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High-dimensional algorithms—Tractability and componentwise constructions

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Abstract

In many applications, as for example physics, economics, finance and computational sciences, high-dimensional integration and approximation are problems which have to be solved numerically. In this thesis we study several aspects of high-dimensional solution algorithms for these problems.

In the first part of the thesis we consider tractability of multivariate continuous problems. This means that we are interested in how much information a numerical algorithm needs to solve the problem with accuracy ε. We study how fast the number of information evaluations required increases if the number of variables goes to infinity or the error demand ε tends to zero. We consider the two examples of a weighted Hermite space and of a hybrid function space.

In the second part of the thesis we investigate the problem of constructing point sets in the s-dimensional unit cube, which are used in a certain type of numerical algorithms, so-called quasi-Monte Carlo algorithms, which are widely used to numerically solve high-dimensional integration problems. We present several fast construction methods which provide point sets having certain good properties.

Kurzfassung

In vielen Anwendungen, etwa in der Physik, den Wirtschaftswissenschaften, der Finanzmathematik oder den Computerwissenschaften, sind multivariate Integration und Approximation Probleme, die häufig auftreten und die numerisch gelöst werden müssen. In der vorliegenden Arbeit betrachten wir diverse Aspekte hochdimensionaler Lösungsalgorithmen für diese Probleme.

Im ersten Teil der Arbeit studieren wir "Tractability" multivariater, stetiger Probleme. Das bedeutet, dass wir uns für das Ausmaß an Information interessieren, welches ein numerischer Algorithmus benötigt um ein gegebenes Problem mit Genauigkeit ε zu lösen. Wir untersuchen die Geschwindigkeit, mit der die benötigte Informationsmenge zunimmt, wenn die Anzahl der Variablen steigt oder die Fehlerschranke ε gegen Null konvergiert. Wir betrachten Tractability anhand der beiden Beispiele eines gewichteten Hermiteraums und eines gemischten Funktionenraums.

Im zweiten Teil der Arbeit betrachten wir das Problem, Punktmengen im s-dimensionalen Einheitsintervall zu konstruieren, welche in sogenannten quasi-Monte Carlo Algorithmen verwendet werden. Quasi-Monte Carlo Algorithmen sind spezielle numerische Algorithmen, die vielfach zur numerischen Lösung hochdimensionaler Integrationsprobleme verwendet werden. Wir analysieren mehrere schnelle Konstruktionsmethoden, welche Punktmengen mit bestimmten guten Eigenschaften liefern.

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List of Abbreviations

CBC	component-by-component
EC-PT	exponential convergence-polynomial tractability
EC-SPT	exponential convergence-strong polynomial tractability
EC-WT	exponential convergence-weak tractability
EXP	exponential convergence
FFT	fast Fourier transform
PT	polynomial tractability
QMC	quasi-Monte Carlo
SPT	strong polynomial tractability
UEXP	uniform exponential convergence
WT	weak tractability

1 Preface

High-dimensional algorithms are widely used in applications to physics, economics, finance, computational sciences and others, see also [55, 60, 63].

Thus high-dimensional algorithms are an extensively studied field, and particularly in finance the number of variables one has to consider can be extremely high or even infinite. In this thesis we want to cover mainly two aspects of the topic, namely tractability of multivariate problems and construction of (polynomial) lattice point sets.

Roughly speaking tractability theory studies how much “effort” one has to make to solve a problem with accuracy ε. In particular it investigates and measures how fast, i.e., with which rate, the amount of effort required increases, if the error demand ε tends to zero, or the dimension of the problem, that is the number of variables, goes to infinity. Tractability properties of several multivariate continuous problems are studied in Section 2 of this thesis.

(Polynomial) lattices are point sets in the s-dimensional unit cube. Such point sets turn out to be a good choice as sample points in so-called quasi-Monte Carlo (QMC) algorithms which are for instance used to numerically approximate the s-dimensional integral of some function f. Properties of lattice point sets influence the quality of the approximation. Thus one wants to have reliable methods at hand to construct lattice point sets with good properties. This is the content of Section 3 of this thesis.

The concepts of both, tractability of multivariate problems, and construction of (polynomial) lattice point sets, are explained in detail in the introductions of the respective Sections 2 and 3.

The content of this thesis is based on the following papers:

- R. Kritzinger, H. Laimer, A reduced fast component-by-component construction of lattice point sets with small weighted star discrepancy, Unif. Distrib. Theory. 10, No.2, (2015) 21–47.
- H. Laimer, On combined component-by-component constructions of lattice point sets, J. Complexity 38 (2017) 22–30.
- R. Kritzinger, H. Laimer, M. Neumüller, A reduced fast construction of polynomial lattice point sets with low weighted star discrepancy, Submitted for publication, 2017.
- P. Kritzer, H. Laimer, F. Pillichshammer, Tractability of $\mathbb{L}_2$-approximation in hybrid function spaces, to appear in Funct. Approx. Comment. Math., 2017.

Furthermore, Section 2.2 contains results which have not been submitted for publication until now:

- C. Irrgeher, P. Kritzer, H. Laimer, On standard tractability notions for integration in Hermite spaces of analytic functions, unpublished notes.

The rest of this thesis is organized as follows. At the beginning of Section 2 we introduce tractability theory in more detail, and subsequently consider two different problem settings and study their tractability properties. In Section 3 we move on to constructing generating vectors of (polynomial) lattice point sets. Here we investigate three different constructions to obtain generating vectors with several good properties. Finally in the Section 4 we briefly summarize the main results and give an outlook on possible future research topics.

2 Tractability

2.1 Introduction

Multivariate continuous problems, defined over function spaces of s variables, can only very seldomly be solved analytically. A multivariate continuous problem could for instance be the approximation of functions in s variables from some suitable function space $\mathcal{H}_s$, or numerical integration of such functions. These are the two main problems we investigate in the following.

Roughly speaking, the field of tractability studies how much information is needed to solve problems at least with a given accuracy. For more detailed information see also [63, 64, 65].

The founders of tractability theory are Joseph Traub, Grzegorz Wasilkowski and Henryk Woźniakowski. After they laid the foundations of this field of research many scientists stepped in, and today tractability theory is a very active field where a lot of research is done all around the world.

The subsequent introduction to tractability theory follows closely the comprehensive book [63] about the topic by Novak and Woźniakowski. In particular we use Chapters 4 and 5 of [63]. Other literature is cited explicitly in the text.

Suppose we have a function space $\mathcal{H}_s$ and further let $S\colon \mathcal{H}_s \to \mathcal{G}$ be some operator, where $\mathcal{G}$ is a normed space. We call S the solution operator. We denote the norm in $\mathcal{H}_s$ with $\|\cdot\|_{\mathcal{H}_s}$ and accordingly the norm in $\mathcal{G}$ with $\|\cdot\|_{\mathcal{G}}$. Similarly, throughout the rest of this thesis, if a norm is indexed with the name of a function space, then this norm is the norm corresponding to the space in question.

It might not be possible to compute $S(f)$ explicitly for $f \in \mathcal{H}_s$. The goal is now, for given $\varepsilon > 0$, to find an algorithm A such that $A(f)$ lies within an ε neighborhood of $S(f)$.

Such an algorithm A uses N pieces of information about f, say $L_0(f), \ldots, L_{N-1}(f)$, as input. That is, A is of the form $A(f) = \varphi(L_0(f), \ldots, L_{N-1}(f))$, where φ is some suitable function. The information $L_0(f), \ldots, L_{N-1}(f)$ usually stems from some class of information $\Lambda \subseteq \mathcal{H}_s^*$, where $\mathcal{H}_s^*$ denotes the dual space of $\mathcal{H}_s$, that is the space of all continuous linear functionals $L\colon \mathcal{H}_s \to \mathbb{K}$, where $\mathbb{K}$ is the underlying field. We distinguish between information from Λ^{all} and from Λ^{std}. Λ^{all} contains all continuous linear functionals in $\mathcal{H}_s^*$, i.e., $\Lambda^{\text{all}} = \mathcal{H}_s^*$, whereas Λ^{std} consists only of function evaluations. This means that for any $L_i \in \Lambda^{\text{std}}$ there exists some x_i such that $L_i(f) = f(x_i)$ for all $f \in \mathcal{H}_s$. In the following sections we will consider both, information from Λ^{std} and from Λ^{all}.

There is another aspect of information worth considering—we categorize whether we use adaptive or non-adaptive information. If the previously chosen pieces of information, $L_0(f), \ldots, L_d(f)$, are taken into account when choosing $L_{d+1}(f) = L_{d+1}(f, L_1(f), \ldots, L_d(f))$, we speak of adaptive or sequential information. If the pieces of information are independent of each other, and thus can be computed at the same time, we have non-adaptive or parallel information. Intuitively it seems to be beneficial to use adaptive information. It turns out, however, that there is almost no gain in using adaptive information, while it is clearly more costly to do so, rather than using non-adaptive information. More precisely Bahvalov proved in 1971 [2] that, given some linear functional as solution operator S and special linear functionals, e.g., function values from Λ^{std}, as information, there is no gain in using adaptive information. For some arbitrary linear solution operator S, however, one can obtain an advantage in using adaptive information, though only a small one. For sets of adaptive and non-adaptive information, N^{ada} and N^{non}, respectively, each consisting of N information evaluations, Gal and Micchelli [24] showed in 1980 that

$$\inf_{\varphi} \sup_{\substack{f\in\mathcal{H}_s \\ \|f\|_{\mathcal{H}_s}\le 1}} \|S(f)-\varphi(N^{\mathrm{non}}(f))\|_{\mathcal{G}} \le 2\inf_{\varphi} \sup_{\substack{f\in\mathcal{H}_s \\ \|f\|_{\mathcal{H}_s}\le 1}} \left\|S(f)-\varphi(N^{\mathrm{ada}}(f))\right\|_{\mathcal{G}}.$$

As explained before, $\|\cdot\|_{\mathcal{H}_s}$ and $\|\cdot\|_{\mathcal{G}}$ denote the norms in $\mathcal{H}_s$ and $\mathcal{G}$, respectively. Roughly speaking, the latter inequality illustrates that the best among all algorithms using non-adaptive information N^{non} evaluated at the function f performing worst of all functions in the unit ball of $\mathcal{H}_s$, is at most twice as bad as the best of all algorithms using adaptive information N^{ada} also evaluated at the worst function in $\mathcal{H}_s$.

This means that adaptive information is at most twice as good as non-adaptive information in this setting. So small a gain might not be worth the extra costs for choosing information adaptively.

Finally we quote one more result in this direction. Creutzig and Wojtaszczyk [6] proved in 2004 that if $S\colon \mathcal{H}_s \to \mathcal{G}$ is linear and if at least one of the conditions

- $\mathcal{G} = \mathbb{R}$,
- $\mathcal{G}$ is the L_∞ space with a measure μ,
- $\mathcal{G}$ is a set of bounded functions on some set K with sup-norm $\|\cdot\|_\infty$,
- S is compact and $\mathcal{G}$ is a set of continuous functions on a compact Hausdorff space K with sup-norm $\|\cdot\|_\infty$,
- $\mathcal{H}_s$ is a pre-Hilbert space,

holds, then

$$\inf_{\varphi} \sup_{\substack{f\in\mathcal{H}_s \\ \|f\|_{\mathcal{H}_s}\le 1}} \|S(f)-\varphi(N^{\mathrm{non}}(f))\|_{\mathcal{G}} \le \inf_{\varphi} \sup_{\substack{f\in\mathcal{H}_s \\ \|f\|_{\mathcal{H}_s}\le 1}} \left\|S(f)-\varphi(N^{\mathrm{ada}}(f))\right\|_{\mathcal{G}}.$$

Hence for a broad range of problems non-adaptive information is at least as good as adaptive information and thus, in this thesis, we only consider non-adaptive information. There exist, however, also many applications where adaption is of great advantage.

We aim at approximating $S(f)$ by $A(f)$ with an error smaller than ε. There are several possibilities to measure this error. Here we present the concept of the worst-case error criterion as this is the criterion we consider in this thesis. It is defined by

$$e^{\mathrm{wor}}_{\mathcal{H}_s}(A) = \sup_{\substack{f\in\mathcal{H}_s \\ \|f\|_{\mathcal{H}_s}\le 1}} \|S(f)-A(f)\|_{\mathcal{G}}. \tag{2.1}$$

Other ways to measure the error one makes when approximating $S(f)$ by $A(f)$, are for example by means of the average-case error, the randomized error or the error in a probabilistic setting as defined on p. 137 of [63]. For more information on different error types see for example [63].

As we do not study other error criteria we omit the superscript "wor" and write $e_{\mathcal{H}_s}(A)$ for the worst-case error instead of $e^{\mathrm{wor}}_{\mathcal{H}_s}(A)$.

Considering an algorithm A which uses no information at all we define the initial worst-case error

$$e_0 = \inf_{g \in \mathcal{G}} \sup_{\substack{f \in \mathcal{H}_s \\ \|f\|_{\mathcal{H}_s} \leq 1}} \|S(f) - g\|_{\mathcal{G}},$$

the smallest worst-case error that can be obtained by approximation with constant algorithms. We measure the quality of our algorithms A either in the normalized error criterion, that means we normalize the worst-case error by the initial worst-case error to $\frac{e_{\mathcal{H}_s}(A)}{e_0}$, or we consider the absolute error criterion which deals with the unnormalized worst-case error $e_{\mathcal{H}_s}(A)$. In all the cases studied in the next sections one can show that $e_0 = 1$, so the normalized and the absolute error criterion coincide. This, however, need not be true for other settings considered elsewhere.

One question we are interested in is how much effort one has to make to solve the problem with accuracy at least ε, that means to obtain a worst-case error that does not exceed ε. We measure this effort by the amount of information used in our algorithms A. To this end we write A_N for algorithms $A(f) = A_N(f) = \varphi(L_0(f), \dots, L_{N-1}(f))$ which use N pieces of information. With this notation we define the N-th minimal worst-case error $e_{\mathcal{H}_s}(N)$ as the smallest among all worst-case errors induced by such algorithms A_N. That is,

$$e_{\mathcal{H}_s}(N) = \inf_{A_N} e_{\mathcal{H}_s}(A_N), \tag{2.2}$$

where the infimum is taken over all admissible algorithms A_N. To clarify which class of information is considered one can write $e_{\mathcal{H}_s,\Lambda}(N)$. The normalized N-th minimal worst-case error is given by

$$\frac{1}{e_0} e_{\mathcal{H}_s}(N).$$

Using this notation, we define the information complexity $N_{\mathcal{H}_s}(\varepsilon)$ as the minimal number N such that there exists an algorithm A_N which uses N pieces of information and has a worst-case error of at most ε. Hence we have

$$N_{\mathcal{H}_s}(\varepsilon) = \min\{N \in \mathbb{N} \colon e_{\mathcal{H}_s}(N) \leq \varepsilon\} \tag{2.3}$$

for the absolute error criterion and

$$N_{\mathcal{H}_s}(\varepsilon) = \min\{N \in \mathbb{N} \colon e_{\mathcal{H}_s}(N) \leq \varepsilon e_0\}$$

for the normalized error criterion. If we need to clarify which class of information is used, we write $N_{\mathcal{H}_s,\Lambda}(\varepsilon)$ for the information complexity.

Note that the two notions of information complexity are the same when the initial error e_0 equals 1. This is the case in all settings we study in this thesis. If it is clear which s-variate function space $\mathcal{H}_s$ we are considering we will frequently replace $\mathcal{H}_s$ by s in the notation of the different notions introduced above. So, for example we write $e_s(A)$ for the worst-case error of algorithm A instead of $e_{\mathcal{H}_s}(A)$.

Tractability theory studies properties and behavior of the information complexity. As the definition of information complexity contains the notion of the N-th minimal worst-case error we start by studying the minimal worst-case error a bit further. We aim at narrowing down the number of algorithms we need to look at in order to compute

$$e^{\text{wor}}_{\mathcal{H}_s}(N) = \inf_{A_N} e_{\mathcal{H}_s}(A_N).$$

Definition 2.1. *Let $S \colon \mathcal{H}_s \to \mathcal{G}$ be a solution operator and let A_N be an algorithm which uses N pieces of information $L_0(.), \dots, L_{N-1}(.)$. A_N is called linear if it is of the form*

$$A_N(f) = \sum_{i=0}^{N-1} a_i L_i(f), \tag{2.4}$$

where $a_0, \dots, a_{N-1} \in \mathcal{G}$.

Smolyak proved in his PhD-thesis [76] in 1965 the following result which was first published by Bahvalov in [1]. Let $S\colon \mathcal{H}_s \to \mathbb{R}$ or $S\colon \widetilde{\mathcal{H}}_s \to \mathbb{R}$ be a linear solution operator, where $\widetilde{\mathcal{H}}_s$ is the unit ball of $\mathcal{H}_s$. Further let A_N^{ada} be an algorithm that uses adaptive information N^{ada}. Then there exists a linear algorithm A_N^{non} of the form

$$A_N^{\text{non}}(f) = \sum_{i=0}^{N-1} a_i L_i(f), \qquad a_0, \ldots, a_{N-1} \in \mathbb{R},$$

which uses non-adaptive information $N^{\text{non}} = [L_0(f), \ldots, L_{N-1}(f)]$, such that

$$e_{\mathcal{H}_s}(A_N^{\text{non}}) \leq e_{\mathcal{H}_s}(A_N^{\text{ada}}).$$

This means that for linear functionals S as solution operator, linear algorithms which use non-adaptive information are optimal. As before, for the question whether to use adaptive or non-adaptive information, there exists a result of Creutzig and Wojtaszczyk [6] from 2004 which states that under some mild conditions linear, non-adaptive algorithms are optimal also for arbitrary linear solution operators. The conditions required are the same as for the result of Creutzig and Wojtaszczyk on p. 5. These conditions are fulfilled for all function spaces we consider throughout the rest of this thesis. Thus, in all our settings we know that we can without loss of generality restrict ourselves to studying only linear algorithms which use non-adaptive information.

Now we are ready to define the different notions of tractability. We call s the dimension of the problem $S_s\colon \mathcal{H}_s \to \mathcal{G}$. Let ε be the error threshold within which we want to approximate the problem. Tractability describes how the information complexity depends on s and ε. For a sequence of problems $S = (S_s)_{s\geq 1}$ we consider the sequence $(N_{\mathcal{H}_s,\Lambda_s}(\varepsilon))_{s\geq 1}$ of their information complexities. Obviously, for growing dimension s and decreasing ε, the information complexity will grow. Tractability measures at what rate $N_{\mathcal{H}_s,\Lambda_s}(\varepsilon)$ grows.

Definition 2.2. *A sequence of problems $S_s\colon \mathcal{H}_s \to \mathcal{G}$ is called*

- *intractable for Λ_s if*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{\mathcal{H}_s,\Lambda_s}(\varepsilon)}{s+\varepsilon^{-1}} > 0,$$

- *weakly tractable for Λ_s if*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{\mathcal{H}_s,\Lambda_s}(\varepsilon)}{s+\varepsilon^{-1}} = 0,$$

- *polynomially tractable for Λ_s if there exist non-negative constants C, p and q such that*

$$N_{\mathcal{H}_s,\Lambda_s}(\varepsilon) \leq C\varepsilon^{-p}s^q \quad \textit{for all } s \in \mathbb{N} \textit{ and for all } \varepsilon \in (0,1), \tag{2.5}$$

- *strongly polynomially tractable for Λ_s if* (2.5) *holds with $q = 0$.*

Remark 2.3. *In the above definition we call the infimum of all p such that* (2.5) *holds with $q = 0$ the exponent of strong polynomial tractability.*

Definition 2.2 means that a problem is at least weakly tractable if the information complexity does not depend exponentially on s and ε^{-1}. Polynomial tractability implies that $N_{s,\Lambda_s}(\varepsilon)$ depends at most polynomially on s and ε^{-1} and strong polynomial tractability means at most polynomial dependence on ε^{-1} and independence of s.

