57), (82) and (103) for $n\ge n_0(\alpha,r,p)$ we arrive at the estimate

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C =$$

$$= e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}}J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)+\Theta^{(3)}_{\alpha,r,p,n}\Big(\frac{1-r}{(\alpha r)^{1+\frac{1}{p}}}J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r}+\frac{1}{n^{\frac{1-r}{p}}}\Big)\Bigg)\times$$

$$(105)\qquad \times\Big(\frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}}\pi^{1+\frac{1}{p'}}}+\delta^{(3)}_n\Big(\frac{1}{\alpha r}\frac{1}{n^r}+\frac{\alpha r}{n^{1-r}}\Big)\Big), \ 1\le p\le\infty,$$

where for $\Theta^{(3)}_{\alpha,r,p,n}$ the estimate (83) takes place, and $|\delta^{(3)}_n| < \frac{10976\pi^2}{117}$.

For $n \geq n_0(\alpha, r, p)$ the following inequality holds

$$|\delta^{(3)}_n| \frac{2^{\frac{1}{p'}}}{(\alpha r)^{\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\Big(\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big) <$$

$$< \frac{21952\pi^2}{117}\Big(\frac{1}{(\alpha r)^{1+\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big), \tag{106}$$

which follows from (87) for $1 \leq p' < \infty$, and it is obvious for $p' = \infty$. Besides, according to (83) and (15) for $n \geq n_0(\alpha, r, p)$

$$\Big|\Theta^{(3)}_{\alpha,r,p,n}\Big|\Big(\frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Big(\frac{\|\cos t\|_{p'}}{2^{\frac{1}{p'}}\pi^{1+\frac{1}{p'}}} + |\delta^{(3)}_n|\Big(\frac{1}{\alpha r}\frac{1}{n^r} + \frac{\alpha r}{n^{1-r}}\Big)\Big) <$$

$$< \frac{363\pi^2}{50}\Big(\frac{1-r}{(\alpha r)^{1+\frac{1}{p}}} J_{p'}\Big(\frac{\pi n^{1-r}}{\alpha r}\Big)\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big). \tag{107}$$

In view of formulas (105)–(107) we arrive at (16). Theorem 1 is proved. $\square$

Proof of the Theorem 2. According to Theorem 1 the following estimate is true for all $1 < p < \infty$, $0 < r < 1$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $n \geq n_0(\alpha, r, p)$

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r} n^{\frac{1-r}{p}}\Bigg(\frac{\|\cos t\|_{p'}}{(\alpha r)^{\frac{1}{p}}\pi^{1+\frac{1}{p'}}}\Bigg(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}} +$$

$$+\gamma^{(1)}_{n,p}\Bigg(\frac{1}{(\alpha r)^{1+\frac{1}{p}}}\Bigg(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}}\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Bigg)\Bigg), \tag{108}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and the quantity $\gamma^{(1)}_{n,p} = \gamma^{(1)}_{n,p}(\alpha, r, \beta)$ is such that $|\gamma^{(1)}_{n,p}| \leq (14\pi)^2$.

By applying the Lagrange theorem, for $n \geq n_0(\alpha, r, p)$ we obtain

$$\Bigg(\int\limits_0^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}} - \Bigg(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}} \leq$$

$$\leq \frac{1}{p'}\Bigg(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Bigg)^{\frac{1}{p'}-1} \int\limits_{\frac{\pi n^{1-r}}{\alpha r}}^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}} \leq$$

$$\leq \frac{1}{p'}\Big(\int\limits_{0}^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t+1)^{p'}}\Big)^{-\frac{1}{p}} \int\limits_{\frac{\pi n^{1-r}}{\alpha r}}^{\infty} \frac{dt}{t^{p'}} =$$

$$= \frac{1}{p'}\frac{1}{(p'-1)}\Big(1-\Big(\frac{\pi n^{1-r}}{\alpha r}+1\Big)^{1-p'}\Big)^{-\frac{1}{p}}\frac{1}{(p'-1)^{\frac{1}{p}}}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1} \leq$$

$$\leq \frac{1}{p'}\frac{1}{(p'-1)}\Big(1-\Big(27\pi^4\frac{p^2}{p-1}+1\Big)^{1-p'}\Big)^{-\frac{1}{p}}\frac{1}{(p'-1)^{\frac{1}{p}}}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1} =$$

$$= \frac{1}{p'-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1}\frac{(p-1)^{\frac{p-1}{p}}}{p}\Big(1-\Big(27\pi^4\frac{p^2}{p-1}+1\Big)^{\frac{1}{1-p}}\Big)^{-\frac{1}{p}} <$$

$$< \frac{2}{p'-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1}. \tag{109}$$