The goal of the remainder of the chapter on tractability is to find out whether and under which conditions certain problems are tractable.

Next we want to have a look at a special, well-studied class of problems, for which we know optimal algorithms and are able to formulate criteria for the different tractability notions to hold. For detailed information see [63, Chapter 5].

We consider linear problems over Hilbert spaces, i.e., $\mathcal{H}_s$ is now assumed to be a Hilbert space.

Definition 2.4. *Let $S\colon \mathcal{H}_s \to \mathcal{G}$ or $S\colon \widetilde{\mathcal{H}_s} \to \mathcal{G}$, respectively. We call the approximation of $S(f)$ by algorithms A a linear problem, if*

1. *the operator S is linear, and $\mathcal{H}_s$ and $\mathcal{G}$ are normed spaces,*
2. *$\widetilde{\mathcal{H}_s}$ is a non-empty subset of $\mathcal{H}_s$,*
3. *$\widetilde{\mathcal{H}_s}$ is convex, i.e., $tf_1 + (1-t)f_2 \in \widetilde{\mathcal{H}_s}$ for all $t \in [0,1]$ as long as $f_1, f_2 \in \widetilde{\mathcal{H}_s}$,*
4. *$\widetilde{\mathcal{H}_s}$ is symmetric, i.e., $f \in \widetilde{\mathcal{H}_s}$ implies $-f \in \widetilde{\mathcal{H}_s}$, and*
5. *algorithms A use information from a class $\Lambda \subseteq \mathcal{H}_s^*$.*

Remark 2.5. *Definition 2.4 is valid for subsets $\widetilde{\mathcal{H}_s} \subseteq \mathcal{H}_s$ other than the unit ball as well. As we only ever consider the unit ball in this thesis which fulfills all the conditions on $\widetilde{\mathcal{H}_s}$ in the definition above, and as our algorithms use information from a class $\Lambda \subseteq \mathcal{H}_s^*$, the problems we consider are linear if S is a linear operator.*

Let $\mathcal{H}_s$ and $\mathcal{G}$ be Hilbert spaces and suppose that $S_s\colon \mathcal{H}_s \to \mathcal{G}$ or $S_s\colon \widetilde{\mathcal{H}_s} \to \mathcal{G}$ is a sequence of linear and compact operators. Here, by compact we mean that each bounded sequence $(\boldsymbol{x}_n)_{n\geq 1} \subseteq \mathcal{H}_s$ or $(\boldsymbol{x}_n)_{n\geq 1} \subseteq \widetilde{\mathcal{H}_s}$, respectively, has a subsequence $(\boldsymbol{x}_{n_k})_{k\geq 1}$ such that $(S_s(\boldsymbol{x}_{n_k}))_{k\geq 1}$ is convergent. Define the adjoint operator $S_s^*\colon \mathcal{G} \to \mathcal{H}_s$ by

$$\langle S_s(f), g\rangle_{\mathcal{G}} = \langle f, S_s^*(g)\rangle_{\mathcal{H}_s}, \text{ for all } f \in \mathcal{H}_s \text{ and all } g \in \mathcal{G},$$

where $\langle\cdot\rangle_{\mathcal{G}}$ and $\langle\cdot\rangle_{\mathcal{H}_s}$ denote the respective inner products of $\mathcal{G}$ and $\mathcal{H}_s$. Then we can define the compact, self-adjoint operator $W_s = S_s^* S_s\colon \mathcal{H}_s \to \mathcal{H}_s$ with eigenpairs $(\lambda_{s,j}, e_{s,j})$. All the eigenvalues $\lambda_{s,j}$ are non-negative reals, as S^*S is a positiv operator, and we can number them such that they are in non-increasing order. That means we have $\lambda_{s,1} \geq \lambda_{s,2} \geq \cdots \geq 0$, $W_s(e_{s,j}) = \lambda_{s,j} e_{s,j}$ and $\langle e_{s,i}, e_{s,j}\rangle_{\mathcal{H}_s} = \delta_{ij}$, where the latter property that the eigenvectors are orthonormal, stems from the spectral theorem for compact operators. In [63, Section 4.2.3] it is proved that within this setting the optimal algorithm A_N using N pieces of information from Λ^{all} is given by

$$A_N^{\text{opt}}(f) = \sum_{j=1}^{N} \langle f, e_{s,j}\rangle_{\mathcal{H}_s} S_s(e_{s,j})$$

and that we have

$$e_{s,\Lambda^{\text{all}}}(A_N^{\text{opt}}) = e_{s,\Lambda^{\text{all}}}(N) = \sqrt{\lambda_{N+1}}.$$

Further the following theorem is true. It is Theorem 5.1 in [63].

Theorem 2.6. *Suppose we have a sequence of linear and compact operators $S = (S_s)_{s\geq 1}$, $S_s\colon \mathcal{H}_s \to \mathcal{G}$, where $\mathcal{H}_s$ and $\mathcal{G}$ are Hilbert spaces. Consider further the absolute worst-case error criterion and information from Λ^{all}.*

- *The problem is polynomially tractable if and only if there exist positive constants C_1, τ and non-negative constants q_1, q_2 such that*

$$C_2 = \sup_{s\in\mathbb{N}} \left(\sum_{j=\lceil C_1 s^{q_1}\rceil}^{\infty} \lambda_{s,j}^{\tau} \right)^{\frac{1}{\tau}} s^{-q_2} < \infty. \tag{2.6}$$

- *If* (2.6) *holds, then*

$$N_s(\varepsilon) \leq (C_1 + C_2^{\tau}) s^{\max\{q_1, q_2\tau\}} \varepsilon^{-2\tau} \quad \text{for all } s \in \mathbb{N} \text{ and for all } \varepsilon \in (0,1].$$

- *The problem is strongly polynomially tractable, iff* (2.6) *holds with $q_1 = q_2 = 0$. The ε-exponent of strong polynomial tractability is then given by $p = \inf\{2\tau \colon \tau$ fulfills (2.6) with $q_1 = q_2 = 0\}$.*

Remark 2.7. *From Theorem 2.6 we know that in this setting the question whether we have (strong) polynomial tractability or not depends solely on the eigenvalues $\lambda_{s,j}$. When considering polynomial tractability we can neglect the behavior of a polynomial number in s of initial eigenvalues, as $\lambda_{s,1}, \dots, \lambda_{s,\lceil C_1 s^{q_1}\rceil - 1}$ do not appear in* (2.6)*. Similarly for strong polynomial tractability we can omit a constant number of initial eigenvalues.*

Similar criteria exist for normalized problems and for weak tractability. Criteria for weak tractability are usually rather complicated, though. Such criteria can for example be found in [63, Theorem 5.2, Theorem 5.3 and Lemma 5.4].

Next we consider another interesting setting, namely linear problems over tensor product spaces. All problems we consider in the following sections are of this type. Suppose we have $\widetilde{\mathcal{H}_1}$, the unit ball of some univariate Hilbert space $\mathcal{H}_1$; $\mathcal{G}_1$ another Hilbert space and $S_1 \colon \mathcal{H}_1 \to \mathcal{G}_1$ or $S_1 \colon \widetilde{\mathcal{H}_1} \to \mathcal{G}_1$ a linear and compact operator. So far, there is no difference to the setting above, except that we are considering strictly only univariate spaces. Thus, as before, we know the optimal algorithm A_N^{opt} using the eigenpairs of the self-adjoint operator W_1. Now build the s-fold tensor products, $\mathcal{H}_s$ and $\mathcal{G}_s$, of $\mathcal{H}_1$ and $\mathcal{G}_1$ and consider the linear and compact operator $S_s \colon \mathcal{H}_s \to \mathcal{G}_s$, given as the s-fold tensor product $S_s = S_1 \otimes \cdots \otimes S_1$. Recall that the eigenvalues and eigenvectors of the self-adjoint operator W_s are now of product structure. So, for $\boldsymbol{j} \in \mathbb{N}^s$ we have $\lambda_{s,\boldsymbol{j}} = \lambda_{1,j_1} \cdots \lambda_{1,j_s}$ for the eigenvalues and $e_{s,\boldsymbol{j}} = e_{1,j_1} \otimes \cdots \otimes e_{s,j_s}$ for the eigenvectors. With this we can write the optimal algorithm as

$$A_N^{\text{opt}}(f) = \sum_{\boldsymbol{j}\in\mathbb{N}^s} \langle f, e_{s,\boldsymbol{j}} \rangle_{\mathcal{H}_s} S_s(e_{s,\boldsymbol{j}}).$$

As before we can formulate criteria for the different tractability notions to hold which depend only on the eigenvalues of W_1. The theorem we want to quote in this direction is a part of the results proved in [63, Theorem 5.5].

Theorem 2.8. *Suppose we have a sequence $S = (S_s)_{s\geq 1}$ of linear tensor product problems as described above with $\lambda_{1,2} > 0$. Consider further the absolute worst-case error criterion and information from Λ^{all}. Then we have:*

- *For $\lambda_{1,1} > 1$ the problem is intractable.*
- *For $\lambda_{1,1} = 1$ the problem is polynomially intractable.*
  - *For $\lambda_{1,1} = \lambda_{1,2} = 1$ the problem is intractable.*
  - *If S is weakly tractable, then we have $\lambda_{1,2} < 1$ and $\lambda_{1,n} = o((\log n)^{-2})$ as $n \to \infty$.*

- *If $\lambda_{1,2} < 1$ and $\lambda_{1,n} = o((\log n)^{-2}(\log\log n)^{-2})$ as $n \to \infty$, then the problem is weakly tractable.*

- *For $\lambda_{1,1} < 1$ we have:*
  - *Weak tractability implies $\lambda_{1,n} = o((\log n)^{-2})$ as $n \to \infty$.*
  - *If $\lambda_{1,n} = o((\log n)^{-2}(\log\log n)^{-2})$ as $n \to \infty$, then the problem is weakly tractable.*
  - *Polynomial and strong polynomial tractability are equivalent and hold if and only if there exists some $r > 0$ such that $\lambda_{1,n} = O(n^{-r})$ as $n \to \infty$.*

Using these, and other, criteria one finds that many problems are intractable over almost all classical spaces. Despite these negative results, linear algorithms often yield very good numerical results even for large dimensions s. In 1998 Sloan and Woźniakowski [74] explained this as follows. For many problems coming from applications, variables and groups of variables do not all have the same influence on the problem. If we do not take into account these differences in the influence of the variables, the problem may seem to be intractable, while numerically the algorithms work well. Thus it can be beneficial to consider weighted spaces. Weights are designed according to the importance of each variable or group of variables.

Weights are described by a sequence of (non-negative) numbers $\boldsymbol{\gamma}$, occurring in the norm of the function space. This means that weights change the norm of the space and thus its unit ball. Considering problems over weighted spaces rather than over their unweighted versions, thus can make the problem easier, as we take the supremum over all function in the unit ball when calculating the worst-case error. The insight of Sloan and Woźniakowski was though, that these simplified problems are indeed often closer to reality than the problems considered over the classical spaces. This is due to the fact, that problems over weighted spaces arise quite naturally from many applications, such as finance or physics.

For some weighted space $\mathcal{H}_{s,\boldsymbol{\gamma}}$, we also indicate in the notation of the worst-case error and the information complexity etc. that we are using their weighted versions by using the subscript $\boldsymbol{\gamma}$. For example we write $e_{s,\boldsymbol{\gamma}}(A_N)$ for the worst-case error of algorithm A_N in the space $\mathcal{H}_{s,\boldsymbol{\gamma}}$. For all the other notions defined above we proceed analogously.

We distinguish between different types of weights $\boldsymbol{\gamma} = \{\gamma_{s,\mathfrak{u}}\}_{s\in\mathbb{N},\mathfrak{u}\subseteq[s]}$, where $[s] = \{1,\dots,s\}$. In this thesis we mostly consider product weights $\boldsymbol{\gamma} = \{\gamma_{s,j}\}_{s\in\mathbb{N},j\geq 1}$ with $0 < \gamma_{s,s} \leq \dots \leq \gamma_{s,1}$, where $\gamma_{s,\mathfrak{u}}$ is defined by

$$\gamma_{s,\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_{s,j}.$$

Product weights are ideal for problems where the influence of the variables decreases as their index j increases. The weight $\gamma_{s,j}$ describes the influence of the j-th variable.

Another type of weights are finite-order weights, where $\gamma_{s,\mathfrak{u}} = 0$ if $\mathfrak{u}$ contains more than w elements, where w is some non-negative integer. Finite-order weights are used for functions of the form

$$f = \sum_{\mathfrak{u}\subseteq[s]} f_{\mathfrak{u}},$$

where f is a sum of functions $f_{\mathfrak{u}}$, which depend on w variables at most. For more details and more different types of weights see [63, Section 5.3]. For many problems which suffer from the curse of dimensionality (that means, which are intractable) one can obtain tractability by introducing weights.

For fine-tuning purposes more precise tractability notions have been introduced. The following definition gives some examples. For further information, see for example [26, 69].

Definition 2.9. *A sequence of problems* $S_s\colon \mathcal{H}_s \to \mathcal{G}$ *is called*

- *quasi-polynomially tractable for* Λ_s *if there exist non-negative constants* C *and* t *such that*

$$N_{s,\Lambda_s}(\varepsilon) \le C \exp(t(1+\log s)(1+\log(\varepsilon^{-1}))) \quad \text{for all } s \in \mathbb{N} \text{ and for all } \varepsilon \in (0,1),$$

- *uniformly weakly tractable if*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{s,\Lambda_s}(\varepsilon)}{s^\alpha + \varepsilon^{-\beta}} = 0 \quad \text{for all } \alpha, \beta \in (0,1],$$

- (t_1, t_2)*-weakly tractable if there exist positive* t_1 *and* t_2 *such that*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_{s,\Lambda_s}(\varepsilon)}{s^{t_1} + \varepsilon^{-t_2}} = 0.$$

We use the following abbreviations.

Weak tractability	WT
(t_1, t_2)-weak tractability	(t_1, t_2)-WT
Uniform weak tractability	UWT
Quasi-polynomial tractability	QPT
Polynomial tractability	PT
Strong polynomial tractability	SPT

For $t_1, t_2 \in (0,1]$ one can easily prove that the following line of implications holds true:

$$\text{SPT} \Rightarrow \text{PT} \Rightarrow \text{QPT} \Rightarrow \text{UWT} \Rightarrow (t_1,t_2)\text{–WT} \Rightarrow \text{WT}.$$

In the following sections we consider different problems over different function spaces and aim at finding necessary and sufficient conditions for the different tractability notions to hold.

2.2 Hermite space

In this section we want to consider tractability of integration in weighted Hermite spaces. They have first been introduced by Irrgeher and Leobacher in [37]. After that Dick, Irrgeher, Kritzer, Leobacher, Pillichshammer and Woźniakowski have done further work in this direction [9, 35, 36].

One of the advantages of considering Hermite spaces is that they allow to tackle the problem of integration and approximation of functions defined on the $\mathbb{R}^s$, whereas many of the classical spaces consist of functions defined on the s-dimensional unit cube. Functions defined on the whole $\mathbb{R}^s$ naturally arise from many problems coming from applications, in particular those from mathematical finance. Although these problems can be transformed to ones on the unit cube, one often ends up with functions that do not belong to spaces for which tractability can be shown. For more detailed information see [35].

Let us start by defining the class of Hermite spaces. We begin by recalling definitions and results about standard Gaussian measure, Hermite polynomials and Hermite expansion, as done in [37].

Definition 2.10. *The Borel probability measure on the $\mathbb{R}^s$ with density $\varphi_s\colon \mathbb{R}^s \to \mathbb{R}$, given by*

$$\varphi_s(\boldsymbol{x}) = (2\pi)^{-\frac{s}{2}} \mathrm{e}^{-\frac{\boldsymbol{x}\cdot\boldsymbol{x}}{2}}$$

with respect to the s-dimensional Lebesgue measure is called the standard Gaussian measure. Here $\boldsymbol{x}\cdot\boldsymbol{y}$ denotes the usual dot product on the $\mathbb{R}^s$.

A measurable function $f\colon \mathbb{R}^s \to \mathbb{R}$ is called Gaussian square integrable if

$$\int_{\mathbb{R}^s} f(\boldsymbol{x})^2 \varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} < \infty.$$

The vector space of Gaussian square integrable functions is denoted by $\mathcal{L}^2(\mathbb{R}^s, \varphi_s)$.

Remark 2.11. *For simplicity we will frequently denote the univariate density function φ_1 by φ.*

The linear space $L^2(\mathbb{R}^s, \varphi_s)$ of all equivalence classes of Gaussian square integrable functions on the $\mathbb{R}^d$ forms a Hilbert space with inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^s,\varphi_s)} = \int_{\mathbb{R}^s} f(\boldsymbol{x}) g(\boldsymbol{x}) \varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Here we say that f and g are equivalent if $f = g$ almost everywhere.

Next we introduce multivariate Hermite polynomials. In the literature there are several related versions of the definition of Hermite polynomials. Here, as done in [37], we use the definition given in [5].

Definition 2.12. *For $k \in \mathbb{N}_0$ the k-th (univariate) Hermite polynomial is given by*

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \mathrm{e}^{\frac{x^2}{2}} \frac{\partial^k}{\partial x^k} \mathrm{e}^{-\frac{x^2}{2}}.$$

For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ *the* $\boldsymbol{k}$*-th Hermite polynomial is defined by*

$$H_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} H_{k_j}(x_j).$$

The (univariate) Hermite polynomials are the Gram-Schmidt orthonormalization of $1, x, x^2, \ldots$ with respect to the standard Gaussian measure. For example, the first three univariate Hermite polynomials are given by $H_0(x) = 1$, $H_1(x) = x$ and $H_2(x) = x^2 - 1$.

In [5, Lemma 1.3.2 and Corollary 1.3.3] Bogachev proves the following theorem.

Theorem 2.13. *The sequence of Hermite polynomials* $(H_{\boldsymbol{k}}(\boldsymbol{x}))_{\boldsymbol{k}\in\mathbb{N}_0^s}$ *forms an orthonormal basis of* $L^2(\mathbb{R}^s, \varphi_s)$*.*

Thus, for functions $f \in L^2(\mathbb{R}^s, \varphi_s)$, the Hermite series

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x}),$$

with Hermite coefficients

$$\hat{f}(\boldsymbol{k}) = \int_{\mathbb{R}^s} f(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x},$$

converges in the $L^2(\mathbb{R}^s, \varphi_s)$ norm.

The Cauchy-Schwarz inequality implies that

$$\int_{\mathbb{R}^s} |f(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \leq \left(\int_{\mathbb{R}^s} f(\boldsymbol{x})^2 \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^s} H_{\boldsymbol{k}}(\boldsymbol{x})^2 \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}} < \infty.$$

Hence, the $\boldsymbol{k}$-th Hermite coefficient exists for every $f \in L^2(\mathbb{R}^s, \varphi_s)$ and every $\boldsymbol{k} \in \mathbb{N}_0^s$ and one can show that the Hermite expansion is unique for continuous f. The Hermite expansion converges pointwise if, additionally, the Hermite coefficients are absolutely summable.

Theorem 2.14. *For continuous* $f \in L^2(\mathbb{R}^s, \varphi_s)$ *with* $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left|\hat{f}(\boldsymbol{k})\right| < \infty$*, we have*

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \hat{f}(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^s$*.*

The proof of this theorem can be found in [37, Proposition 2.6].

Now we are ready to define Hermite spaces as done in [37].

Definition 2.15. *Let* $r\colon \mathbb{N}_0^s \to \mathbb{R}^+$ *be summable, i.e.,* $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r(\boldsymbol{k}) < \infty$*. For* $f \in L^2(\mathbb{R}^s, \varphi_s)$ *let*

$$\|f\|_r = \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r(\boldsymbol{k})^{-1} \left|\hat{f}(\boldsymbol{k})\right|^2 \right)^{\frac{1}{2}}.$$

Then

$$\mathcal{H}_r = \{ f \in \mathcal{L}^2(\mathbb{R}^s, \varphi_s) \cap \mathcal{C}(\mathbb{R}^s) \colon \ \|f\|_r < \infty \}$$

is called a Hermite space.

On $\mathcal{L}^2(\mathbb{R}^s, \varphi_s)$, $\|\cdot\|_r$ is only a semi-norm, but if we consider only continuous functions f, it is a norm, see [37]. Then, a Hermite space $\mathcal{H}_r$ is a Hilbert space with inner product

$$\langle f, g\rangle_r = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r(\boldsymbol{k})^{-1}\hat{f}(\boldsymbol{k})\hat{g}(\boldsymbol{k}).$$

In a Hermite space, the Hermite expansion converges pointwise. Proof of this fact can be found in [37, Theorem 3.2]. Irrgeher and Leobacher also show that a Hermite space is a reproducing kernel Hilbert space with reproducing kernel $K_r\colon \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$,

$$K_r(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{y}).$$

Irrgeher and Leobacher in [37] move on to studying weighted Hermite spaces. They consider two examples, one with polynomially decaying Hermite coefficients and one with exponentially decaying coefficients. In the following we study a weighted Hermite space with exponentially decaying coefficients as well. This space was also studied by Irrgeher, Kritzer, Leobacher, Pillichshammer and Woźniakowski in [35, 36]. It is defined below.

We study standard notions of tractability of integration in a weighted Hermite space $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ of analytic functions, constructed as follows. Let $\boldsymbol{a} = (a_j)_{j\geq 0}$, $\boldsymbol{b} = (b_j)_{j\geq 1}$ be two weight sequences of real numbers, such that

$$a_0 = 0, \quad 1 \leq a_1 \leq a_2 \leq \dots \qquad \text{and} \qquad 1 \leq b_1 \leq b_2 \leq \dots . \tag{2.7}$$

Let $\omega \in (0, 1)$ and for any $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ let

$$\omega^{|\boldsymbol{k}|_{\boldsymbol{a},\boldsymbol{b}}} = \omega^{\sum\limits_{j=1}^{s} a_j k_j^{b_j}}.$$

We consider the reproducing kernel Hilbert space $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ with kernel

$$K_{s,\boldsymbol{a},\boldsymbol{b},\omega}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \omega^{|\boldsymbol{k}|_{\boldsymbol{a},\boldsymbol{b}}} H_{\boldsymbol{k}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{y})$$

and an inner product

$$\langle f, g\rangle_{H(K_{s,\alpha,\gamma})} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \omega^{-|\boldsymbol{k}|_{\boldsymbol{a},\boldsymbol{b}}} \hat{f}(\boldsymbol{k})\hat{g}(\boldsymbol{k}).$$

This is the weighted Hermite space with $r\colon \mathbb{N}_0^s \to \mathbb{R}^+$ given by

$$r(\boldsymbol{k}) = \omega^{\sum\limits_{j=1}^{s} a_j k^{b_j}}.$$

In [35] it is shown that the Hermite coefficients of this particular Hermite space are decreasing very fast. Furthermore, we achieve exponential convergence, which is defined as follows.

Definition 2.16. *If there exists some $q \in (0, 1)$ and functions $p, C_1, C_2\colon \mathbb{N} \to (0, \infty)$ such that for all $s, N \in \mathbb{N}$*

$$e_s(N) \leq C_1(s) q^{(N/C_2(s))^{p(s)}}, \tag{2.8}$$

we say that we achieve exponential convergence (EXP) for $e_s(N)$. If $p(s)$ from (2.8) can be taken as a constant function $p(s) = p$ for all $s \in \mathbb{N}$, we speak of uniform exponential convergence (UEXP).

For more information about (U)EXP we refer to [14, 35, 45, 46]. For problems with this nice behavior one can study exponential convergence-tractability (EC-tractability), as defined in [14, 17, 35, 45, 46].

Definition 2.17. *We speak of*

- *exponential convergence-weak tractability (EC-WT) if*

$$\lim_{s+\log \varepsilon^{-1}} \frac{\log N_s(\varepsilon)}{s+\log \varepsilon^{-1}} = 0 \text{ with the convention that } \log 0 = 0,$$

- *exponential convergence-polynomial tractability (EC-PT) if there exist constants* $c, \tau_1, \tau_2 \geq 0$ *such that*

$$N_s(\varepsilon) \leq c(1+\log \varepsilon^{-1})^{\tau_1} s^{\tau_2} \text{ for all } s \in \mathbb{N} \text{ and all } \varepsilon \in (0,1),$$

- *exponential convergence-strong polynomial tractability (EC-SPT) if the latter condition is true for* $\tau_2 = 0$.

Note that the difference to the standard tractability notions is that for EC-tractability we consider $\log \varepsilon^{-1}$ rather than ε^{-1}. It is possible to consider these more demanding notions, while still obtaining good results, because functions in this function space are very smooth and thus the error tends to zero very fast.

We are interested in integration,

$$I_s(f) = \int_{\mathbb{R}^s} f(\boldsymbol{x}) \varphi_s \boldsymbol{x} \, \mathrm{d}\boldsymbol{x},$$

of functions $f \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$. As integrals are linear functionals themselves, it is obviously only interesting to consider information from the class Λ^{std}. We know from a result of Creutzig and Wojtaszczyk [6] (cf. p. 7), that it is enough to consider linear algorithms

$$A_{n,s}(f) = \sum_{i=1}^{n} q_k f(\boldsymbol{x}_k),$$

with $q_k \in \mathbb{R}$ and $\boldsymbol{x}_k \in \mathbb{R}^s$, when numerically approximating the above integrals. For this setting, in [35, Theorem 1] the following was proven.

Theorem 2.18. *Consider integration over* $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ *with weight sequences* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, given as in* (2.7)*. Then we have:*

1. *EXP holds for all* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *satisfying* (2.7)*.*
2. *The following statements are equivalent:*
   (a) *The weight sequence* $\boldsymbol{b}$ *is summable, i.e.,* $\sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$,
   (b) *UEXP holds,*
   (c) *EC-PT holds,*
   (d) *EC-SPT holds.*
3. *A necessary condition for EC-WT is that* $\lim_{j\to\infty} a_j 2^{b_j} = \infty$.

4. A suffictient condition for EC-WT is that there exist positive constants η and β such that

$$a_j 2^{b_j} \geq \beta j^{1+\eta}$$

for all $j \in \mathbb{N}$.

What we are interested in now is what happens if we consider standard notions of tractability, although we are in a space of analytic functions. We study this for the integration problem as well as for the approximation problem in the Hermite space. The approximation problem is given as follows. We want to approximate the embedding operator EMB: $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \to L_2(\mathbb{R}^s, \varphi_s)$, $\mathrm{EMB}(f) = f$, by linear algorithms which use information from Λ^{std} or from Λ^{all}.

The hope is that necessary and sufficient conditions for the standard notions to hold are milder than they are for the EC-tractability notions. This is the content of the rest of this section. Unfortunately we were not able to prove necessary conditions for many of the standard tractability notions. The results of this section are yet unpublished and are joint work with Christian Irrgeher and Peter Kritzer [34].

2.2.1 Tractability of integration and approximation in $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$

To provide an overview of the known results on standard notions of tractability of integration as well as approximation in this particular Hermite space, we first present them in the following tables. In these tables, for conditions which contain a limit, we always assume that this limit exists. This assumption is mentioned in the theorems on the following pages, where the results presented in the tables are summarized, but we do not explicitly mention it in the tables to preserve good readability.

Approximation using Λ^{all}:
The following table summarizes the results of Theorems 2,3,4 and 5 of [36].

Tractability notion	sufficient conditions	necessary conditions
SPT	$\liminf_{j\to\infty} \frac{a_j}{\log j} > 0$	$\liminf_{j\to\infty} \frac{a_j}{\log j} > 0$
PT	$\liminf_{j\to\infty} \frac{a_j}{\log j} > 0$	$\liminf_{j\to\infty} \frac{a_j}{\log j} > 0$
QPT	no conditions	no conditions
UWT	no conditions	no conditions
(t_1, t_2)-WT	$t_1 > 1$	$t_1 > 1$
WT	no conditions	no conditions

Approximation using Λ^{std}:
The results outlined in the following table are summarized in Theorem 2.20. The proof stems from [34].

Tractability notion	sufficient conditions	necessary conditions
SPT	$\lim\limits_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}}$	$a_j 2^{b_j} \gtrsim \frac{\log j}{\log \omega^{-1}}$ or $\lim\limits_{j\to\infty} \frac{a_j}{\log j} > 0$
PT	$\frac{a_j}{\log j} \geq \frac{1}{\log \omega^{-1}}$ for all sufficiently large j	$\lim\limits_{j\to\infty} \frac{a_j}{\log j} > 0$
WT	$\lim\limits_{j\to\infty} a_j = \infty$	nothing known

Integration:
The results presented in the following table stem from [34]. They are summarized in Theorem 2.23.

Tractability notion	sufficient conditions	necessary conditions
SPT	$\lim\limits_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}}$ or $a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	$a_j 2^{b_j} \gtrsim \frac{\log j}{\log \omega^{-1}}$
PT	$\frac{a_j}{\log j} \geq \frac{1}{\log \omega^{-1}}$ for all sufficiently large j or $a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	nothing known
QPT	$\frac{a_j}{\log j} \geq \frac{1}{\log \omega^{-1}}$ for all sufficiently large j or $a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	nothing known
UWT	$\lim\limits_{j\to\infty} \frac{a_j}{\log j} \geq \frac{1}{\log \omega^{-1}}$ or $a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	nothing known
(t_1, t_2)-WT	$t_1 > 1$ or $t_1, t_2 \in (0, 1] \wedge a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	nothing known
WT	$\lim\limits_{j\to\infty} a_j = \infty$ or $a_j 2^{b_j} \geq \beta j^{1+\eta}$ for some $\beta > 0, \eta > 0$	nothing known

2.2.2 Approximation using Λ^{all} in Hermite spaces of analytic functions

Summarizing the conditions for the different tractability notions of approximation using the class Λ^{all} from the above table we have the following Theorem 2.19. Its content stems from Theorems 2,3,4 and 5 of [36].

Theorem 2.19. *Consider L_2-approximation using information from $\Lambda^{\rm all}$ defined over the Hermite space $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ introduced above. Then the following results hold:*

- *PT and SPT are equivalent.*
- *A sufficient and necessary condition for SPT is given by $\liminf_{j\to\infty} \frac{a_j}{\log j} > 0$.*
  If we have SPT, the exponent of SPT *is*
  $$\tau^*_{\rm all} = \frac{2}{A \log \omega^{-1}}$$
- *QPT, UWT, and* WT *hold for all considered $\boldsymbol{a}$ and $\boldsymbol{b}$.*
- *(t_1, t_2)-WT holds for $t_1 > 1$.*

2.2.3 Approximation using $\Lambda^{\rm std}$ in Hermite spaces of analytic functions

In this section we outline the known conditions for the different notions of tractability of the approximation problem using the class $\Lambda^{\rm std}$.

Theorem 2.20. *Consider L_2-approximation using information from $\Lambda^{\rm std}$ defined over $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$. Assume that*
$$A = \lim_{j\to\infty} \frac{a_j}{\log j}$$
exists. Then the following statements are true:

- SPT *holds if*
  $$\lim_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}}.$$
  In this case the exponent τ^* of SPT *satisfies*
  $$\tau^*_{\rm all} \le \tau^*_{\rm std} = \tau^*_{\rm all} + \frac{1}{2}\left(\tau^*_{\rm all}\right)^2 < \tau^*_{\rm all} + 2,$$
  where $\tau^*_{\rm all}$ and $\tau^*_{\rm std}$ denote the exponents of strong polynomial tractability for the cases of using information from $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$, respectively.

  Necessary conditions are
  $$a_j 2^{b_j} \gtrsim \frac{\log j}{\log \omega^{-1}} \qquad \text{and} \qquad \lim_{j\to\infty} \frac{a_j}{\log j} > 0,$$
  respectively. (Here, for two functions f and g we say $f \gtrsim g$, if there exists some constant $c > 0$ such that $f(x) \ge c g(x)$ for all x.)
- PT *holds if*
  $$\frac{a_j}{\log j} \ge \frac{1}{\log \omega^{-1}} \text{ for all sufficiently large } j.$$
  If we have PT*, then*
  $$\lim_{j\to\infty} \frac{a_j}{\log j} > 0.$$
- WT *holds if*
  $$\lim_{j\to\infty} a_j = \infty.$$

Proof. • First we consider SPT. Under the assumption that $A = \lim_{j\to\infty} \frac{a_j}{\log j}$ exists we know from [36] and Theorem 2.19, respectively that $A = \lim_{j\to\infty} \frac{a_j}{\log j} > 0$ implies SPT for approximation using Λ^{all}. The exponent of SPT is then

$$\tau^*_{\text{all}} = \frac{2}{A \log \omega^{-1}}.$$

This follows from [36, Theorem 5]. Here if $A > \frac{1}{\log \omega^{-1}}$, we have $\tau^*_{\text{all}} < 2$. Then [65, Theorem 26.20] implies that we also have SPT for Λ^{std} with

$$\tau^*_{\text{all}} \le \tau^*_{\text{std}} = \tau^*_{\text{all}} + \frac{1}{2}\left(\tau^*_{\text{all}}\right)^2 < \tau^*_{\text{all}} + 2.$$

Approximation in Λ^{all} is not harder than in Λ^{std}. According to Theorem 2.19, SPT and PT are equivalent for approximation in Λ^{all} and hence they have the same necessary condition, $\lim_{j\to\infty} \frac{a_j}{\log j}$. Integration is not harder than approximation in Λ^{std}. Therefore we get the same necessary condition, $a_j 2^{b_j} \gtrsim \frac{\log j}{\log \omega^{-1}}$, as for integration, cf. Theorem 2.23.

- To achieve the sufficient condition for PT we follow exactly the same lines as for the proof of [46, Theorem 5.2]. We can employ the same argumentation as for SPT to find the necessary condition for PT.

- The last to consider is the sufficient condition for WT which again is implied by [36, Theorem 7], where we have that $\lim_{j\to\infty} a_j = \infty$ implies EC-WT.

□

2.2.4 A lower bound

To obtain necessary conditions for strong polynomial tractability of integration we use some lower bound on the minimal worst-case error $e_s(n)$. In the following we provide this lower bound proceeding analogously to [46].

In order to establish this lower bound we will frequently apply Lemma 1 from [35] which is given by

Lemma 2.21. *Let $k, l \in \mathbb{N}_0$. Then*

$$\int_{\mathbb{R}} H_k(x) H_l(x) \varphi(x) \, \mathrm{d}x = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Further for $k, l, m \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} H_k(x) H_l(x) H_m(x) \varphi(x) \, \mathrm{d}x = \begin{cases} \frac{\sqrt{k!l!m!}}{(t-k)!(t-l)!(t-m)!} & \text{if } k + l + m = 2t \text{ and } k, l, m \le t \\ 0 & \text{otherwise.} \end{cases}$$

The lower bound we want to prove is stated in the following lemma.

Lemma 2.22. *The n-th minimal worst-case error of integration satisfies*

$$e_s(n) \ge \frac{1}{\sqrt{1 + 2\omega^{-a_s 2^{b_s}}}} \quad \text{for all} \quad n \le s \text{ and for all } s \in \mathbb{N}. \tag{2.9}$$

Proof. As said before, without loss of generality, we only consider linear algorithms

$$A_{n,s}(f) = \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k),$$

with $q_k \in \mathbb{R}$ and $\boldsymbol{x}_k \in \mathbb{R}^s$ to approximate the integral of f. Thus using the definition of the n-th minimal worst-case error we get

$$\begin{aligned} e_s(n) &= \inf_{\substack{q_k, \boldsymbol{x}_k, \\ k=1,\dots,n}} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \le 1}} \left| I_s(f) - \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k) \right| \\ &\ge \inf_{\substack{q_k, \boldsymbol{x}_k, \\ k=1,\dots,n}} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \le 1 \\ f(\boldsymbol{x}_k)=0, k=1,\dots,n}} \left| I_s(f) - \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k) \right| \\ &= \inf_{\boldsymbol{x}_k, k=1,\dots,n} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \le 1 \\ f(\boldsymbol{x}_k)=0, k=1,\dots,n}} |I_s(f)|. \end{aligned} \tag{2.10}$$

Our next goal is to try and find a function g such that $\frac{g}{\|g\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}}$ satisfies the conditions in the supremum above for further estimation of $e_s(n)$. To this end we define

$$\boldsymbol{h}^{(j)} = \begin{cases} (0,\dots,0) \in \mathbb{N}_0^s, & \text{if} \quad j = 0 \\ (0,\dots,0,1,0,\dots,0) \in \mathbb{N}_0^s, & \text{if} \quad j \in \{1,\dots,s\}, \end{cases}$$

where the component 1 in the vector above is meant to be on the j-th position.