As follows from (109)

$$\Big(\int\limits_{0}^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} =$$

$$= \Big(\int\limits_{0}^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} + \frac{\Theta^{(1)}_{\alpha,r,p,n}}{p'-1}\Big(\frac{\alpha r}{\pi n^{1-r}}\Big)^{p'-1}, \quad |\Theta^{(1)}_{\alpha,r,p,n}|<2. \tag{110}$$

In the work [26] (see formula (27)) it was showed, that for arbitrary $1<p'<\infty$ the following equality takes place

$$\Big(\int\limits_{0}^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} = F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big). \tag{111}$$

Taking into account

$$\Big(\int\limits_{0}^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} \leq \Big(\int\limits_{0}^{\infty} \frac{dt}{(t^2+1)^{\frac{p'}{2}}}\Big)^{\frac{1}{p'}} < \Big(1+\int\limits_{1}^{\infty}\frac{dt}{t^{p'}}\Big)^{\frac{1}{p'}} < (p)^{\frac{1}{p'}}, \tag{112}$$

from formulas (108), (110), (111) and (112) we have

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,p})_C = e^{-\alpha n^r} n^{\frac{1-r}{p}}\Big(\frac{\|\cos t\|_{p'}}{\pi^{1+\frac{1}{p'}}(\alpha r)^{\frac{1}{p}}}F^{\frac{1}{p'}}\Big(\frac{1}{2},\frac{3-p'}{2};\frac{3}{2};1\Big)+$$

$$(113) \qquad +\gamma^{(3)}_{n,p}\Big(\frac{1}{p'-1}\frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{(p)^{\frac{1}{s}}}{(\alpha r)^{1+\frac{1}{p}}}\frac{1}{n^r} + \frac{1}{n^{\frac{1-r}{p}}}\Big)\Big),$$

where for quantity $\gamma^{(3)}_{n,p} = \gamma^{(3)}_{n,p}(\alpha, r, \beta)$ the inequality holds $|\gamma^{(3)}_{n,p}| \le (14\pi)^2$.

Since

$$\frac{1}{n^{\frac{1-r}{p}}} = \frac{1}{n^{(1-r)\frac{p'-1}{p'}}} > \frac{1}{n^{(1-r)(p'-1)}},$$

then

$$(114) \qquad \frac{1}{p'-1}\frac{(\alpha r)^{\frac{p'-1}{p}}}{n^{(1-r)(p'-1)}} + \frac{1}{n^{\frac{1-r}{p}}} \le \Big(1 + \frac{(\alpha r)^{\frac{p'-1}{p}}}{p'-1}\Big)\frac{1}{n^{\frac{1-r}{p}}}.$$

From formulas (113) and (114) we obtain (17).

Formula (18) can be obtained from the equality (16) as consequence of substitution $p = 1$ and elementary transformations. Theorem 2 is proved. □

Proof of the Theorem 3. From definitions (29) and (15) it follows that $n_1(\alpha, r) > n_0(\alpha, r, \infty)$. So, applying the equality (16) for $p = \infty$ $(p' = 1)$, we get for $n \ge n_1(\alpha, r)$

$$(115) \quad \mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = e^{-\alpha n^r}\Bigg(\frac{4}{\pi^2}\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}} + \gamma^{(1)}_{n,\infty}\Big(\frac{1}{\alpha r}\frac{1}{n^r}\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}} + 1\Big)\Bigg).$$

Since

$$\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}} = \int\limits_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{t} + \Bigg(\int\limits_0^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{\sqrt{t^2+1}} - \int\limits_1^{\frac{\pi n^{1-r}}{\alpha r}} \frac{dt}{t}\Bigg) =$$

$$(116) \qquad = \ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) + \Theta^{(3)}_{\alpha,r,n}, \quad 0 < \Theta^{(3)}_{\alpha,r,n} < 1,$$

by virtue of (115) and (116) for $n \ge n_1(\alpha, r)$

$$\mathcal{E}_n(C^{\alpha,r}_{\beta,\infty})_C = e^{-\alpha n^r}\Bigg(\frac{4}{\pi^2}\ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) + \frac{4}{\pi^2}\Theta^{(3)}_{\alpha,r,n} +$$

$$(117) \qquad +\gamma^{(1)}_{n,\infty}\Big(\frac{1}{\alpha r}\frac{1}{n^r}\ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) + \frac{\Theta^{(3)}_{\alpha,r,n}}{\alpha r n^r} + 1\Big)\Bigg).$$