For arbitrary $\boldsymbol{h} \in \mathbb{N}_0^s$, $\boldsymbol{h} = (h_1, \dots, h_s)$, let

$$c_{\boldsymbol{h}}(\boldsymbol{x}) = H_{\boldsymbol{h}}(\boldsymbol{x}) = \prod_{j=1}^{s} H_{h_j}(x_j) \text{ for all } \boldsymbol{x} = (x_1, \dots, x_s).$$

Clearly $c_{\boldsymbol{h}} \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ for all $\boldsymbol{h} \in \mathbb{N}_0^s$. For the vectors $\boldsymbol{h}^{(j)}$ defined above we obtain

$$c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) = \begin{cases} \prod_{j=1}^{s} H_0(x_j) = 1, & \text{if } j = 0 \\ H_1(x_j) = x_j, & \text{if } j \in \{1,\dots,s\}. \end{cases}$$

We proceed by choosing arbitrary $\boldsymbol{x}_1, \dots, \boldsymbol{x}_s \in \mathbb{R}^s$ and constructing the following auxiliary function $\tilde{g}$ according to the subsequent rule. Let

$$\tilde{g}(\boldsymbol{x}) = \sum_{j=0}^{s} \alpha_j c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^s,$$

where we define the α_j's such that

$$\tilde{g}(\boldsymbol{x}_k) = 0 \text{ for all } k \in \{1,\dots,s\}$$

and

$$\sum_{j=0}^{s} \alpha_j^2 = 1.$$

To do so, we have to solve a system of s homogeneous linear equations in $s+1$ unknowns, which is always possible. With the aid of $\tilde{g}$ we are able to define our desired function g by

$$g(\boldsymbol{x}) = (\tilde{g}(\boldsymbol{x}))^2 = \left(\sum_{j=0}^{s} \alpha_j c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x})\right)\left(\sum_{j=0}^{s} \alpha_j c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x})\right) = \sum_{j,k=0}^{s} \alpha_j \alpha_k c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k)}}(\boldsymbol{x}).$$

Obviously $g(\boldsymbol{x}_k) = 0$ for all $k = 1, \ldots, s$. We calculate $I_s(g)$ as follows, using Lemma 2.21: For $j \neq k$ and $j, k \neq 0$, we have, assuming without loss of generality $j < k$,

$$\begin{aligned}
I_s(c_{\boldsymbol{h}^{(j)}} c_{\boldsymbol{h}^{(k)}}) &= \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k)}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}} \varphi(x_1) \,\mathrm{d}x_1 \cdots \int_{\mathbb{R}} \varphi(x_{j-1}) \,\mathrm{d}x_{j-1} \int_{\mathbb{R}} H_1(x_j) \varphi(x_j) \,\mathrm{d}x_j \int_{\mathbb{R}} \varphi(x_{j+1}) \,\mathrm{d}x_{j+1} \cdots \\
&\quad \cdots \int_{\mathbb{R}} \varphi(x_{k-1}) \,\mathrm{d}x_{k-1} \int_{\mathbb{R}} H_1(x_k) \varphi(x_k) \,\mathrm{d}x_k \int_{\mathbb{R}} \varphi(x_{k+1}) \,\mathrm{d}x_{k+1} \cdots \int_{\mathbb{R}} \varphi(x_s) \,\mathrm{d}x_s = 0,
\end{aligned}$$

whereas for $j = k \neq 0$ we obtain

$$\begin{aligned}
I_s(c_{\boldsymbol{h}^{(j)}} c_{\boldsymbol{h}^{(j)}}) &= \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}} \varphi(x_1) \,\mathrm{d}x_1 \cdots \int_{\mathbb{R}} \varphi(x_{j-1}) \,\mathrm{d}x_{j-1} \int_{\mathbb{R}} H_1(x_j) H_1(x_j) \varphi(x_j) \,\mathrm{d}x_j \int_{\mathbb{R}} \varphi(x_{j+1}) \,\mathrm{d}x_{j+1} \cdots \\
&\quad \cdots \int_{\mathbb{R}} \varphi(x_s) \,\mathrm{d}x_s = 1.
\end{aligned}$$

Similar considerations for j and/or k equal to zero yield

$$I_s(c_{\boldsymbol{h}^{(j)}} c_{\boldsymbol{h}^{(k)}}) = \begin{cases} 0 \text{ if } j \neq k, \\ 1 \text{ if } j = k. \end{cases}$$

Thus

$$I_s(g) = \int_{\mathbb{R}^s} \sum_{j,k=0}^{s} \alpha_j \alpha_k c_{\boldsymbol{h}^{(j)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k)}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \sum_{j=0}^{s} \alpha_j^2 = 1$$

and we conclude from (2.10) that

$$e_s(s) \geq \inf_{\boldsymbol{x}_k, k=1,\ldots,s} \sup_{\substack{f \in H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \\ \|f\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \leq 1 \\ f(\boldsymbol{x}_k)=0, k=1,\ldots,s}} |I_s(f)| \geq I_s\left(\frac{g}{\|g\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}}\right) = \frac{1}{\|g\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}}. \tag{2.11}$$

To find our desired lower bound we thus have to calculate the norm of g.

$$\begin{aligned}
\|g\|^2_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= \langle g, g \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = \left\langle \sum_{j,k=0}^{s} \alpha_j \alpha_k c_{\boldsymbol{h}^{(j)}} c_{\boldsymbol{h}^{(k)}}, \sum_{j,k=0}^{s} \alpha_j \alpha_k c_{\boldsymbol{h}^{(j)}} c_{\boldsymbol{h}^{(k)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{j_1,k_1,j_2,k_2=0}^{s} \alpha_{j_1} \alpha_{k_1} \alpha_{j_2} \alpha_{k_2} \left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}.
\end{aligned} \tag{2.12}$$

Let us first investigate $\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}$.

$$
\begin{aligned}
\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= \sum_{\boldsymbol{l}\in\mathbb{N}_0^s} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} (\widehat{c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}})(\boldsymbol{l}) (\widehat{c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}}})(\boldsymbol{l}) \\
= \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\dots,l^{(s)})}} & \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j_1)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k_1)}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j_2)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k_2)}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \\
= \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\dots,l^{(s)})}} & \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j_1)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k_1)}}(\boldsymbol{x}) \prod_{m=1}^{s} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}\boldsymbol{x} \\
& \times \int_{\mathbb{R}^s} c_{\boldsymbol{h}^{(j_2)}}(\boldsymbol{x}) c_{\boldsymbol{h}^{(k_2)}}(\boldsymbol{x}) \prod_{n=1}^{s} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}\boldsymbol{x}.
\end{aligned} \tag{2.13}
$$

We distinguish the following four basic cases and split each of them into further subcases.

1. $j_1 \neq k_1$, $j_2 \neq k_2$,
2. $j_1 \neq k_1$, $j_2 = k_2$,
3. $j_1 = k_1$, $j_2 \neq k_2$,
4. $j_1 = k_1$, $j_2 = k_2$.

Case 1: $j_1 \neq k_1$, $j_2 \neq k_2$.
Subcase 1.1: We further assume $j_1, k_1, j_2, k_2 \neq 0$. Then (2.13) yields

$$
\begin{aligned}
&\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\dots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m\neq j_1 \\ m\neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{j_1}) H_{l^{(j_1)}}(x_{j_1}) \varphi(x_{j_1}) \,\mathrm{d}x_{j_1} \\
&\qquad \times \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \,\mathrm{d}x_{k_1} \left(\prod_{\substack{n=1 \\ n\neq j_2 \\ n\neq k_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) \\
&\qquad \times \int_{\mathbb{R}} H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \,\mathrm{d}x_{j_2} \int_{\mathbb{R}} H_1(x_{k_2}) H_{l^{(k_2)}}(x_{k_2}) \varphi(x_{k_2}) \,\mathrm{d}x_{k_2}.
\end{aligned} \tag{2.14}
$$

For any $\boldsymbol{l} \in \mathbb{N}^s$, the corresponding addend in the above sum reduces to zero, unless all its factors are simultaneously not equal to zero. That is we only have to consider those $\boldsymbol{l} = (l^{(1)}, \dots, l^{(s)}) \in \mathbb{N}^s$, which simultaneously satisfy

$$
\begin{cases}
l^{(m)} = 0, \forall m \in \{1, \dots, s\} \setminus \{j_1, k_1\} \\
l^{(m)} = 0, \forall m \in \{1, \dots, s\} \setminus \{j_2, k_2\} \\
l^{(j_1)} = 1 \\
l^{(k_1)} = 1 \\
l^{(j_2)} = 1 \\
l^{(k_2)} = 1.
\end{cases}
$$

This leaves the only possibilities $j_1 = j_2, k_1 = k_2$ or $j_1 = k_2, k_1 = j_2$, respectively, and $\boldsymbol{l}$ of the form $\boldsymbol{l} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$. Inserting this into (2.14) yields

$$\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = \omega^{-a_{j_1}-a_{k_1}} \text{ for } j_1 \neq k_1, j_2 \neq k_2 \text{ and all } j_1, k_1, j_2, k_2 \neq 0.$$

Subcase 1.2: We still have the assumption $j_1 \neq k_1$, $j_2 \neq k_2$ from Case 1. Now we consider the case where exactly one of the four indices equals zero. We can without loss of generality assume $j_1 = 0$. This leads to

$$\begin{aligned}
&\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l} = (l^{(1)}, \dots, l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \,\mathrm{d}x_{k_1} \\
&\qquad \times \left(\prod_{\substack{n=1 \\ n \neq j_2 \\ n \neq k_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \,\mathrm{d}x_{j_2} \\
&\qquad \times \int_{\mathbb{R}} H_1(x_{k_2}) H_{l^{(k_2)}}(x_{k_2}) \varphi(x_{k_2}) \,\mathrm{d}x_{k_2}.
\end{aligned}$$

Using basically the same argumentation as above we find that this sum equals zero, since some factors of the summands need $\boldsymbol{l} = (0, \dots, 0, 1, 0, \dots, 0)$ in order not to vanish, whereas others require $\boldsymbol{l}$'s with two components equal to 1, which is not simultaneously possible.

Subcase 1.3: We now investigate the case where two indices are equal to zero. Again without loss of generality we can assume them to be j_1 and j_2. Then we have

$$\begin{aligned}
&\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l} = (l^{(1)}, \dots, l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \,\mathrm{d}x_{k_1} \\
&\qquad \times \left(\prod_{\substack{n=1 \\ n \neq k_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{k_2}) H_{l^{(k_2)}}(x_{k_2}) \varphi(x_{k_2}) \,\mathrm{d}x_{k_2}.
\end{aligned}$$

Clearly, we obtain the only non-vanishing summand considering $k_1 = k_2$ and $\boldsymbol{l} = (0, \dots, 0, 1, 0, \dots, 0)$. As $j_1 = 0$ and $a_0 = 0$ (2.7) we write

$$\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = \omega^{-a_{j_1}-a_{k_1}} \text{ for all } j_1 \neq k_1, j_2 \neq k_2, \text{ with } k_1 = k_2 \text{ and } j_1 = j_2 = 0.$$

With that we are done with Case 1 as it requires $j_1 \neq k_1$ and $j_2 \neq k_2$ which obviously cannot be true with more than two indices being equal to zero. We move on to

Case 2: $j_1 \neq k_1$, $j_2 = k_2$.

Subcase 2.1: We further assume $j_1, k_1, j_2, k_2 \neq 0$. Then (2.13) yields

$$\begin{aligned}
&\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq j_1 \\ m \neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \, \mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{j_1}) H_{l^{(j_1)}}(x_{j_1}) \varphi(x_{j_1}) \, \mathrm{d}x_{j_1} \\
&\quad \times \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \, \mathrm{d}x_{k_1} \left(\prod_{\substack{n=1 \\ n \neq j_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \, \mathrm{d}x_n \right) \\
&\quad \times \int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \, \mathrm{d}x_{j_2}.
\end{aligned}$$

This sum equals zero, as can be seen from a similar argumentation as in Subcase 1.2.

Subcase 2.2: Let now $j_1, k_1 \neq 0$, and $j_2 = k_2 = 0$. Then we have

$$\begin{aligned}
&\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq j_1 \\ m \neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \, \mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{j_1}) H_{l^{(j_1)}}(x_{j_1}) \varphi(x_{j_1}) \, \mathrm{d}x_{j_1} \\
&\quad \times \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \, \mathrm{d}x_{k_1} \left(\prod_{n=1}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \, \mathrm{d}x_n \right).
\end{aligned}$$

With an analogous argument as above we see that this sum equals zero as well.

Subcase 2.3: Assume $j_1 = 0$, $j_2 = k_2 \neq 0$. Then

$$\begin{aligned}
&\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq k_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \, \mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{k_1}) H_{l^{(k_1)}}(x_{k_1}) \varphi(x_{k_1}) \, \mathrm{d}x_{k_1} \\
&\quad \times \left(\prod_{\substack{n=1 \\ n \neq j_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \, \mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \, \mathrm{d}x_{j_2}.
\end{aligned}$$

We first investigate the last factor $\int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \, \mathrm{d}x_{j_2}$ of this sum. Taking into account Lemma 2.21 this integral vanishes, except when $2 + l^{(j_2)} = 2t$ and $1, l^{(j_2)} \leq t$. Thus we have $t = 1 + \frac{l^{(j_2)}}{2} \leq 1 + \frac{t}{2}$, or equivalently $t \leq 2$ and consequently $l^{(j_2)} \leq 2$. As $2 + l^{(j_2)}$ has to be even, $l^{(j_2)}$ cannot be 1 and we can once again use an analogous argument as in Subcase 1.2 to find that the above sum equals zero.

Note that the case where $k_1 = 0$, $j_2 = k_2 \neq 0$ works completely analogously to Subcase 2.3.

Subcase 2.4: Let $j_1 = 0$, $j_2 = k_2 = 0$. We can once again argue in the same way to obtain

$$\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = 0.$$

Case 3: $j_1 = k_1$, $j_2 \neq k_2$. This case can be treated entirely analogously to Case 2.

Case 4: $j_1 = k_1$, $j_2 = k_2$.
Subcase 4.1: Let $j_1 \neq j_2$ and $j_1, k_1, j_2, k_2 \neq 0$. Then

$$\begin{aligned}
&\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m \neq j_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{j_1}) H_1(x_{j_1}) H_{l^{(j_1)}}(x_{j_1}) \varphi(x_{j_1}) \,\mathrm{d}x_{j_1} \\
&\qquad \times \left(\prod_{\substack{n=1 \\ n \neq j_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \,\mathrm{d}x_{j_2}.
\end{aligned}$$

As $j_1 \neq j_2$ all the addends vanish except for $\boldsymbol{l} = \boldsymbol{0}$ and thus the sum equals 1.

Subcase 4.2: Assume $j_1 \neq j_2$ and $j_1 = 0$. We then have

$$\begin{aligned}
\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{m=1}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \\
&\times \left(\prod_{\substack{n=1 \\ n \neq j_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2}) \varphi(x_{j_2}) \,\mathrm{d}x_{j_2},
\end{aligned}$$

which again is equal to 1, as only the addend for $\boldsymbol{l} = \boldsymbol{0}$ is non-zero. Of course the case where $j_1 \neq j_2$ and $j_2 = 0$ works altogether analogously.

Subcase 4.3: Let $j_1 = k_1 = j_2 = k_2 = 0$. Then

$$\begin{aligned}
\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= \sum_{\substack{\boldsymbol{l} \in \mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\ldots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{m=1}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m) \varphi(x_m) \,\mathrm{d}x_m \right) \\
&\times \left(\prod_{n=1}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n) \varphi(x_n) \,\mathrm{d}x_n \right) = 1,
\end{aligned}$$

again, as only the addend for $\boldsymbol{l} = \boldsymbol{0}$ is non-zero.

Subcase 4.4: Assume $j_1 = k_1 = j_2 = k_2 \neq 0$. Then

$$\begin{aligned}
&\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= \sum_{\substack{\boldsymbol{l}\in\mathbb{N}_0^s \\ \boldsymbol{l}=(l^{(1)},\dots,l^{(s)})}} \omega^{-|\boldsymbol{l}|_{\boldsymbol{a},\boldsymbol{b}}} \left(\prod_{\substack{m=1 \\ m\neq j_1}}^{s} \int_{\mathbb{R}} H_{l^{(m)}}(x_m)\varphi(x_m)\,\mathrm{d}x_m \right) \int_{\mathbb{R}} H_1(x_{j_1}) H_1(x_{j_1}) H_{l^{(j_1)}}(x_{j_1})\varphi(x_{j_1})\,\mathrm{d}x_{j_1} \\
&\qquad \times \left(\prod_{\substack{n=1 \\ n\neq j_2}}^{s} \int_{\mathbb{R}} H_{l^{(n)}}(x_n)\varphi(x_n)\,\mathrm{d}x_n \right) \int_{\mathbb{R}} H_1(x_{j_2}) H_1(x_{j_2}) H_{l^{(j_2)}}(x_{j_2})\varphi(x_{j_2})\,\mathrm{d}x_{j_2}.
\end{aligned}$$

With an analogous argumentation as in Subcase 2.3 we find that all addends vanish, except for $\boldsymbol{l} = (0,\dots,0)$ and $\boldsymbol{l} = (0,\dots 0,2,0\dots,0)$. Consequently, using Lemma 2.21,

$$\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = 1 + \sqrt{2}\sqrt{2}\omega^{-|(0,\dots 0,2,0\dots,0)|_{\boldsymbol{a},\boldsymbol{b}}} = 1 + 2\omega^{-a_{j_1}2^{b_{j_1}}}.$$

The above case analysis summarizes to

$$\left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} = \begin{cases} \omega^{-a_{j_1}-a_{k_1}}, & \text{if } j_1 \neq k_1, j_1 = j_2 \text{ and } k_1 = k_2 \\ & \text{or } j_1 \neq k_1, j_1 = k_2 \text{ and } k_1 = j_2, \text{ respectively} \\ 1, & \text{if } j_1 = k_1, j_2 = k_2 \text{ and } j_1 \neq j_2 \\ 1, & \text{if } j_1 = k_1 = j_2 = k_2 = 0 \\ 1 + 2\omega^{-a_{j_1}2^{b_{j_1}}}, & \text{if } j_1 = k_1 = j_2 = k_2 \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Inserting this into (2.12) we find the following. Note that, as we now only need two indices any more, we switch from using j_1, k_1, j_2, k_2 to using only j, k from the first to the second line of the following equation.

$$\begin{aligned}
\|g\|^2_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= \sum_{j_1,k_1,j_2,k_2=0}^{s} \alpha_{j_1}\alpha_{k_1}\alpha_{j_2}\alpha_{k_2} \left\langle c_{\boldsymbol{h}^{(j_1)}} c_{\boldsymbol{h}^{(k_1)}}, c_{\boldsymbol{h}^{(j_2)}} c_{\boldsymbol{h}^{(k_2)}} \right\rangle_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \\
&= 2\sum_{j=0}^{s}\sum_{\substack{k=0\\k\neq j}}^{s} \alpha_j^2\alpha_k^2\omega^{-a_j-a_k} + \sum_{j=0}^{s}\sum_{\substack{k=0\\k\neq j}}^{s} \alpha_j^2\alpha_k^2 + \sum_{j=1}^{s}\alpha_j^4\left(1 + 2\omega^{-a_j 2^{b_j}}\right) + \alpha_0^4.
\end{aligned} \tag{2.15}$$

Studying the latter expression we find

$$\begin{aligned}
\|g\|^2_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &= 2\sum_{j=0}^{s}\sum_{\substack{k=0\\k\neq j}}^{s} \alpha_j^2\alpha_k^2\omega^{-a_j-a_k} + \sum_{j=0}^{s}\sum_{\substack{k=0\\k\neq j}}^{s} \alpha_j^2\alpha_k^2 + \sum_{j=0}^{s}\alpha_j^4 + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-a_j 2^{b_j}} \\
&= 2\sum_{j=0}^{s}\sum_{\substack{k=0\\k\neq j}}^{s} \alpha_j^2\alpha_k^2\omega^{-a_j-a_k} + \sum_{j=0}^{s}\alpha_j^2\left(\alpha_j^2 + \sum_{\substack{k=0\\k\neq j}}^{s}\alpha_k^2\right) + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-a_j 2^{b_j}} \\
&= 2\sum_{j=0}^{s}\alpha_j^2\omega^{-a_j}\left(-\alpha_j^2\omega^{-a_j} + \sum_{k=0}^{s}\alpha_k^2\omega^{-a_k}\right) + 1 + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-a_j 2^{b_j}}
\end{aligned}$$

$$= 2\left(\sum_{j=0}^{s}\alpha_j^2\omega^{-a_j}\right)^2 + 1 - 2\alpha_0^4\omega^{-2a_0} + 2\sum_{j=1}^{s}\left(\alpha_j^4\omega^{-a_j 2^{b_j}} - \alpha_j^4\omega^{-2a_j}\right)$$

$$= 2\left(\sum_{j=0}^{s}\alpha_j^2\omega^{-a_j}\right)^2 + 1 - 2\alpha_0^4 + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-2a_j}\left(\omega^{-a_j 2^{b_j}+2a_j} - 1\right). \tag{2.16}$$

To estimate the latter expression we have to be sure that $\omega^{-a_j 2^{b_j}+2a_j} - 1 \geq 0$ in the above sum. We sum only over j's which are greater than or equal 1, therefore the corresponding a_j's and b_j's are not smaller than 1 as well and hence

$$2^{b_j-1} \geq 1 \Leftrightarrow a_j 2^{b_j} \geq 2a_j \Leftrightarrow a_j 2^{b_j} - 2a_j \geq 0 \Leftrightarrow 1 \geq \omega^{a_j 2^{b_j}-2a_j} \Leftrightarrow$$
$$\Leftrightarrow \omega^{-a_j 2^{b_j}+2a_j} \geq 1 \Leftrightarrow \omega^{-a_j 2^{b_j}+2a_j} - 1 \geq 0.$$

Using this fact and (2.16) we bound $\|g\|^2_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})}$ by

$$\begin{aligned}
\|g\|^2_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} &\leq 2\left(\sum_{j=0}^{s}\alpha_j^2\omega^{-a_s}\right)^2 + 1 - 2\alpha_0^4 + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-2a_s}\left(\omega^{-a_j 2^{b_j}+2a_j} - 1\right)\\
&= 2\left(\sum_{j=0}^{s}\alpha_j^2\omega^{-a_s}\right)^2 + 1 - 2\alpha_0^4 + 2\sum_{j=1}^{s}\alpha_j^4\omega^{-2a_s}\left(\omega^{-a_j(2^{b_j}-2)} - 1\right)\\
&\leq 2\omega^{-2a_s}\left(\sum_{j=0}^{s}\alpha_j^2\right)^2 + 1 - 2\alpha_0^4 + 2\omega^{-2a_s}\left(\omega^{-a_s(2^{b_s}-2)} - 1\right)\sum_{j=1}^{s}\alpha_j^4\\
&\leq 2\omega^{-2a_s} + 1 + 2\omega^{-2a_s}\left(\omega^{-a_s(2^{b_s}-2)} - 1\right)\\
&= 1 + 2\omega^{-a_s 2^{b_s}},
\end{aligned}$$

where we used that $2^{b_j} - 2 \geq 0$ to proceed from the second line of the equation to the third.

Thus

$$\|g\|_{H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})} \leq \sqrt{1 + 2\omega^{-a_s 2^{b_s}}}$$

and with (2.11)

$$e_s(n) \geq e_s(s) \geq \frac{1}{\sqrt{1 + 2\omega^{-a_s 2^{b_s}}}} \text{ for all } n \leq s \text{ and for all } s \in \mathbb{N}.$$

Here we obtain $e_s(n) \geq e_s(s)$ for $n \leq s$ by assuming $q_{n+1} = \cdots = q_s = 0$ in the infimum in (2.10).

This completes the proof. □

2.2.5 Integration in Hermite spaces of analytic functions

Finally we outline the conditions for tractability of integration.

Here we use the notation "$\gtrsim$" several times, by which we indicate that an inequality holds up to constants.

Theorem 2.23. *Consider intergration defined over the Hermite space $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ with weight sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ and assume that*

$$A = \lim_{j\to\infty}\frac{a_j}{\log j}$$

exists.

- SPT *holds if*

$$A = \lim_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}} \qquad \textit{or} \qquad a_j 2^{b_j} \geq \beta j^{1+\eta} \textit{ for some } \beta > 0, \eta > 0.$$

In this case the exponent τ^* *of* SPT *satisfies*

$$\tau^* \leq \max\left\{1, \min\left\{2, \frac{2}{A \log \omega^{-1}}\left(1 + \frac{1}{A\log\omega^{-1}}\right)\right\}\right\}.$$

A necessary condition, on the other hand, is

$$a_j 2^{b_j} \gtrsim \frac{\log j}{\log \omega^{-1}}.$$

- PT *as well as* QPT *hold if*

$$\frac{a_j}{\log j} \geq \frac{1}{\log \omega^{-1}} \textit{ for all sufficiently large } j$$

or

$$a_j 2^{b_j} \geq \beta j^{1+\eta} \textit{ for some } \beta > 0, \eta > 0.$$

- UWT *holds if*

$$\lim_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}} \qquad \textit{or} \qquad a_j 2^{b_j} \geq \beta j^{1+\eta} \textit{ for some } \beta > 0, \eta > 0.$$

- (t_1, t_2)-WT *is achieved for all weight sequences* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *as long as*

$$t_1 > 1.$$

Assuming $t_1, t_2 \in (0,1]$ *we again have the sufficient condition*

$$a_j 2^{b_j} \geq \beta j^{1+\eta} \textit{ for some } \beta > 0, \eta > 0.$$

- WT *holds if*

$$\lim_{j\to\infty} a_j = \infty \qquad \textit{or} \qquad a_j 2^{b_j} \geq \beta j^{1+\eta} \textit{ for some } \beta, \eta > 0.$$

Remark 2.24. *In the theorem above the sufficient condition "*$a_j 2^{b_j} \geq \beta j^{1+\eta}$ *for some* $\beta > 0, \eta > 0$*" can be replaced by "*$2^{b_j} \geq \beta j^{1+\eta}$ *for some* $\beta > 0, \eta > 0$*" for every considered tractability notion. This is due to our assumption that* $a_j \geq 1$ *for all* $j \geq 1$*.*

Proof. As we can apply [63, Theorem 5.2] and [65, Theorem 26.11] we proceed analogously to the proof of [46, Theorem 4.2] and [46, Theorem 5.2], respectively to obtain the first sufficient condition for SPT, and the upper bound on the exponent of SPT.

The second sufficient condition is proved in Theorem 2.25 below. This implies the second sufficient condition for all tractability notions considered in Theorem 2.23.

To establish the necessary condition for SPT we proceed as follows. Assume SPT with exponent τ^*, i.e.

$$\forall \delta > 0 \; \exists C_\delta > 0 \text{ such that } n(\varepsilon, s) \leq C_\delta \, \varepsilon^{-(\tau^*+\delta)} \text{ for all } \varepsilon \in (0,1) \text{ and for all } s \in \mathbb{N}.$$

Defining $n = \left\lfloor C_\delta\, \varepsilon^{-(\tau^*+\delta)} \right\rfloor$ yields $e_s(n) \leq \varepsilon$ for all $s \in \mathbb{N}$. Next we apply Lemma 2.22 for $s = n$ and obtain

$$\frac{1}{\sqrt{1 + 2\omega^{-a_s 2^{b_s}}}} \leq e_s(s) \leq \varepsilon,$$

which leads to

$$\frac{1}{2}\left(1 - \varepsilon^2\right)\omega^{a_s 2^{b_s}} \leq \varepsilon^2 \quad \text{for all} \quad \varepsilon \in (0,1).$$

Taking logarithms we find

$$\log\frac{1}{2} + \log\left(1 - \varepsilon^2\right) - a_s 2^{b_s}\log\omega^{-1} \leq -\log\varepsilon^{-2},$$

thus

$$\begin{aligned} a_s 2^{b_s} &\geq \frac{-\log 2 + \log\left(1-\varepsilon^2\right) + \log\varepsilon^{-2}}{\log\omega^{-1}} \\ &= \frac{\log\left(\varepsilon^{-2} - 1\right) - \log 2}{\log\omega^{-1}} \\ &\approx \frac{2\log\varepsilon^{-1} - \log 2}{\log\omega^{-1}} \end{aligned}$$

for sufficiently small $\varepsilon \in (0,1)$ and for all $s \in \mathbb{N}$. As in the proof of Theorem 4.2 in [46] we find

$$\log\varepsilon^{-1} = \frac{1 + o(1)}{\tau^* + \delta}\log s$$

and hence

$$a_s 2^{b_s} \gtrsim \frac{\log s}{\log\omega^{-1}},$$

as claimed.

Now we consider PT and WT. We have already seen, that the eigenvalues of

$$W = \mathrm{EMB}^*\mathrm{EMB} : H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \to H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$$

are the same as for the corresponding operator in the Korobov space. Thus we proceed once more analogously as in the proofs of [46, Theorem 4.2] and [46, Theorem 5.2], respectively to establish the first sufficient condition for PT as well as the condition for WT.
As PT implies QPT we have also established the first sufficient condition for QPT. Of course the first sufficient condition for UWT is also clear by now, as PT implies UWT as well. Nonetheless we want to briefly state an alternative proof. The following technique also yields the first sufficient condition for (t_1, t_2)-WT as well as an alternative method to obtain the first sufficient condition for WT.

Once again we use the fact, that linear integration rules

$$A_{n,s}(f) = \sum_{k=1}^{n} q_k f(\boldsymbol{x}_k),$$

where $q_k \in \mathbb{R}$ and $\boldsymbol{x}_k \in \mathbb{R}^s$ for $k = 1, \ldots, n$ are optimal for our integration problem. Then, using arguments as in [75, Equation (3)], [16, Theorem 3.5] or [20, Proposition 2.11], we know that the worst case error for $A_{n,s}$ is given by

$$e_s(A_{n,s})^2 = 1 - 2\sum_{k=1}^{n} q_k + \sum_{k,i=1}^{n} q_k q_i K_{s,\boldsymbol{a},\boldsymbol{b},\omega}(\boldsymbol{x}_k, \boldsymbol{x}_i)$$

or, in the special case, where the weights sum up to 1,

$$e_s(A_{n,s})^2 = -1 + \sum_{k,i=1}^{n} q_k q_i K_{s,\boldsymbol{a},\boldsymbol{b},\omega}(\boldsymbol{x}_k, \boldsymbol{x}_i),$$

respectively. Let the Gaussian-weighted mean-square error be defined as

$$\bar{e}_s(A_{n,s})^2 = \int_{\mathbb{R}^s} \cdots \int_{\mathbb{R}^s} [e_s(A_{n,s})(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)]^2 \varphi(\boldsymbol{x}_1) \cdots \varphi(\boldsymbol{x}_n) \, \mathrm{d}\boldsymbol{x}_1 \cdots \mathrm{d}\boldsymbol{x}_n,$$

where $e_s(A_{n,s})(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ indicates the worst case error of $A_{n,s}$ which uses integration nodes $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$. If we restrict ourselves to quasi-Monte Carlo (QMC) algorithms, i.e. $q_k = 1/n$, $k = 1, \dots, n$, (see Section 3.1 for detailed information) we get the following formula,

$$\bar{e}_s(A_{n,s})^2 = \frac{1}{n} \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \omega^{|\boldsymbol{k}|_{\boldsymbol{a},\boldsymbol{b}}} = \frac{1}{n} \prod_{j=1}^{s} \sum_{k=0}^{\infty} \omega^{a_j k^{b_j}}.$$

Thus we can estimate the nth minimal worst case error by

$$\begin{aligned} e_s(n)^2 &\leq \frac{1}{n} \prod_{j=1}^{s} \sum_{k=0}^{\infty} \omega^{a_j k^{b_j}} \\ &\leq \frac{1}{n} \prod_{j=1}^{s} \sum_{k=0}^{\infty} \omega^{a_j k} \\ &= \frac{1}{n} \prod_{j=1}^{s} \frac{1}{1 - \omega^{a_j}}. \end{aligned}$$

From this we derive upper bound

$$N_s(\varepsilon) \leq \left\lceil \varepsilon^{-2} \prod_{j=1}^{s} \frac{1}{1 - \omega^{a_j}} \right\rceil$$

on the information complexity and consequently,

$$\log N_s(\varepsilon) \leq 2 \log \varepsilon^{-1} + \sum_{j=1}^{s} \log \frac{1}{1 - \omega^{a_j}} + c.$$

We use this upper bound to obtain sufficient conditions for UWT, (t_1, t_2)-WT and WT, starting with UWT. Let $t_1, t_2 \in (0, 1]$ and further assume that $\lim_{j \to \infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}}$. So there exists an index $j_0 \geq 2$ such that $\omega^{a_j} < 1/j$ for all $j \geq j_0$. Then

$$\begin{aligned} \lim_{s+\varepsilon^{-1} \to \infty} \frac{\log N_s(\varepsilon)}{s^{t_1} + \varepsilon^{-t_2}} &\leq \lim_{\varepsilon^{-1} \to \infty} \frac{2 \log \varepsilon^{-1}}{\varepsilon^{-t_2}} + \lim_{s \to \infty} \sum_{j=1}^{\min\{s, j_0 - 1\}} \frac{\log (1 - \omega^{a_j})^{-1}}{s^{t_1}} \\ &\quad + \lim_{s \to \infty} \sum_{j=\min\{s+1, j_0\}}^{s} \frac{\log (1 - \omega^{a_j})^{-1}}{s^{t_1}} \\ &\leq \lim_{s \to \infty} \frac{1}{s^{t_1}} \sum_{j=\min\{s+1, j_0\}}^{s} \log \frac{1}{1 - 1/j} \\ &= \lim_{s \to \infty} \frac{1}{s^{t_1}} \sum_{j=\min\{s+1, j_0\}}^{s} \log \frac{j}{j - 1} \end{aligned}$$

$$\leq \lim_{s\to\infty} \frac{1}{s^{t_1}} \sum_{j=2}^{s} (\log j - \log(j-1))$$
$$= \lim_{s\to\infty} \frac{1}{s^{t_1}} \log s = 0$$

We have shown that UWT holds, if $\lim_{j\to\infty} \frac{a_j}{\log j} > \frac{1}{\log \omega^{-1}}$.

As for (t_1, t_2)-WT, assume that $t_1 > 1$. Then

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_s(\varepsilon)}{s^{t_1} + \varepsilon^{-t_2}} \leq \lim_{\varepsilon^{-1}\to\infty} \frac{2\log\varepsilon^{-1}}{\varepsilon^{-t_2}} + \lim_{s\to\infty} \sum_{j=1}^{s} \frac{\log(1-\omega^{a_j})^{-1}}{s^{t_1}}$$
$$\leq \lim_{s\to\infty} \sum_{j=1}^{s} \frac{\log(1-\omega^{a_1})^{-1}}{s^{t_1}}$$
$$= \lim_{s\to\infty} \log(1-\omega^{a_1})^{-1} \frac{1}{s^{t_1-1}} = 0.$$

Therefore, if $t_1 > 1$, (t_1, t_2)-weak tractability holds for arbitrary weight sequences $\boldsymbol{a}$ and $\boldsymbol{b}$, satisfying (2.7).

Finally we consider again WT. Assume that $\lim_{j\to\infty} a_j = \infty$. It follows that $\omega^{a_j} \to 0$ as well as $\log(1-\omega^{a_j})^{-1} \to 0$ as $j \to \infty$. Thus, from Cauchy's limit theorem we know that the Cesàro means converge to zero as well. Hence, $\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} \log(1-\omega^{a_j})^{-1} = 0$. Therefore,

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log N_s(\varepsilon)}{s+\varepsilon^{-1}} \leq \lim_{\varepsilon^{-1}\to\infty} \frac{2\log\varepsilon^{-1}}{\varepsilon^{-1}} + \lim_{s\to\infty} \sum_{j=1}^{s} \frac{\log(1-\omega^{a_j})^{-1}}{s} = 0.$$

Thus we have weak tractability, if $\lim_{j\to\infty} a_j = \infty$ and $\boldsymbol{b}$ arbitrary and the proof is complete. □

In the following theorem we show the second sufficient condition for SPT, given in Theorem 2.23.

Theorem 2.25. *Assume that there exist $\eta, \beta > 0$ such that*

$$a_j 2^{b_j} \geq \beta j^{1+\eta} \text{ for all } j \in \mathbb{N}.$$

Then we have SPT *with exponent $\tau^* \leq 1$, and there exists an explicit Gauss-Hermite rule achieving the corresponding error.*

Proof. Assume that the condition in the theorem is satisfied. Then in the proof of Item 4 of Theorem 1 in [35], an explicit Gauss-Hermite rule $A_{n,s}$ using n points was given such that

$$e_s(A_{n,s}) \leq \varepsilon$$

and

$$\log n \leq c_1 (\log\varepsilon^{-1})^{\frac{1}{1+\eta}} \left(2 + \log\left(\frac{c_2 + 2\log\varepsilon^{-1}}{\log\omega^{-1}}\right)\right)$$

for some positive c_1, c_2. Hence there exists a constant $c_3 > 0$ such that

$$\log(N_s(\varepsilon)) \leq c_3 (\log\varepsilon^{-1})^{\frac{1}{1+\eta}} \log\log\varepsilon^{-1}.$$

However, for sufficiently small ε,

$$\log\log\varepsilon^{-1} \leq (\log\varepsilon^{-1})^{\frac{\eta}{1+\eta}},$$

so

$$\log(N_s(\varepsilon)) \leq c_3 (\log\varepsilon^{-1})^{\frac{1}{1+\eta}} (\log\varepsilon^{-1})^{\frac{\eta}{1+\eta}} = c_3 \log\varepsilon^{-1}.$$

It follows that

$$N_s(\varepsilon) \leq c\varepsilon^{-1}$$

for some positive $c > 0$ for sufficiently small ε. This implies SPT with $\tau^* \leq 1$. □

Finally we compare standard tractability results for integration to EC-tractability results for integration in $H(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$. We see that the (sufficient) conditions are indeed eased in a sense for the standard tractability notions, as has been our hope (cf. Theorems 2.23 and 2.18).

For WT we have gained an alternative condition which does not demand a growing rate for $\boldsymbol{a}$ and is independent of $\boldsymbol{b}$.

Conditions for PT and SPT are relaxed in the following sense. On the one hand we have now separate conditions for PT and SPT and on the other hand we have conditions depending only on $\boldsymbol{a}$, whereas for EC-(S)PT we have conditions which depend exclusively on $\boldsymbol{b}$. The conditions on $\boldsymbol{a}$ for (S)PT are a lot less restrictive than the ones on $\boldsymbol{b}$ for EC-(S)PT.

2.3 Hybrid approximation

In this section we consider tractability of approximation in hybrid function spaces. The results of this section are joint work with Peter Kritzer and Friedrich Pillichshammer and are based on [43]. Kritzer and Pillichshammer introduced hybrid function spaces in [44] before, where they worked on tractability of QMC-integration in these spaces.

We want to start by recalling the definition of these spaces and we also want to give some reasons why it can be beneficial to have results for hybrid function spaces.

2.3.1 The hybrid function space

The hybrid function space we study is a specific reproducing kernel Hilbert space that was introduced in [44], namely the tensor product of a Korobov space and a Walsh space.

Fix a prime number b and let $\mathrm{i} = \sqrt{-1}$. For $k \in \mathbb{N}_0$ with b-adic expansion $k = \kappa_a b^a + \cdots + \kappa_1 b + \kappa_0$ with $\kappa_j \in \{0, \ldots, b-1\}$ and $\kappa_a \neq 0$ we define the k-th Walsh function $\mathrm{wal}_k : [0,1) \to \mathbb{C}$ by

$$\mathrm{wal}_k(x) = \exp\left(2\pi\mathrm{i}\frac{\xi_1\kappa_0 + \cdots + \xi_{a+1}\kappa_a}{b}\right),$$

for $x \in [0,1)$ with b-adic expansion $x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \cdots$ (unique in the sense that infinitely many of the ξ_i are different from $b-1$). Note that $a = \lfloor \log_b k \rfloor$.