The results of our calculations show that for $n \ge n_1(\alpha, r)$

$$(118) \qquad \frac{4}{\pi^2}\big(\ln\pi + \Theta^{(3)}_{\alpha,r,n}\big) + |\gamma^{(1)}_{n,\infty}|\Big(\frac{1}{\alpha r}\frac{1}{n^r}\ln\Big(\frac{\pi n^{1-r}}{\alpha r}\Big) + \frac{\Theta^{(3)}_{\alpha,r,n}}{\alpha r n^r} + 1\Big) \le 20\pi^4,$$

and therefore, in view of (117) and (118) we obtain (30).

Theorem 3 is proved. □

4 Proof of Lemmas 1-2

Proof of Lemma 1. It is obvious that for $1 \le s \le \infty$

$$\inf_{\lambda\in\mathbb{R}} \|\phi(t)-\lambda\|_s \le \|\phi\|_s,$$

$$\frac{1}{2}\|\phi\big(t+\frac{\pi}{n}\big)-\phi(t)\|_s \le \sup_{h\in\mathbb{R}} \frac{1}{2}\|\phi\big(t+h\big)-\phi(t)\|_s$$

and

$$\sup_{h\in\mathbb{R}} \frac{1}{2}\|\phi\big(t+h\big)-\phi(t)\|_s \le \inf_{\lambda\in\mathbb{R}} \|\phi(t)-\lambda\|_s.$$

Hence, in order to proof Lemma it suffices to verify the validity of formula (34) and relation

$$\frac{1}{2}\|\phi\big(t+\frac{\pi}{n}\big)-\phi(t)\|_s \ge \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - 14\pi\frac{M}{n}\Big). \tag{119}$$

First, we consider the case $1 \le s < \infty$. Let verify the validity of equality (34). Setting

$$\phi_k(t) = g\Big(\frac{k\pi}{n}\Big)\cos(nt+\gamma) + h\Big(\frac{k\pi}{n}\Big)\sin(nt+\gamma), \quad k=-n+1...n, \tag{120}$$

we get

$$\|\phi\|_s = \Big(\sum_{k=-n+1}^{n} \int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t)|^s dt\Big)^{\frac{1}{s}} = \Big(\sum_{k=-n+1}^{n} \int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi_k(t)|^s dt\Big)^{\frac{1}{s}} +$$

$$+\Theta_n^{(1)}\Big(\sum_{k=-n+1}^{n} \int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi(t)-\phi_k(t)|^s dt\Big)^{\frac{1}{s}}, \quad |\Theta_n^{(1)}| \le 1. \tag{121}$$

Let us find the estimate of first term in (121). It is obvious, that according to (120)

$$\Big(\sum_{k=-n+1}^{n} \int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} |\phi_k(t)|^s dt\Big)^{\frac{1}{s}} =$$

$$= \Big(\sum_{k=-n+1}^{n} r^s\Big(\frac{k\pi}{n}\Big) \int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} \Big| \cos\Big(nt+\gamma - \arg\Big(g\Big(\frac{k\pi}{n}\Big)+ih\Big(\frac{k\pi}{n}\Big)\Big)\Big)\Big|^s dt\Big)^{\frac{1}{s}} =$$

$$= \Big(\sum_{k=-n+1}^{n} r^s\Big(\frac{k\pi}{n}\Big)\frac{1}{n}\int\limits_0^{\pi} |\cos t|^s dt\Big)^{\frac{1}{s}} = \frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}}\Big(\sum_{k=-n+1}^{n} r^s\Big(\frac{k\pi}{n}\Big)\frac{\pi}{n}\Big)^{\frac{1}{s}}, \tag{122}$$

where $r(t)$ is defined by formula (31), and i is imaginary unit.

Let us show that for any collection of points ξ_k, $k=-n+1...n$, such that $\frac{(k-1)\pi}{n} \le \xi_k \le \frac{k\pi}{n}$, for $n \ge 4\pi s M$ the following estimate is true

$$\Big(\sum_{k=-n+1}^{n} r^s(\xi_k)\frac{\pi}{n}\Big)^{\frac{1}{s}} = \|r\|_s\Big(1+\Theta_n^{(2)}\frac{M}{n}\Big), \quad |\Theta_n^{(2)}| \le 4. \tag{123}$$

Indeed, since

$$\sum_{k=-n+1}^{n} r^s(\xi_k)\frac{\pi}{n} = \int\limits_{-\pi}^{\pi} r^s(t)dt + \Theta_n^{(3)}\frac{\bigvee\limits_{-\pi}^{\pi}(r^s)}{n}, \ |\Theta_n^{(3)}| \le \pi,$$

and under the condition

$$n \ge \frac{2\pi \bigvee\limits_{-\pi}^{\pi}(r^s)}{\|r\|_s^s} \tag{124}$$

$$\Big(\int\limits_{-\pi}^{\pi} r^s(t)dt + \Theta_n^{(3)}\frac{\bigvee\limits_{-\pi}^{\pi}(r^s)}{n}\Big)^{\frac{1}{s}} = \|r\|_s\Big(1+\Theta_n^{(4)}\frac{\bigvee\limits_{-\pi}^{\pi}(r^s)}{ns\|r\|_s^s}\Big), \quad |\Theta_n^{(4)}| \le 2, \tag{125}$$

hence

$$\Big(\sum_{k=-n+1}^{n} r^s(\xi_k)\frac{\pi}{n}\Big)^{\frac{1}{s}} = \|r\|_s\Big(1+\Theta_n^{(4)}\frac{\bigvee\limits_{-\pi}^{\pi}(r^s)}{ns\|r\|_s^s}\Big), \quad |\Theta_n^{(4)}| \le 2. \tag{126}$$

It is easy to verify that

$$\bigvee_{-\pi}^{\pi}(r^s) = s\int\limits_{-\pi}^{\pi} r^{s-1}(t)|r'(t)|dt \le s\|r\|_s^s\Big\|\frac{r'(t)}{r(t)}\Big\|_\infty, \tag{127}$$

$$\left|\frac{r'(t)}{r(t)}\right| = \left|\frac{g(t)g'(t)+h(t)h'(t)}{r^2(t)}\right| \leq \frac{|g'(t)|+|h'(t)|}{r(t)} \leq 2M, \quad t\in\mathbb{R}, \tag{128}$$

therefore

$$\frac{\bigvee\limits_{-\pi}^{\pi}(r^s)}{\|r\|_s^s} \leq s\left\|\frac{r'(t)}{r(t)}\right\|_\infty \leq 2sM. \tag{129}$$

By virtue of (129), for $n \geq 4\pi sM$ the condition (124) is satisfied. Therefore, according to (126), the estimate (123) takes place. Setting in (123) $\xi_k = \frac{k\pi}{n}$, $k=-n+1...n$, in view of (122) we obtain

$$\Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}|\phi_k(t)|^s dt\Big)^{\frac{1}{s}} = \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}}+\Theta_n^{(5)}\frac{M}{n}\Big), \quad |\Theta_n^{(5)}|\leq 4. \tag{130}$$

Let us find upper estimate of the second term in (121). On the basis of (33) and (120)

$$\phi(t)-\phi_k(t) =$$

$$= \Big(r(t)-r\Big(\frac{k\pi}{n}\Big)\Big)\Big(\frac{g(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\cos(nt+\gamma)+\frac{h(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\sin(nt+\gamma)\Big)+$$

$$+r(t)\Big(\Big(\frac{g(t)}{r(t)}-\frac{g(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big)\cos(nt+\gamma)+\Big(\frac{h(t)}{r(t)}-\frac{h(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big)\sin(nt+\gamma)\Big), \tag{131}$$

therefore

$$\Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}|\phi(t)-\phi_k(t)|^s dt\Big)^{\frac{1}{s}} \leq I_n^{(1)}+I_n^{(2)}, \tag{132}$$

where

$$I_n^{(1)} := \Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\Big|r(t)-r\Big(\frac{k\pi}{n}\Big)\Big|^s(|\cos(nt+\gamma)|+|\sin(nt+\gamma)|)^s dt\Big)^{\frac{1}{s}},$$

$$I_n^{(2)} := \Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}} r^s(t)\Big(\Big|\frac{g(t)}{r(t)}-\frac{g(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big||\cos(nt+\gamma)|+$$

$$+\Big|\frac{h(t)}{r(t)}-\frac{h(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big||\sin(nt+\gamma)|\Big)^{s}dt\Big)^{\frac{1}{s}}.$$