For $\boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \ldots, x_s) \in [0,1)^s$ the $\boldsymbol{k}$-th s-variate Walsh function $\mathrm{wal}_{\boldsymbol{k}} : [0,1)^s \to \mathbb{C}$ is given by

$$\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} \mathrm{wal}_{k_j}(x_j).$$

Some crucial properties of Walsh functions that we are going to use in the following are that for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in [0,1)^s$ and $\boldsymbol{k}, \boldsymbol{h} \in \mathbb{N}_0^s$ it is true that

$$\begin{aligned}
\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_1)\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_2) &= \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_1 \oplus \boldsymbol{x}_2),\\
\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_1)\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_2)} &= \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_1 \ominus \boldsymbol{x}_2) \text{ and}\\
\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_1)\mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{x}_1) &= \mathrm{wal}_{\boldsymbol{k}\oplus\boldsymbol{h}}(\boldsymbol{x}_1),
\end{aligned}$$

where $\oplus$ denotes digit-wise addition modulo b, and is defined component-wise for vectors; by $\ominus$ we denote (component-wise) digit-wise subtraction modulo b.

We remark that Walsh functions could also be defined for arbitrary integer bases $b \geq 2$ (see, e.g., [20]), but for the use of our approximation algorithms we additionally require that b is prime.

Further, for $\boldsymbol{l} \in \mathbb{Z}^t$ we define the t-variate $\boldsymbol{l}$-th trigonometric function $\mathrm{e}_{\boldsymbol{l}} \colon [0,1)^t \to \mathbb{C}$ as

$$\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) = \exp(2\pi\mathrm{i}\boldsymbol{l} \cdot \boldsymbol{y}),$$

where $\cdot$ denotes the usual Euclidean inner product.

Let now $s,t \in \mathbb{N}$, $\alpha,\beta > 1$ and let $\boldsymbol{\gamma}^{(1)},\boldsymbol{\gamma}^{(2)}$ be two non-increasing sequences $\boldsymbol{\gamma}^{(i)} = (\gamma_j^{(i)})_{j\geq 1}$ for $i \in \{1,2\}$, where $0 < \gamma_j^{(i)} \leq 1$. We define two functions $\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}$ and $r_{\beta,\boldsymbol{\gamma}^{(2)}}$ as follows: For $\boldsymbol{k} = (k_1,\dots,k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{l} = (l_1,\dots,l_t) \in \mathbb{Z}^t$ let

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) = \prod_{j=1}^{s} \rho_{\alpha,\gamma_j^{(1)}}(k_j) \quad \text{and} \quad r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) = \prod_{j=1}^{t} r_{\beta,\gamma_j^{(2)}}(l_j),$$

where

$$\rho_{\alpha,\gamma_j^{(1)}}(k_j) = \begin{cases} 1 & \text{if } k_j = 0, \\ \gamma_j^{(1)} b^{-\alpha\lfloor \log_b(k_j)\rfloor} & \text{if } k_j \neq 0, \end{cases}$$

and

$$r_{\beta,\gamma_j^{(2)}}(l_j) = \begin{cases} 1 & \text{if } l_j = 0, \\ \gamma_j^{(2)} |l_j|^{-\beta} & \text{if } l_j \neq 0. \end{cases}$$

With the help of these functions we start by defining Walsh spaces [15, 19] and Korobov spaces [16, 56, 64] and subsequently move on to defining the hybrid function spaces we want to study in this section.

We begin with the Walsh space which was introduced in [19] (see also [20] for further details). Its reproducing kernel is given by

$$K_{s,\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{x},\boldsymbol{x}') = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}')} \quad \text{for} \quad \boldsymbol{x},\boldsymbol{x}' \in [0,1)^s,$$

and its inner product by

$$\langle f,g\rangle_{s,\alpha,\boldsymbol{\gamma}^{(1)}} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1} \hat{f}_{\mathrm{wal}}(\boldsymbol{k})\overline{\hat{g}_{\mathrm{wal}}(\boldsymbol{k})},$$

where

$$\hat{f}_{\mathrm{wal}}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x}$$

is the $\boldsymbol{k}$-th Walsh coefficient of f. The Walsh space is then defined as the space of all functions that can be expressed as absolutely convergent Walsh series with finite norm,

$$H(K_{s,\alpha,\boldsymbol{\gamma}^{(1)}}) = \left\{ f\colon f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \hat{f}_{\mathrm{wal}}(\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}), ||f||_{s,\alpha,\boldsymbol{\gamma}^{(1)}} < \infty \right\},$$

where $||\cdot||_{s,\alpha,\boldsymbol{\gamma}^{(1)}}$ denotes the norm induced by the inner product $\langle\cdot,\cdot\rangle_{s,\alpha,\boldsymbol{\gamma}^{(1)}}$ defined above.

The Korobov space which we are going to introduce next has been studied in many papers. We refer to [64] for detailed information. The reproducing kernel of the Korobov space is

$$K_{t,\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{y},\boldsymbol{y}') = \sum_{\boldsymbol{l}\in\mathbb{Z}^t} r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})\overline{\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}')} \quad \text{for} \quad \boldsymbol{y},\boldsymbol{y}' \in [0,1)^t.$$

Its inner product is given by

$$\langle f,g\rangle_{t,\beta,\boldsymbol{\gamma}^{(2)}} = \sum_{\boldsymbol{l}\in\mathbb{Z}^t} \left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1} \hat{f}_{\mathrm{trig}}(\boldsymbol{l})\overline{\hat{g}_{\mathrm{trig}}(\boldsymbol{l})},$$

where

$$\hat{f}_{\mathrm{trig}}(\boldsymbol{l}) = \int_{[0,1]^t} f(\boldsymbol{y}) \overline{\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})} \, \mathrm{d}\boldsymbol{y}$$

is the $\boldsymbol{l}$-th Fourier coefficient of f. The Korobov space is then defined as the space of all functions that can be expressed as absolutely convergent Fourier series with finite norm,

$$H(K_{t,\beta,\boldsymbol{\gamma}^{(2)}}) = \left\{ f \colon f(\boldsymbol{y}) = \sum_{\boldsymbol{l} \in \mathbb{Z}^t} \hat{f}_{\mathrm{trig}}(\boldsymbol{l}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}), ||f||_{t,\beta,\boldsymbol{\gamma}^{(2)}} < \infty \right\},$$

where $|| \cdot ||_{t,\beta,\boldsymbol{\gamma}^{(2)}}$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_{t,\beta,\boldsymbol{\gamma}^{(2)}}$.

Now we are ready to define our hybrid function space as the tensor product of the Walsh and Korobov spaces. The hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$, where $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^{(1)}, \boldsymbol{\gamma}^{(2)})$, is the reproducing kernel Hilbert space with kernel function given by $K_{s,t,\alpha,\beta,\boldsymbol{\gamma}} : [0,1)^{s+t} \times [0,1)^{s+t} \to \mathbb{C}$,

$$K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}((\boldsymbol{x},\boldsymbol{y}),(\boldsymbol{x}',\boldsymbol{y}')) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \sum_{\boldsymbol{l} \in \mathbb{Z}^t} \rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}')} \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \overline{\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}')}$$

and inner product

$$\langle f, g \rangle_{s,t,\alpha,\beta,\boldsymbol{\gamma}} = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \sum_{\boldsymbol{l} \in \mathbb{Z}^t} \frac{1}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \overline{\widehat{g}(\boldsymbol{k},\boldsymbol{l})},$$

with

$$\widehat{f}(\boldsymbol{k},\boldsymbol{l}) = \int_{[0,1]^s} \int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})} \, \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y}.$$

The space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is the tensor product of a Walsh space and a Korobov space. If $s = 0$, then we obtain the Korobov space, if $t = 0$, then we obtain the Walsh space.

Remark 2.26. For convenience we will in the following use the notation $\int_{[0,1]^d} f(\boldsymbol{x},\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$, where $d = s + t$, by which we mean $\int_{[0,1]^s} \int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$.

The hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is the space of all absolutely convergent series f of the form

$$f(\boldsymbol{x},\boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t} \widehat{f}(\boldsymbol{k},\boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \quad \text{with} \quad \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} < \infty,$$

where $\|\cdot\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}$ denotes the norm in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. For further information on the space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ we refer to [44, Section 2.2].

We consider $\mathbb{L}_2$-approximation of functions in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$, which is embedded into $\mathbb{L}_2([0,1]^{s+t})$. To be more precise, we approximate the embedding operator

$$\mathrm{EMB}_{s+t} : \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}) \to \mathbb{L}_2([0,1]^{s+t}), \quad \mathrm{EMB}_{s+t}(f) = f,$$

and measure the approximation error in the $\mathbb{L}_2$-norm. As before, the theorem of Creutzig and Wojtaszczyk from [6] (cf. also pages 5 and 7) applies, and there is no loss of generality when we restrict ourselves to linear approximation algorithms of the form $A_{N,s,t}(f) = \sum_{k=1}^N a_k L_k(f)$ with coefficients $a_k \in \mathbb{L}_2([0,1]^{s+t})$ and continuous linear functionals L_k on $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ from a permissible class of information Λ. Here N is the number of information evaluations.

In previous papers, several authors have studied approximation problems, similar to the one we consider in this section, in various reproducing kernel Hilbert spaces, see, e.g., [3, 11, 14, 51, 62, 79]. These investigations have in common that the reproducing kernel Hilbert spaces considered are tensor products of one-dimensional spaces whose kernels are all of the same type (but maybe equipped with different weights). In this section we consider the case where the reproducing kernel is a product of kernels of different type. We call such spaces hybrid spaces. Some results on tractability in general hybrid spaces can be found in the literature. For example, in [64] multivariate integration is studied for arbitrary reproducing kernels K_d without relation to K_{d+1}. Here we consider as a special instance the tensor product of Walsh and Korobov spaces. The problem of numerical integration in such spaces was recently considered in [44]. The study of a hybrid of Korobov and Walsh spaces could be important in view of functions which are periodic with respect to some of the components and, for example, piece-wise constant with respect to the remaining components. Moreover, it has been pointed out by several scientists (see, e.g., [39, 53]) that hybrid problems may be relevant for certain applications. Indeed, communication with the authors of [39] and [53] have motivated our idea for considering function spaces where we may have very different properties of the elements with respect to different components, as for example regarding smoothness.

From the analytical point of view, it is very challenging to deal with hybrid spaces. The reason for this is the rather complex interplay between the different analytic and algebraic structures of the kernel functions. In the present study we are concerned with Fourier analysis carried out simultaneously with respect to the Walsh and the trigonometric function systems. The problem is also closely related to the study of hybrid point sets which received much attention in recent years (see, for example, [28, 33]). Hence we also have considerable theoretical interest in studying this problem.

2.3.2 $\mathbb{L}_2$-approximation

Our goal is to approximate the embedding from the hybrid space $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ to the space $\mathbb{L}_2([0,1]^{s+t})$, i.e.,

$$\mathrm{EMB}_{s,t} : \mathcal{H}(K_{s,t,\alpha,\beta,\gamma}) \to \mathbb{L}_2([0,1]^{s+t}), \quad \mathrm{EMB}_{s,t}(f) = f. \tag{2.17}$$

As already mentioned, it is enough to consider linear algorithms $A_{N,s,t}$ of the form

$$A_{N,s,t}(f) = \sum_{k=1}^{N} a_k L_k(f), \tag{2.18}$$

with $a_k \in \mathbb{L}_2([0,1]^{s+t})$ and continuous linear functionals L_k on $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ from a permissible class of information Λ. As already explained in the introduction we consider the two classes Λ^{all} and Λ^{std}:

- $\Lambda = \Lambda^{\mathrm{all}}$, the class of all continuous linear functionals defined on $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$. Since $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ is a Hilbert space, for every $L_k \in \Lambda^{\mathrm{all}}$ there exists a function f_k from $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ such that $L_k(f) = \langle f, f_k\rangle_{d,\alpha,\beta,\gamma}$ for all $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.
- $\Lambda = \Lambda^{\mathrm{std}}$, the class of standard information consisting only of function evaluations. That is, $L_k \in \Lambda^{\mathrm{std}}$ if there exists $(\boldsymbol{x}_k, \boldsymbol{y}_k) \in [0,1]^{s+t}$ such that $L_k(f) = f(\boldsymbol{x}_k, \boldsymbol{y}_k)$ for all $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

Since $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ is a reproducing kernel Hilbert space, function evaluations are continuous linear functionals, and therefore $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$. More precisely,

$$L_k(f) = f(\boldsymbol{x}_k, \boldsymbol{y}_k) = \langle f, K_{s,t,\alpha,\beta,\gamma}(\cdot, (\boldsymbol{x}_k, \boldsymbol{y}_k))\rangle_{s,t,\alpha,\beta,\gamma}$$

and

$$\|L_k\| = \|K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} = K^{1/2}_{s,t,\alpha,\beta,\boldsymbol{\gamma}}((\boldsymbol{x}_k,\boldsymbol{y}_k),(\boldsymbol{x}_k,\boldsymbol{y}_k)).$$

Recall that the worst-case error in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ of a linear algorithm as in (2.18) is

$$e_{s+t}(A_{N,s,t}) = \sup_{\substack{f\in\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})\\ \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}\le 1}} \|\mathrm{EMB}_{s,t}(f) - A_{N,s,t}(f)\|_{\mathbb{L}_2([0,1]^{s+t})}.$$

As we will sometimes consider the error of the integration problem in the following analysis, we will use the notation $e^{\mathrm{app}}_{s+t}(A_{N,s,t})$ for the worst-case error to avoid ambiguities.

Similarly, the N-th minimal worst-case error is given by

$$e^{\mathrm{app}}_{s+t,\Lambda}(N) = \inf_{A_{N,s,t}} e^{\mathrm{app}}_{s+t}(A_{N,s,t}),$$

where the infimum is extended over all linear algorithms $A_{N,s,t}$ using information from the class Λ. The information complexity is given as

$$N^{\mathrm{app}}_{s+t,\Lambda}(\varepsilon) = \min\{N : e^{\mathrm{app}}_{s+t,\Lambda}(N) \le \varepsilon\}.$$

Since $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$, it follows that $N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{all}}}(\varepsilon) \le N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon)$.

For $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^{(1)},\boldsymbol{\gamma}^{(2)})$ we define the sum exponent

$$s_{\boldsymbol{\gamma}} = \inf\left\{\kappa > 0 \ : \ \sum_{j=1}^{\infty}(\gamma_j^{(1)})^{\kappa} < \infty \text{ and } \sum_{j=1}^{\infty}(\gamma_j^{(2)})^{\kappa} < \infty\right\} \tag{2.19}$$

with the convention that $\inf\emptyset = \infty$.

Our main goal in this section is to show the following theorem.

Theorem 2.27. *Consider the approximation problem* EMB *as defined in* (2.17). *Then we have:*

1. *Strong polynomial tractability and polynomial tractability in the class* Λ^{all} *are equivalent, and they hold if and only if* $s_{\boldsymbol{\gamma}} < \infty$, *where* $s_{\boldsymbol{\gamma}}$ *is defined in* (2.19). *In this case the exponent of strong polynomial tractability is* $\tau^*(\Lambda^{\mathrm{all}}) = 2\max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}, \frac{1}{\beta})$.

2. *The problem is weakly tractable in the class* Λ^{all} *if and only if*

$$\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s}\gamma_j^{(1)} + \sum_{j=1}^{t}\gamma_j^{(2)}}{s+t} = 0. \tag{2.20}$$

3. *The problem is strongly polynomially tractable in the class* Λ^{std} *if*

$$\sum_{j=1}^{\infty}\gamma_j^{(1)} < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty}\gamma_j^{(2)} < \infty.$$

The exponent of strong polynomial tractability in the class Λ^{std} *satisfies*

$$\tau^*(\Lambda^{\mathrm{std}}) \in [2\max(\tfrac{1}{\alpha},\tfrac{1}{\beta},s_{\boldsymbol{\gamma}}), 4 + 2\max(\tfrac{1}{\alpha},\tfrac{1}{\beta},s_{\boldsymbol{\gamma}})].$$

4. The problem is polynomially tractable in the class Λ^{std} *if*

$$\limsup_{s\to\infty}\frac{\sum_{j=1}^{s}\gamma_j^{(1)}}{\log s}<\infty \quad \textit{and} \quad \limsup_{t\to\infty}\frac{\sum_{j=1}^{t}\gamma_j^{(2)}}{\log t}<\infty.$$

5. The problem is weakly tractable in the class Λ^{std} *if and only if*

$$\lim_{s+t\to\infty}\frac{\sum_{j=1}^{s}\gamma_j^{(1)}+\sum_{j=1}^{t}\gamma_j^{(2)}}{s+t}=0.$$

Remark 2.28. Since it can easily be verified that integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ is not harder than approximation using Λ^{std}, all sufficifent conditions stated in Theorem 2.27 for approximation are sufficient for integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ as well. These conditions coincide with the ones given in [44] for QMC integration.

2.3.3 Proof of Theorem 2.27

We recall that strong polynomial tractability implies polynomial tractability which in turn implies weak tractability. Furthermore, all sufficient conditions for the class Λ^{std} are also sufficient for the class Λ^{all} with $\tau^*(\Lambda^{\mathrm{all}})\le\tau^*(\Lambda^{\mathrm{std}})$ in the case of strong polynomial tractability. All necessary conditions for the class Λ^{all} are also necessary for the class Λ^{std}.

Proof of Item 1 In order to give a necessary and sufficient condition for strong polynomial tractability for Λ^{all} we use a criterion from [63, Section 5.1]. Let us consider the self-adjoint operator $W_{s,t}:=\mathrm{EMB}_{s,t}^{*}\mathrm{EMB}_{s,t}:\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})\to\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$, which in our case is given by

$$W_{s,t}f=\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t}\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\widehat{f}(\boldsymbol{k},\boldsymbol{l})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}).$$

The eigenvalues are then given by the collection of the numbers

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \quad \text{for } (\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t.$$

Furthermore, the largest eigenvalue is $\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{0})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{0})=1$.

From [63, Theorem 5.2] we find that the problem EMB is polynomially tractable for Λ^{all} if and only if there exist $\nu>0$ and $q\ge 0$ such that

$$\sup_{s,t}\left(\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t}(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}))^{\nu}\right)^{1/\nu}(s+t)^{-q}<\infty. \tag{2.21}$$

Furthermore, we have strong polynomial tractability if and only if (2.21) holds with $q=0$.

It is easy to check that we require $\nu>\max(\frac{1}{\alpha},\frac{1}{\beta})$ in order for (2.21) to hold with $q=0$. Let us now assume that ν is indeed bigger than $\max(\frac{1}{\alpha},\frac{1}{\beta})$. For the sum in (2.21) we have

$$\sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t}(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}))^{\nu} \;=\; \prod_{j=1}^{s}\Big(1+(\gamma_j^{(1)})^{\nu}\mu(\alpha\nu)\Big)\prod_{j=1}^{t}\Big(1+(\gamma_j^{(2)})^{\nu}2\zeta(\beta\nu)\Big), \tag{2.22}$$

where $\mu(x)=\frac{b^x(b-1)}{b^x-b}$ for $x>1$ and $\zeta(\cdot)$ is the Riemann zeta function.