Using obvious inequality

$$|\cos t|+|\sin t|\le\sqrt{2}, \tag{133}$$

Lagrange theorem and relation (128), we have

$$I_n^{(1)}\le\sqrt{2}\Big(\sum_{k=-n+1}^{n}\max_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}}\Big|r(t)-r\Big(\frac{k\pi}{n}\Big)\Big|^{s}\frac{\pi}{n}\Big)^{\frac{1}{s}}\le$$

$$\le\frac{\sqrt{2}\pi}{n}\sup_{t\in\mathbb{R}}\Big|\frac{r'(t)}{r(t)}\Big|\Big(\sum_{k=-n+1}^{n}\max_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}}r^{s}(t)\frac{\pi}{n}\Big)^{\frac{1}{s}}. \tag{134}$$

It follows from (123), (128) and (134), that for $n\ge4\pi sM$

$$I_n^{(1)}\le2\sqrt{2}\pi\frac{M}{n}(1+4\frac{M}{n})\|r\|_s\le2\sqrt{2}\pi\frac{M}{n}\Big(1+\frac{1}{\pi}\Big)\|r\|_s=$$

$$=\frac{2\sqrt{2}M(1+\pi)}{n}\|r\|_s. \tag{135}$$

It is easy to see that

$$I_n^{(2)}\le\Big(\sum_{k=-n+1}^{n}\Big(\max_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}}\Big\{\Big|\frac{g(t)}{r(t)}-\frac{g(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big||\cos(nt+\gamma)|+$$

$$+\Big|\frac{h(t)}{r(t)}-\frac{h(\frac{k\pi}{n})}{r(\frac{k\pi}{n})}\Big||\sin(nt+\gamma)|\Big\}\Big)^{s}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}r^{s}(t)dt\Big)^{\frac{1}{s}}. \tag{136}$$

For any $t_1,\ t_2\in\mathbb{R}$ such that $|t_1-t_2|\le\frac{\pi}{n}$ the following inequalities take place

$$\Big|\frac{g(t_1)}{r(t_1)}-\frac{g(t_2)}{r(t_2)}\Big|\le\frac{3\pi M}{n}, \tag{137}$$

$$\Big|\frac{h(t_1)}{r(t_1)}-\frac{h(t_2)}{r(t_2)}\Big|\le\frac{3\pi M}{n}. \tag{138}$$

Indeed, by virtue of Lagrange theorem, taking into account (32) and (128), we have

$$\Big|\frac{g(t_1)}{r(t_1)}-\frac{g(t_2)}{r(t_2)}\Big|\le\frac{\pi}{n}\sup_{t\in\mathbb{R}}\Big|\frac{g'(t)r(t)-g(t)r'(t)}{r^2(t)}\Big|\le$$

$$\leq \frac{\pi}{n}\sup_{t\in\mathbb{R}}\frac{|g'(t)|}{r(t)}+\frac{\pi}{n}\sup_{t\in\mathbb{R}}\frac{|r'(t)|}{r(t)}\leq\frac{3\pi M}{n}. \tag{139}$$

By analogy, we prove the inequality (138). In view of (133), (137), (138) and (136) we obtain

$$I_n^{(2)}\leq\frac{3\sqrt{2}\pi M}{n}\|r\|_s,\quad n\in\mathbb{N}. \tag{140}$$

Combining (132), (135) and (140), we arrive at the estimate

$$\Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}|\phi(t)-\phi_k(t)|^s dt\Big)^{\frac{1}{s}}\leq\sqrt{2}(5\pi+2)\|r\|_s\frac{M}{n},\quad n\geq 4\pi sM. \tag{141}$$

By comparing estimates (121), (130) and (141) we conclude that for $n\geq 4\pi sM$

$$\|\phi\|_s=\|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}}+\delta^{(1)}_{s,n}\frac{M}{n}\Big),\ |\delta^{(1)}_{s,n}|\leq\sqrt{2}(5\pi+2)+4,\ 1\leq s<\infty. \tag{142}$$