Now, using arguments outlined in [75] (see also [56, Section 4.5]), it is easy to see that the existence of some $\nu > \max(\frac{1}{\alpha}, \frac{1}{\beta})$ with

$$\sum_{j=1}^{\infty} (\gamma_j^{(1)})^{\nu} < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} (\gamma_j^{(2)})^{\nu} < \infty$$

is a necessary and sufficient condition for (2.21) with $q = 0$ and therefore for strong polynomial tractability of the problem EMB.

Again according to [63, Theorem 5.2], the exponent of strong polynomial tractability is $2\max(\frac{1}{\alpha}, \frac{1}{\beta}, s_{\boldsymbol{\gamma}})$, where $s_{\boldsymbol{\gamma}}$ is defined in (2.19).

It remains to show the equivalence of strong polynomial and polynomial tractability. Of course, it suffices to show that polynomial tractability implies strong polynomial tractability. So assume that the problem EMB is polynomially tractable for the class Λ^{all}. Then we obtain polynomial tractability also for the embedding problem in the pure Walsh space $\mathcal{H}(K_{s,0,\alpha,\beta,\boldsymbol{\gamma}})$ and in the pure Korobov space $\mathcal{H}(K_{0,t,\alpha,\beta,\boldsymbol{\gamma}})$. According to [77, Theorem 2] this is equivalent to strong polynomial tractability for the embedding problem in the pure Walsh space $\mathcal{H}(K_{s,0,\alpha,\beta,\boldsymbol{\gamma}})$ and in the pure Korobov space $\mathcal{H}(K_{0,t,\alpha,\beta,\boldsymbol{\gamma}})$. According to [11] and [51] this implies the existence of $\nu_1 > 0$ such that $\sum_{j\geq 1}(\gamma_j^{(1)})^{\nu_1} < \infty$ and the existence of $\nu_2 > 0$ such that $\sum_{j\geq 1}(\gamma_j^{(2)})^{\nu_2} < \infty$. Hence we have $s_{\boldsymbol{\gamma}} < \infty$ and this in turn implies strong polynomial tractability for the class Λ^{all}, as shown above. This completes the proof of Item 1.

Proof of Item 2 Sufficiency of Condition (2.20) follows by Item 5 of Theorem 2.27. Item 5 is proved on pp. 40.

For showing necessity of Condition (2.20), we use [63, Theorem 5.3] in the following. To keep notation simple, we shall frequently write d instead of $s + t$. Theorem 5.3 in [63] states that our approximation problem is weakly tractable for Λ^{all} if and only if

- $\lim_{j\to\infty} \lambda_{d,j} \log^2 j = 0$ for all $d \in \mathbb{N}$ and
- there exists some function $f\colon (0, \frac{1}{2}] \to \mathbb{N}$ such that

$$\sup_{\eta\in(0,\frac{1}{2}]} \frac{1}{\eta^2} \sup_{d\geq f(\eta)} \sup_{j\geq\lceil\exp(d\sqrt{\eta})\rceil+1} \lambda_{d,j} \log^2 j < \infty, \tag{2.23}$$

where $\lambda_{d,j} = \lambda_{s+t,j}$ denotes the j^{th} eigenvalue of $W_{s,t}$ in non-increasing order.

Let us now assume that the approximation problem is weakly tractable for Λ^{all}. This then in particular implies that

$$\lim_{j\to\infty} \lambda_{d,j} \log^2 j = 0 \quad \text{for all} \quad d \in \mathbb{N}. \tag{2.24}$$

We are now going to show that (2.24) implies (2.20). To this end, recall that the eigenvalues of $W_{s,t}$ are of the form

$$\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \quad \text{for } (\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t.$$

Note that we have $\lambda_{d,1} = 1$; furthermore, note that $\rho_{\alpha,\gamma_j^{(1)}}(1) = \gamma_j^{(1)}$ for any $j \in \mathbb{N}$, and $r_{\beta,\gamma_i^{(2)}}(1) = \gamma_i^{(2)}$ for any $i \in \mathbb{N}$. Hence, by choosing all components of $(\boldsymbol{k},\boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t$ but one equal to zero, and the remaining equal to one, we see that

$$\gamma_1^{(1)}, \ldots, \gamma_s^{(1)} \quad \text{and} \quad \gamma_1^{(2)}, \ldots, \gamma_t^{(2)}$$

are eigenvalues of $W_{s,t}$. Consequently,

$$\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)} \le \sum_{j=1}^{d} \lambda_{d,j},$$

and hence

$$\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)}}{s+t} \le \lim_{d\to\infty} \frac{\sum_{j=1}^{d} \lambda_{d,j}}{d}.$$

However, due to (2.24), it follows that the latter limit is 0, which shows that indeed (2.20) holds.

Proof of Items 3–5 Any $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ can be displayed as

$$f(\boldsymbol{x}, \boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t} \widehat{f}(\boldsymbol{k}, \boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}).$$

The idea is now to choose some suitable (finite) subset $\mathcal{A}$ of $\mathbb{N}_0^s \times \mathbb{Z}^t$ and approximate f by a truncated series over $\mathcal{A}$, where we approximate $\hat{f}(\boldsymbol{k}, \boldsymbol{l})$ for every $(\boldsymbol{k}, \boldsymbol{l}) \in \mathcal{A}$.

In order to approximate $\widehat{f}(\boldsymbol{k}, \boldsymbol{l})$, we are going to use quasi-Monte Carlo algorithms based on classical and on polynomial lattice point sets.

Classical lattice point sets. For a detailed definition see also Definition 3.1 and, e.g., [60, Chapter 5].

For $N \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \ldots, z_t) \in \mathcal{Z}_N^t$, where $\mathcal{Z}_N := \{z \in \{1, \ldots, N-1\} : \gcd(z, N) = 1\}$, the lattice point set $\{\boldsymbol{q}_v\}_{v=0}^{N-1}$ with generating vector $\boldsymbol{z}$ is defined by

$$\boldsymbol{q}_v = \left(\left\{ \frac{vz_1}{N} \right\}, \ldots, \left\{ \frac{vz_t}{N} \right\} \right) \quad \text{for all } \ 0 \le v \le N-1.$$

Here $\{\cdot\}$ denotes the fractional part of a real number.

Polynomial lattice point sets. Polynomial lattice point sets are introduced in greater detail in Section 3.1 on p. 60. Fur further information see also, e.g., [20, Chapter 10]. What follows here, is a short introduction.

Let $\mathbb{F}_b$ be the finite field of prime order b, $\mathbb{F}_b[x]$ be the set of polynomials over $\mathbb{F}_b$, and let $\mathbb{F}_b((x^{-1}))$ be the field of formal Laurent series over $\mathbb{F}_b$. The latter contains the field of rational functions as a subfield. Given $m \in \mathbb{N}$, set $G_{b,m} := \{a \in \mathbb{F}_b[x] \,:\, \deg(a) < m\}$ and define a mapping $\phi_m : \mathbb{F}_b((x^{-1})) \to [0, 1)$ by

$$\phi_m \left(\sum_{l=w}^{\infty} t_l x^{-l} \right) := \sum_{l=\max\{1,w\}}^{m} t_l b^{-l}.$$

Let $f \in \mathbb{F}_b[x]$ with $\deg(f) = m$ and $\boldsymbol{g} = (g_1, \ldots, g_s) \in \mathbb{F}_b[x]^s$. The polynomial lattice point set $(\boldsymbol{p}_v)_{v\in G_{b,m}}$ with generating vector $\boldsymbol{g}$ is defined by

$$\boldsymbol{p}_v := \left(\phi_m \left(\frac{v(x)g_1(x)}{f(x)} \right), \ldots, \phi_m \left(\frac{v(x)g_s(x)}{f(x)} \right) \right) \quad \text{for all } \ v \in G_{b,m}.$$

Note that we can associate the polynomial $v(x) = \sum_{r=0}^{m-1} v_r x^r \in G_{b,m}$ with the integer $v = \sum_{r=0}^{m-1} v_r b^r$, where, with a slight abuse of notation, the element $v_r \in \mathbb{F}_b$ is associated with the integer $v_r \in \{0, 1 \ldots, b-1\}$. In this way we can index the points of a polynomial lattice point set by integers ranging from 0 to $b^m - 1$.

Now suppose that N is of the form b^m for some $m \in \mathbb{N}$, and let $\mathrm{PL} = \{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{N-1}\} \subseteq [0,1)^s$ be a polynomial lattice point set and $\mathrm{L} = \{\boldsymbol{q}_0, \ldots, \boldsymbol{q}_{N-1}\} \subseteq [0,1)^t$ be a lattice point set. We consider the point set $(\mathrm{PL}, \mathrm{L}) = \{(\boldsymbol{p}, \boldsymbol{q})_v = (\boldsymbol{p}_v, \boldsymbol{q}_v) \colon v = 0, \ldots, N-1\}$.

For instance Algorithm 1 in [44] provides a component-by-component method to find a point set (PL, L) such that a QMC algorithm based on (PL,L) has a low worst-case integration error in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$. The same point set can also be used for approximation in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$, as we will see below.

For $M \geq 0$ define the set

$$\mathcal{A}_M = \{(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : (\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}))^{-1} (r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}))^{-1} \leq M\}. \tag{2.25}$$

As $\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}))^{-1}$ and $(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}))^{-1}$ are always greater than or equal to 1, the set $\mathcal{A}_M$ is empty for all $0 < M < 1$ and we approximate f by 0 for any such M. Thus we only consider $M \geq 1$ throughout the rest of this section.

In order to approximate the embedding $\mathrm{EMB}_{s,t}(f) = f$ for $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ we use the algorithm

$$A_{N,s,t,M}(f)(\boldsymbol{x}, \boldsymbol{y}) = \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \left(\frac{1}{N} \sum_{v=0}^{N-1} f((\boldsymbol{p}, \boldsymbol{q})_v) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q}_v)} \right) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}). \tag{2.26}$$

By rearranging (2.26) to

$$A_{N,s,t,M}(f)(\boldsymbol{x}, \boldsymbol{y}) = \sum_{v=0}^{N-1} \left(\frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y} - \boldsymbol{q}_v) \right) f((\boldsymbol{p}, \boldsymbol{q})_v),$$

one can easily see that $A_{N,s,t,M}$ is a linear algorithm of the form (2.18) with

$$a_v(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y} - \boldsymbol{q}_v) \quad \text{and} \quad L_v(f) = f((\boldsymbol{p}, \boldsymbol{q})_v),\ 0 \leq v \leq N-1.$$

The error of approximation for given $f \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ is then

$$\begin{aligned}(f - A_{N,s,t,M}(f))(\boldsymbol{x}, \boldsymbol{y}) = &\sum_{(\boldsymbol{k},\boldsymbol{l}) \notin \mathcal{A}_M} \widehat{f}(\boldsymbol{k}, \boldsymbol{l}) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \\ &+ \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \left(\widehat{f}(\boldsymbol{k}, \boldsymbol{l}) - \frac{1}{N} \sum_{v=0}^{N-1} f((\boldsymbol{p}, \boldsymbol{q})_v) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q}_v)} \right) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}).\end{aligned} \tag{2.27}$$

We use (2.27) and Parseval's identity to obtain

$$\|\mathrm{EMB}_{s,t}(f) - A_{N,s,t,M}(f)\|^2_{\mathbb{L}_2([0,1]^{s+t})} = S_1 + S_2,$$

where

$$S_1 := \sum_{(\boldsymbol{k},\boldsymbol{l}) \notin \mathcal{A}_M} |\widehat{f}(\boldsymbol{k}, \boldsymbol{l})|^2,$$

and

$$S_2 := \sum_{(\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A}_M} \left| \int_{[0,1]^{\boldsymbol{s}}} f(\boldsymbol{x}, \boldsymbol{y}) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})} \,\mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) - \frac{1}{N} \sum_{v=0}^{N-1} f((\boldsymbol{p}, \boldsymbol{q})_v) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q})} \right|^2$$

$$= \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \left| \int_{[0,1]^{s+t}} f_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{v=0}^{N-1} f_{(\boldsymbol{k},\boldsymbol{l})}((\boldsymbol{p},\boldsymbol{q})_v) \right|^2,$$

with

$$f_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{x},\boldsymbol{y}) = f(\boldsymbol{x},\boldsymbol{y})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})}.$$

We bound S_1 from above by writing

$$\begin{aligned} S_1 &= \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_M} \left|\hat{f}(\boldsymbol{k},\boldsymbol{l})\right|^2 \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1} \left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1} \rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}) r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}) \\ &< \frac{1}{M} \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathbb{N}_0^s\times\mathbb{Z}^t} \left|\hat{f}(\boldsymbol{k},\boldsymbol{l})\right|^2 \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1} \left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1} \\ &= \frac{1}{M}\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}. \end{aligned}$$

Let us now consider S_2. The term in-between the absolute value signs in S_2 is the integration error of the QMC rule using the nodes (PL,L) for the function $f_{(\boldsymbol{k},\boldsymbol{l})}$. Since the product of two Walsh functions is again a Walsh function, and the analogue is true for trigonometric functions, it can easily be verified that $f_{(\boldsymbol{k},\boldsymbol{l})} \in \mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. Hence we can bound S_2 by

$$S_2 \le (e^{\mathrm{int}}_{s+t}(\mathrm{PL},\mathrm{L}))^2 \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})},$$

where $e^{\mathrm{int}}_{s+t}(\mathrm{PL},\mathrm{L})$ is the worst-case integration error in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ of the QMC rule based on the nodes (PL,L), i.e.,

$$e^{\mathrm{int}}_{s+t}(\mathrm{PL},\mathrm{L}) = \sup_{\substack{f\in\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}) \\ \|f\|_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}\le 1}} \left| \int_{[0,1]^{s+t}} f(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} - \frac{1}{N}\sum_{v=0}^{N-1} f((\boldsymbol{p},\boldsymbol{q})_v) \right|.$$

From [44, Theorem 3] it then follows that

$$\begin{aligned} S_2 &\le \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \left(-1 + \frac{1}{N^2}\sum_{k,k'=0}^{N-1} K_{s,t,\alpha,\beta,\boldsymbol{\gamma}}((\boldsymbol{p},\boldsymbol{q})_k, (\boldsymbol{p},\boldsymbol{q})_{k'})\right) ||f_{(\boldsymbol{k},\boldsymbol{l})}||^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} \\ &\le \frac{2}{N}\left(\prod_{j=1}^{s}(1+\gamma_j^{(1)}2\mu(\alpha))\right)\left(\prod_{j=1}^{t}(1+\gamma_j^{(2)}4\zeta(\beta))\right) \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M} \|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}. \end{aligned} \tag{2.28}$$

Next we find an estimate for $\|f_{(\boldsymbol{k},\boldsymbol{l})}\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})}$ for $(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_M$.

By definition we have

$$||f_{(\boldsymbol{k},\boldsymbol{l})}||^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})} = \sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t} \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})\right)^{-1} \left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})\right)^{-1} \left|\widehat{f}_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{h},\boldsymbol{m})\right|^2.$$

We start by considering

$$\begin{aligned} \left|\widehat{f}_{(\boldsymbol{k},\boldsymbol{l})}(\boldsymbol{h},\boldsymbol{m})\right|^2 &= \left| \int_{[0,1]^s}\int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y})\overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y})\mathrm{wal}_{\boldsymbol{h}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{m}}(\boldsymbol{y})} \,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right|^2 \\ &= \left| \int_{[0,1]^s}\int_{[0,1]^t} f(\boldsymbol{x},\boldsymbol{y})\overline{\mathrm{wal}_{\boldsymbol{k}\oplus\boldsymbol{h}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}+\boldsymbol{m}}(\boldsymbol{y})} \,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} \right|^2 \end{aligned}$$

$$= \left|\hat{f}(\boldsymbol{k} \oplus \boldsymbol{h}, \boldsymbol{l}+\boldsymbol{m})\right|^2.$$

Therefore

$$\begin{aligned}
||f_{(\boldsymbol{k},\boldsymbol{l})}||^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})} &= \sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t} \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})\right)^{-1}\left|\hat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})\right|^2 \\
&= \sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t} \left|\hat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})\right|^2\left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})\right)^{-1} \\
&\quad\times\left(\frac{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}\right)^{-1}.
\end{aligned}$$

From [51] we know that

$$\left(\frac{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}\right)^{-1} \le \left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1}\prod_{j=1}^t \max\{1, 2^\beta\gamma_j^{(2)}\},$$

and from [11] that

$$\left(\frac{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})}\right)^{-1} \le \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1}.$$

Altogether we find

$$\left(\frac{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{m})}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})}\right)^{-1} \le \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1}\prod_{j=1}^t \max\{1, 2^\beta\gamma_j^{(2)}\},$$

and

$$\begin{aligned}
||f_{(\boldsymbol{k},\boldsymbol{l})}||^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})} &\le \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1}\prod_{j=1}^t \max\{1, 2^\beta\gamma_j^{(2)}\} \\
&\quad\times\sum_{\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{m}\in\mathbb{Z}^t} \left|\hat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})\right|^2\left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})\right)^{-1} \\
&= \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1}\prod_{j=1}^t \max\{1, 2^\beta\gamma_j^{(2)}\} \\
&\quad\times\sum_{\boldsymbol{k}\oplus\boldsymbol{h}\in\mathbb{N}_0^s}\sum_{\boldsymbol{l}+\boldsymbol{m}\in\mathbb{Z}^t} \left|\hat{f}(\boldsymbol{k}\oplus\boldsymbol{h},\boldsymbol{l}+\boldsymbol{m})\right|^2\left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k}\oplus\boldsymbol{h})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l}+\boldsymbol{m})\right)^{-1} \\
&= \left(\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})\right)^{-1}||f||^2_{\boldsymbol{s},\boldsymbol{\alpha},\boldsymbol{\gamma}}\prod_{j=1}^t \max\{1, 2^\beta\gamma_j^{(2)}\} \\
&\le M\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})}\prod_{j=1}^t \max(1, 2^\beta\gamma_j^{(2)}).
\end{aligned}$$

Plugging this into (2.28) we obtain

$$S_2 \le \frac{2}{N}\left(\prod_{j=1}^s(1+\gamma_j^{(1)}2\mu(\alpha))\right)\left(\prod_{j=1}^t(1+\gamma_j^{(2)}4\zeta(\beta))\right)\|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})}M|\mathcal{A}_M|\prod_{j=1}^t \max(1, 2^\beta\gamma_j^{(2)}). \tag{2.29}$$

Next we study the cardinality of the set $\mathcal{A}_M$.