Further, we prove the relation (119) for $1\leq s<\infty$. In view of definition (33)

$$|\phi\big(t+\frac{\pi}{n}\big)-\phi(t)|=$$

$$=\Big|2\phi(t)+g\Big(t+\frac{\pi}{n}\Big)\cos(nt+\gamma)+h\Big(t+\frac{\pi}{n}\Big)\sin(nt+\gamma)-$$

$$-(g(t)\cos(nt+\gamma)+h(t)\sin(nt+\gamma))\Big|=$$

$$=\Big|2\phi(t)+\Big(r\Big(t+\frac{\pi}{n}\Big)-r(t)\Big)\Big(\frac{g(t+\frac{\pi}{n})}{r(t+\frac{\pi}{n})}\cos(nt+\gamma)+\frac{h(t+\frac{\pi}{n})}{r(t+\frac{\pi}{n})}\sin(nt+\gamma)\Big)+$$

$$+r(t)\Big(\Big(\frac{g(t+\frac{\pi}{n})}{r(t+\frac{\pi}{n})}-\frac{g(t)}{r(t)}\Big)\cos(nt+\gamma)+\Big(\frac{h(t+\frac{\pi}{n})}{r(t+\frac{\pi}{n})}-\frac{h(t)}{r(t)}\Big)\sin(nt+\gamma)\Big)\Big|, \tag{143}$$

therefore for any $1\leq s\leq\infty$ by virtue of (133), (137) and (138), we get

$$\frac{1}{2}\|\phi\big(t+\frac{\pi}{n}\big)-\phi(t)\|_s\geq$$

$$\geq\|\phi\|_s-\frac{1}{\sqrt{2}}\Big(\Big\|r\Big(t+\frac{\pi}{n}\Big)-r(t)\Big\|_s+3\pi\|r\|_s\frac{M}{n}\Big). \tag{144}$$

By applying the Lagrange theorem, we obtain

$$\Big\|r\Big(t+\frac{\pi}{n}\Big)-r(t)\Big\|_s=\Big(\sum_{k=-n+1}^{n}\int\limits_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\Big|r\Big(t+\frac{\pi}{n}\Big)-r(t)\Big|^s dt\Big)^{\frac{1}{s}}\leq$$

$$\leq \Big(\sum_{k=-n+1}^{n} \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \Big| r\Big(t+\frac{\pi}{n}\Big) - r(t)\Big|^s \frac{\pi}{n}\Big)^{\frac{1}{s}} \leq$$

$$(145) \qquad \leq \frac{\pi}{n} \sup_{t\in\mathbb{R}} \Big|\frac{r'(t)}{r(t)}\Big| \Big(\sum_{k=-n+1}^{n} \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r^s(t)\frac{\pi}{n}\Big)^{\frac{1}{s}}, \ 1 \leq s < \infty.$$

It follows from (123), (128) and (145) that for $n \geq 4\pi sM$

$$(146) \qquad \Big\| r\Big(t+\frac{\pi}{n}\Big) - r(t)\Big\|_s \leq (2\pi+2)\|r\|_s \frac{M}{n}.$$

In view of (142), (144) and (146) for $n \geq 4\pi sM$ we arrive at the estimate

$$\frac{1}{2}\|\phi(t+\frac{\pi}{n}) - \phi(t)\|_s \geq \|\phi\|_s - \frac{5\pi+2}{\sqrt{2}}\|r\|_s\frac{M}{n} \geq$$

$$\geq \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - \Big(\frac{15\pi+6}{\sqrt{2}}+4\Big)\frac{M}{n}\Big) >$$

$$> \|r\|_s\Big(\frac{\|\cos t\|_s}{(2\pi)^{\frac{1}{s}}} - 14\pi\frac{M}{n}\Big), \ 1 \leq s < \infty.$$

Thus, the validity of formula (119) is established for $1 \leq s < \infty$.

Let us prove the relation (34) for $s = \infty$. Consider a function $\phi^*(t)$ such that

$$\phi^*(t) = \phi^*_k(t), \quad \frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}, \ k = -n+1...n,$$

where

$$(147) \qquad \phi^*_k(t) = g(t^*_k)\cos(nt+\gamma) + h(t^*_k)\sin(nt+\gamma),$$

and points t^*_k, $t^*_k \in [\frac{(k-1)\pi}{n}, \frac{k\pi}{n}]$ are chosen from the condition

$$r(t^*_k) = \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} r(t).$$

For the function $\phi^*(t)$ the following equality takes place

$$(148) \qquad \|\phi^*\|_\infty = \|r\|_C.$$

Indeed,

$$\|\phi^*\|_\infty = \max_{-n+1\leq k\leq n} \operatorname*{ess\,sup}_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} |\phi^*(t)| =$$

$$= \max_{-n+1\leq k\leq n} r\big(t^*_k\big) \max_{\frac{(k-1)\pi}{n} \leq t \leq \frac{k\pi}{n}} \Big|\frac{g(t^*_k)}{r(t^*_k)}\cos(nt+\gamma) + \frac{h(t^*_k)}{r(t^*_k)}\sin(nt+\gamma)\Big| =$$

$$= \max_{-n+1\le k\le n} r\big(t_k^*\big) \max_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}} \Big|\cos\Big(nt+\gamma-arg(g(t_k^*)+ih(t_k^*))\Big)\Big| =$$

$$= \max_{-n+1\le k\le n} r\big(t_k^*\big)\ \|\cos t\|_C = \|r\|_C.$$

It is obvious that in view of (148) we obtain

$$\|\phi\|_\infty = \|\phi^*\|_\infty + \Theta_n^{(6)}\|\phi-\phi^*\|_\infty = \|r\|_C + \Theta_n^{(6)}\|\phi-\phi^*\|_\infty,\quad |\Theta_n^{(6)}|\le 1. \tag{149}$$

Let us find upper estimate for the quantity $\|\phi-\phi^*\|_\infty$. By virtue of (33) and (147), for any $t\in[\frac{(k-1)\pi}{n},\frac{k\pi}{n}]$ the following equality takes place

$$|\phi(t)-\phi_k^*(t)| = \Big|(r(t)-r(t_k^*))\Big(\frac{g(t_k^*)}{r(t_k^*)}\cos(nt+\gamma)+\frac{h(t_k^*)}{r(t_k^*))}\sin(nt+\gamma)\Big)+$$

$$+r(t)\Big(\Big(\frac{g(t)}{r(t)}-\frac{g(t_k^*)}{r(t_k^*)}\Big)\cos(nt+\gamma)+\Big(\frac{h(t)}{r(t)}-\frac{h(t_k^*)}{r(t_k^*)}\Big)\sin(nt+\gamma)\Big)\Big|. \tag{150}$$

By using (133), the Lagrange theorem and inequality (128), we get

$$\operatorname*{ess\,sup}_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}} \Big|(r(t)-r(t_k^*))\Big(\frac{g(t_k^*)}{r(t_k^*)}\cos(nt+\gamma)+\frac{h(t_k^*)}{r(t_k^*))}\sin(nt+\gamma)\Big|\le$$

$$\le\sqrt{2}\operatorname*{ess\,sup}_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}}\Big|r(t)-r(t_k^*)\Big|\le\frac{\sqrt{2}\pi}{n}\sup_{t\in\mathbb{R}}\Big|\frac{r'(t)}{r(t)}\Big|\|r\|_C\le$$

$$\le\frac{2\sqrt{2}\pi M}{n}\|r\|_C. \tag{151}$$

Further, it follows from (133), (137) and (138) that

$$\operatorname*{ess\,sup}_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}} r(t)\Big(\Big|\frac{g(t)}{r(t)}-\frac{g(t_k^*)}{r(t_k^*)}\Big||\cos(nt+\gamma)|+\Big|\frac{h(t)}{r(t)}-\frac{h(t_k^*)}{r(t_k^*)}\Big||\sin(nt+\gamma)|\Big)\le$$

$$\le 3\sqrt{2}\pi\frac{M}{n}\|r\|_C. \tag{152}$$

In view of (150)–(152), we arrive at the estimate

$$\|\phi-\phi^*\|_\infty = \max_{-n+1\le k\le n}\ \operatorname*{ess\,sup}_{\frac{(k-1)\pi}{n}\le t\le\frac{k\pi}{n}} |\phi(t)-\phi_k^*(t)|\le 5\sqrt{2}\pi\frac{M}{n}\|r\|_C,\ n\in\mathbb{N}. \tag{153}$$

It follows from (149), (148) and (153) that

$$\|\phi\|_\infty = \|r\|_C\Big(1+\delta_{\infty,n}^{(1)}\frac{M}{n}\Big),\quad |\delta_{\infty,n}^{(1)}|\le 5\sqrt{2}\pi. \tag{154}$$

Let us prove inequality (119) for $s=\infty$. By using the inequality (144) for $s=\infty$, by applying Lagrange theorem, formulas (128) and (154), we obtain

$$\frac{1}{2}\|\phi(t+\frac{\pi}{n})-\phi(t)\|_\infty \geq$$
$$\geq \|\phi\|_\infty - \frac{1}{\sqrt{2}}\Big(\Big\|r\Big(t+\frac{\pi}{n}\Big)-r(t)\Big\|_\infty + 3\pi\|r\|_C\frac{M}{n}\Big) \geq$$
$$\geq \|\phi\|_\infty - \frac{1}{\sqrt{2}}\Big(\frac{\pi}{n}\sup_{t\in\mathbb{R}}\Big|\frac{r'(t)}{r(t)}\Big|\|r\|_C + 3\pi\|r\|_C\frac{M}{n}\Big) >$$
$$> \|r\|_C\Big(1-\frac{15\pi}{\sqrt{2}}\frac{M}{n}\Big).$$

Lemma 1 is proved. □

Remark 1. *In proof of Lemma 1 we established more exact, than (37) estimates of quantities* $\delta_{s,n}^{(i)}, i=1...3$. *Namely, we showed that for*

$$n \geq \begin{cases} 4\pi sM, & 1\leq s<\infty,\\ 1, & s=\infty,\end{cases}$$

the following estimates hold

$$|\delta_{s,n}^{(1)}| \leq \begin{cases} \sqrt{2}(5\pi+2)+4, & 1\leq s<\infty,\\ 5\sqrt{2}\pi, & s=\infty,\end{cases}$$

$$-\frac{15\pi+6}{\sqrt{2}}-4 \leq \delta_{s,n}^{(i)} \leq \sqrt{2}(5\pi+2)+4, \quad i=2,3,\ 1\leq s<\infty,$$

$$-\frac{15\pi}{\sqrt{2}} \leq \delta_{s,n}^{(i)} \leq 5\sqrt{2}\pi, \quad i=2,3,\ s=\infty.$$

Proof of Lemma 2. We use the scheme of the proof of the estimate (2.4.31) from the work [27, p. 93]. Let, e.g., consider the case $v>0$. Using the method of integration by parts, we have

$$\int_0^\infty \psi(\tau+u)\cos vu du = \frac{-1}{v}\int_0^\infty \psi'(\tau+u)\sin vu du. \tag{155}$$

We set

$$I(x)=I(\psi;\tau;v;x) := -\int_x^\infty \psi'(\tau+u)\sin vu du,\ x\geq 0,\ v>0,\ \tau\in\mathbb{N}.$$

The function $I(x)$, obviously, is continuous for every fixed v, and on every interval between the consecutive zeros $u_m = \frac{\pi m}{v}$ and $u_{m+1} = \frac{\pi(m+1)}{v}$ of the integrand has one simple zero x_m. Existance of zeros x_m of the function $I(x)$ is a consequence of the Leibniz theorem on alternating series, and uniqueness of zero x_m on the interval (u_m, u_{m+1}) follows from the equality

$$\operatorname{sign} I'(x) = -\operatorname{sign} \sin xv, \ \ x \in (u_m, u_{m+1}) \ m \in \mathbb{Z}_+.$$

Let x_0 be the zero closest from the right to the point $x = 0$. It is obvious that

$$0 \le x_0 \le \frac{\pi}{v}.$$

Taking into account this fact and also monotone decreasing of the function $-\psi'(t)$ on the interval $[1, \infty)$, we have

$$\frac{-1}{v} \int\limits_0^\infty \psi'(\tau + u) \sin vudu = \frac{1}{v} \int\limits_0^{x_0} |\psi'(\tau + u)| \sin vudu \le$$

$$\le \frac{1}{v} \int\limits_0^{\frac{\pi}{v}} |\psi'(\tau + u)| du \le \frac{\pi}{v^2} |\psi'(\tau)|. \tag{156}$$

For $v > 0$ inequality (39) follows from the formulas (155) and (156). For $v < 0$ the proof of inequality (39) is analogous. Lemma 2 is proved. $\square$

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