Lemma 2.29. *Let $\theta = \min(\alpha, \beta)$. For arbitrary $\kappa > 1/\theta = \max(\frac{1}{\alpha}, \frac{1}{\beta})$ we have*

$$|\mathcal{A}_M| \le M^\kappa \prod_{j=1}^{s} \left(1 + 2\zeta(\theta\kappa)(b^\alpha \gamma_j^{(1)})^\kappa\right) \prod_{j=1}^{t} \left(1 + 2\zeta(\theta\kappa)(\gamma_j^{(2)})^\kappa\right).$$

Proof. For $k \in \mathbb{N}$ we have

$$\frac{1}{\rho_{\alpha,\gamma}(k)} = \frac{b^{\alpha\lfloor \log_b k \rfloor}}{\gamma} \ge \frac{b^{\alpha(-1+\log_b k)}}{\gamma} = \frac{k^\alpha}{\gamma b^\alpha} = \frac{1}{r_{\alpha,\gamma b^\alpha}(k)}.$$

Then we have

$$\begin{aligned}
\mathcal{A}_M &= \left\{(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : \frac{1}{\rho_{\alpha,\boldsymbol{\gamma}^{(1)}}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\} \\
&\subseteq \left\{(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{N}_0^s \times \mathbb{Z}^t : \frac{1}{r_{\alpha,\boldsymbol{\gamma}^{(1)} b^\alpha}(\boldsymbol{k})} \frac{1}{r_{\beta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\} \\
&\subseteq \left\{(\boldsymbol{k}, \boldsymbol{l}) \in \mathbb{Z}^s \times \mathbb{Z}^t : \frac{1}{r_{\theta,\boldsymbol{\gamma}^{(1)} b^\alpha}(\boldsymbol{k})} \frac{1}{r_{\theta,\boldsymbol{\gamma}^{(2)}}(\boldsymbol{l})} \le M\right\}
\end{aligned}$$

from which the result follows immediately from [51, Lemma 1]. □

Considering Lemma 2.29, for any $\kappa > 1/\theta$ we obtain

$$S_2 \le c_{s,t,\alpha,\beta,\gamma,\kappa} \frac{M^{1+\kappa}}{N} \|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})},$$

where

$$\begin{aligned}
c_{s,t,\alpha,\beta,\gamma,\kappa} := 2 &\left(\prod_{j=1}^{s}(1 + \gamma_j^{(1)} 2\mu(\alpha))\right) \left(\prod_{j=1}^{t}(1 + \gamma_j^{(2)} 4\zeta(\beta))\right) \prod_{j=1}^{t} \max(1, 2^\beta \gamma_j^{(2)}) \\
&\times \prod_{j=1}^{s} \left(1 + 2\zeta(\theta\kappa)(b^\alpha \gamma_j^{(1)})^\kappa\right) \prod_{j=1}^{t} \left(1 + 2\zeta(\theta\kappa)(\gamma_j^{(2)})^\kappa\right). \qquad (2.30)
\end{aligned}$$

Summing up we have

$$\|\mathrm{EMB}_{s,t}(f) - A_{N,s,t,M}(f)\|^2_{\mathbb{L}_2([0,1]^{s+t})} \le \left(\frac{1}{M} + c_{s,t,\alpha,\beta,\gamma,\kappa} \frac{M^{1+\kappa}}{N}\right) \|f\|^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})}.$$

This leads to the following proposition and its corollary, which then concludes the proof of Theorem 2.27.

Proposition 2.30. *Let $\kappa > 1/\min(\alpha, \beta)$ and let $c_{s,t,\alpha,\beta,\gamma,\kappa}$ be defined as in (2.30). The worst-case error of the algorithm $A_{N,s,t,M}$ as defined in (2.26) using a point set (PL, L) constructed by [44, Algorithm 1] and with $M = (N/c_{s,t,\alpha,\beta,\gamma,\kappa})^{1/(2+\kappa)}$ satisfies*

$$e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t,M}) \le \sqrt{2} \left(\frac{c_{s,t,\alpha,\beta,\gamma,\kappa}}{N}\right)^{\frac{1}{4+2\kappa}}.$$

Proof. We have

$$\left(e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t,M})\right)^2 \le \frac{1}{M} + \frac{c_{s,t,\alpha,\beta,\gamma,\kappa} M^{1+\kappa}}{N}.$$

As we want to have

$$\frac{1}{M} = \frac{c_{s,t,\alpha,\beta,\gamma,\kappa} M^{1+\kappa}}{N}$$

we end up with

$$M(N) = \left(\frac{N}{c_{s,t,\alpha,\beta,\gamma,\kappa}} \right)^{\frac{1}{2+\kappa}}$$

and consequently

$$\begin{aligned}
\left(e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t,M}) \right)^2 &\leq \frac{1}{\left(\frac{N}{c_{s,t,\alpha,\beta,\gamma,\kappa}} \right)^{\frac{1}{2+\kappa}}} + \frac{\left(\frac{N}{c_{s,t,\alpha,\beta,\gamma,\kappa}} \right)^{\frac{1+\kappa}{2+\kappa}} c_{s,t,\alpha,\beta,\gamma,\kappa}}{N} \\
&= \left(\frac{c_{s,t,\alpha,\beta,\gamma,\kappa}}{N} \right)^{\frac{1}{2+\kappa}} + \left(c_{s,t,\alpha,\beta,\gamma,\kappa} \right)^{\frac{1}{2+\kappa}} N^{-\frac{1}{2+\kappa}} \\
&= 2 \left(\frac{c_{s,t,\alpha,\beta,\gamma,\kappa}}{N} \right)^{\frac{1}{2+\kappa}}
\end{aligned}$$

and the result follows. □

Corollary 2.31. *Consider the approximation problem* EMB *with information from the class* Λ^{std}*.*

- *If* $\sum_{j=1}^{\infty} \gamma_j^{(1)} < \infty$ *and* $\sum_{j=1}^{\infty} \gamma_j^{(2)} < \infty$*, then* EMB *is strongly polynomially tractable with* ε*-exponent at most* $4 + 2\max(s_\gamma, \frac{1}{\alpha}, \frac{1}{\beta})$*;*

- *if* $\limsup_{s\to\infty} \sum_{j=1}^{s} \frac{\gamma_j^{(1)}}{\log s} < \infty$ *and* $\limsup_{t\to\infty} \sum_{j=1}^{t} \frac{\gamma_j^{(2)}}{\log t} < \infty$*, then* EMB *is polynomially tractable;*

- *if* $\lim_{s+t\to\infty} \frac{\sum_{j=1}^{s} \gamma_j^{(1)} + \sum_{j=1}^{t} \gamma_j^{(2)}}{s+t} = 0$*, then* EMB *is weakly tractable.*

Proof. Employing Proposition 2.30, we know that, if

$$2 \left(\frac{c_{s,t,\alpha,\beta,\gamma,\kappa}}{N} \right)^{\frac{1}{2+\kappa}} \leq \varepsilon^2 \tag{2.31}$$

holds, then we have

$$e^{\mathbb{L}_2-\mathrm{app}}(A_{N,s,t,M}) < \varepsilon.$$

Hence we need

$$N \geq \frac{2^{2+\kappa} c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}}. \tag{2.32}$$

Let $pp(x)$ denote the smallest prime power greater than or equal to x. We define

$$N = N(\varepsilon) := pp\left(\frac{2^{2+\kappa} c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}} \right) \leq \frac{2^{3+\kappa} c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}}.$$

This leads to

$$\frac{2}{\varepsilon^2} \leq M(N,\varepsilon) \leq \left(\frac{\frac{2^{3+\kappa} c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}}}{c_{s,t,\alpha,\beta,\gamma,\kappa}} \right)^{\frac{1}{2+\kappa}} = \frac{2^{\frac{3+\kappa}{2+\kappa}}}{\varepsilon^2}.$$

To find the sufficient conditions stated in Corollary 2.31 we study the upper bound on $N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon)$ which we just found:

$$N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon) \le N(\varepsilon) \le \frac{2^{3+\kappa} c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}}.$$

We estimate $c_{s,t,\alpha,\beta,\gamma,\kappa}$ a bit further.

$$\begin{aligned}
c_{s,t,\alpha,\beta,\gamma,\kappa} &= 2\left(\prod_{j=1}^{s}(1+\gamma_j^{(1)}2\mu(\alpha))\right)\left(\prod_{j=1}^{t}(1+\gamma_j^{(2)}4\zeta(\beta))\right)\prod_{j=1}^{t}\max(1,2^{\beta}\gamma_j^{(2)}) \\
&\quad\times\prod_{j=1}^{s}\left(1+2\zeta(\theta\kappa)(b^{\alpha}\gamma_j^{(1)})^{\kappa}\right)\prod_{j=1}^{t}\left(1+2\zeta(\theta\kappa)(\gamma_j^{(2)})^{\kappa}\right) \\
&\le 2\exp\left(2\mu(\alpha)\sum_{j=1}^{\infty}\gamma_j^{(1)}\right)\exp\left(4\zeta(\beta)\sum_{j=1}^{\infty}\gamma_j^{(2)}\right)\exp\left(2^{\beta}\sum_{j=1}^{\infty}\gamma_j^{(2)}\right)\exp\left(2\zeta(\theta\kappa)b^{\alpha\kappa}\sum_{j=1}^{\infty}\gamma_j^{(1)}\right) \qquad (2.33) \\
&\quad\times\exp\left(2\zeta(\theta\kappa)\sum_{j=1}^{\infty}\gamma_j^{(2)}\right),
\end{aligned}$$

where we have used the well-known estimate

$$\prod_{j=1}^{n}(1+x_j) = \exp\left(\sum_{j=1}^{n}\log(1+x_j)\right) \le \exp\left(\sum_{j=1}^{n}x_j\right).$$

Thus $c_{s,t,\alpha,\beta,\gamma,\kappa}$ is uniformly bounded in s if $\sum_{j=1}^{\infty}\gamma_j^{(1)} < \infty$ and $\sum_{j=1}^{\infty}\gamma_j^{(2)} < \infty$. Consequently $\sum_{j=1}^{\infty}\gamma_j^{(1)} < \infty$ and $\sum_{j=1}^{\infty}\gamma_j^{(2)} < \infty$ are sufficient conditions for strong polynomial tractability.

From (2.33) we see that we can bound $c_{s,t,\alpha,\beta,\gamma,\kappa}$ by s^q if

$$\limsup_{s\to\infty}\sum_{j=1}^{s}\frac{\gamma_j^{(1)}}{\log s} < \infty \quad \text{and} \quad \limsup_{t\to\infty}\sum_{j=1}^{t}\frac{\gamma_j^{(2)}}{\log t} < \infty.$$

Hence these are sufficient conditions for polynomial tractability.

Finally we show that

$$\lim_{s+t\to\infty}\frac{\sum_{j=1}^{s}\gamma_j^{(1)}}{s} + \frac{\sum_{j=1}^{t}\gamma_j^{(2)}}{t} = 0 \qquad (2.34)$$

is a sufficient condition for weak tractability, i.e. we show that, provided this condition holds, we always obtain

$$\lim_{s+t+\varepsilon^{-1}\to\infty}\frac{\log N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon)}{s+t+\varepsilon^{-1}} = 0.$$

Using again (2.33) we have

$$\log N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon) \le \log\frac{2^{3+\kappa}c_{s,t,\alpha,\beta,\gamma,\kappa}}{\varepsilon^{2(2+\kappa)}}$$

$$\leq (3+\kappa)\log 2 - 2(2+\kappa)\log\varepsilon + \log 2 + 2\mu(\alpha)\sum_{j=1}^{s}\gamma_j^{(1)} + 4\zeta(\beta)\sum_{j=1}^{t}\gamma_j^{(2)}$$
$$+ 2^{\beta}\sum_{j=1}^{t}\gamma_j^{(2)} + 2\zeta(\theta\kappa)b^{\alpha\kappa}\sum_{j=1}^{s}\gamma_j^{(1)} + 2\zeta(\theta\kappa)\sum_{j=1}^{t}\gamma_j^{(2)}. \tag{2.35}$$

We consider the second summand $2(2+\kappa)\log\varepsilon$ more closely. We would like to have

$$\lim_{s+t+\varepsilon^{-1}\to\infty}\frac{2(2+\kappa)\log\varepsilon}{s+t+\varepsilon^{-1}} = 0. \tag{2.36}$$

If $s\to\infty$ or $t\to\infty$ this clearly is true. So we study the case where s and t are bounded and only ε^{-1} tends to infinity. Of course $\varepsilon^{-1}\to\infty$ implies $\varepsilon\to 0$. We use L'Hôpital's rule to find

$$\lim_{\varepsilon^{-1}\to\infty}\frac{2(2+\kappa)\log\varepsilon}{s+t+\varepsilon^{-1}} = \lim_{\varepsilon^{-1}\to\infty}\frac{2(2+\kappa)\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -2(2+\kappa)\lim_{\varepsilon^{-1}\to\infty}\varepsilon = 0.$$

Thus (2.36) holds in any case and, keeping in mind our assumption that (2.34) is fulfilled, we see that each summand in (2.35) tends to zero as $s+t+\varepsilon^{-1}\to\infty$. This completes the proof. □

2.3.4 Necessary conditions in the class Λ^{std}

We know already that necessary conditions in the class Λ^{all} are also necessary in the class Λ^{std}. In the case of weak tractability we thus have matching necessary and sufficient conditions for Λ^{all} and Λ^{std} due to Theorem 2.27. As for polynomial and strong polynomial tractability, we follow a different track of argumentation to find other necessary conditions than the ones implied by Theorem 2.27. Even though we conjecture that the sufficient conditions presented in Corollary 2.31 are also necessary, we currently only have partial results in this direction.

First we show that approximation by a linear algorithm using information from the class Λ^{std} is not easier than integration by a linear algorithm of the same order.

Proposition 2.32. *$\mathbb{L}_2$-Approximation in the space $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ by linear algorithms using N information evaluations from Λ^{std} is not easier than integration by quadratures using N function values, i.e.,*

$$e^{\mathrm{int}}_{s+t,\Lambda^{\mathrm{std}}}(N) \leq e^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(N),$$

where $e^{\mathrm{int}}_{s+t,\Lambda^{\mathrm{std}}}(N)$ denotes the N-th minimal worst-case error of integration using linear algorithms in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

Proof. Consider a linear approximation algorithm

$$A_{N,s,t}(f) = \sum_{v=1}^{N} a_v f(\boldsymbol{x}_v,\boldsymbol{y}_v)$$

with $a_v\in L_2([0,1]^{s+t})$ and $(\boldsymbol{x}_v,\boldsymbol{y}_v)\in[0,1]^{s+t}$. Now define an integration algorithm

$$Q_{N,s,t}(f) := \sum_{v=1}^{N} b_v f(\boldsymbol{x}_v,\boldsymbol{y}_v),$$

where $b_v := \int_{[0,1]^{s+t}} a_v(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$. Then,

$$|I_{s+t}(f) - Q_{N,s,t}(f)| \;=\; \left|\int_{[0,1]^{s+t}} f(\boldsymbol{x},\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - Q_{N,s,t}(f)\right|$$

$$
\begin{aligned}
&= \left| \int_{[0,1]^{s+t}} f(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - \int_{[0,1]^{s+t}} \sum_{v=1}^{N} a_v(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{x}_v,\boldsymbol{y}_v) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right| \\
&\leq \left(\int_{[0,1]^{s+t}} \left| f(\boldsymbol{x},\boldsymbol{y}) - \sum_{v=1}^{N} a_v(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{x}_v,\boldsymbol{y}_v) \right|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right)^{1/2} \\
&= ||f - A_{N,s,t}(f)||_{L_2}.
\end{aligned}
$$

The result follows. □

Remark 2.33. *Note that Proposition 2.32 implies that, given $\varepsilon > 0$,*

$$N^{\mathrm{int}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon) \leq N^{\mathrm{app}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon),$$

where $N^{\mathrm{int}}_{s+t,\Lambda^{\mathrm{std}}}(\varepsilon)$ denotes the information complexity of integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

The following result can be viewed as a special case of Proposition 2.32. It implies that certain linear approximation algorithms, including the algorithm defined in (2.26), cannot have a worst-case error lower than the worst-case error of arbitrary QMC integration algorithms. As mentioned before on p. 30 QMC algorithms are equal weight algorithms and they are described in detail in Section 3.1. Hence we can say that QMC integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ is easier than approximation by means of (2.26).

Proposition 2.34. *Let $N \in \mathbb{N}$. Let $\Xi_{N,s,t}$ be the class of all approximation algorithms $A_{N,s,t}$ of the form*

$$A_{N,s,t}(f)(\boldsymbol{x},\boldsymbol{y}) := \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}} P_{N,s,t}\left(f, \overline{\mathrm{wal}_{\boldsymbol{k}}}, \overline{\mathrm{e}_{\boldsymbol{l}}}\right) \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}),$$

where $\mathcal{A} \subseteq \mathbb{N}_0^s \times \mathbb{Z}^t$ such that $(\boldsymbol{0},\boldsymbol{0}) \in \mathcal{A}$, and where

$$P_{N,s,t}\left(f, \overline{\mathrm{wal}_{\boldsymbol{k}}}, \overline{\mathrm{e}_{\boldsymbol{l}}}\right) = \frac{1}{N} \sum_{v=1}^{N} f((\boldsymbol{p},\boldsymbol{q})_v) \overline{\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{q}_v)} \quad \text{for } (\boldsymbol{k},\boldsymbol{l}) \in \mathcal{A},$$

with $((\boldsymbol{p},\boldsymbol{q})_v)_{v=1}^N = (\boldsymbol{p}_v,\boldsymbol{q}_v)_{v=1}^N \subseteq [0,1)^s \times [0,1)^t$. Furthermore let $\mathcal{Q}_{N,s,t}$ be the class of all QMC algorithms with N points for integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

Then it is true that

$$\inf_{Q_{N,s,t}\in\mathcal{Q}_{N,s,t}} e^{\mathrm{int}}_{s+t}(Q_{N,s,t}) \leq \inf_{A_{N,s,t}\in\Xi_{N,s,t}} e^{\mathrm{app}}_{s+t}(A_{N,s,t}).$$

Proof. The proof is similar to that of Proposition 2.32. Consider $A_{N,s,t} \in \Xi_{N,ss,t}$. Then $A_{N,s,t}$ is a linear approximation algorithm with

$$A_{N,s,t}(f) = \sum_{v=1}^{N} a_v f(\boldsymbol{x}_v,\boldsymbol{y}_v)$$

where

$$a_v(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y} - \boldsymbol{q}_v).$$

Then, using the elementary properties of Walsh and trigonometric functions, and the fact that $(\boldsymbol{0},\boldsymbol{0}) \in \mathcal{A}$,

$$\int_{[0,1]^{s+t}} a_v(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} = \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}} \int_{[0,1]^{s+t}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x} \ominus \boldsymbol{p}_v) \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y} - \boldsymbol{q}_v) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$$

$$
\begin{aligned}
&= \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}} \mathrm{wal}_{\boldsymbol{k}}(\ominus \boldsymbol{p}_v)\mathrm{e}_{\boldsymbol{l}}(-\boldsymbol{q}_v) \int_{[0,1]^{s+t}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
&= \frac{1}{N} \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}} \mathrm{wal}_{\boldsymbol{k}}(\ominus \boldsymbol{p}_v)\mathrm{e}_{\boldsymbol{l}}(-\boldsymbol{q}_v) \int_{[0,1]^{s}} \mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} \int_{[0,1]^{t}} \mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}) \,\mathrm{d}\boldsymbol{y} \\
&= \frac{1}{N} \mathrm{wal}_{\boldsymbol{0}}(\ominus \boldsymbol{p}_v)\mathrm{e}_{\boldsymbol{0}}(-\boldsymbol{q}_v) \\
&= \frac{1}{N}.
\end{aligned}
$$

Then the integration algorithm

$$
Q_{N,s,t}(f) := \sum_{v=1}^{N} b_v f(\boldsymbol{x}_v, \boldsymbol{y}_v),
$$

where $b_v := \int_{[0,1]^{s+t}} a_v(\boldsymbol{x},\boldsymbol{y}) \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$, is a QMC algorithm

$$
Q_{N,s,t}(f) := \frac{1}{N} \sum_{v=1}^{N} f(\boldsymbol{x}_v, \boldsymbol{y}_v).
$$

In the same way as in the proof of Proposition 2.32 we see that

$$
|I_{s+t}(f) - Q_{N,s,t}(f)| \le ||f - A_{N,s,t}(f)||_{L_2}.
$$

This yields the result. □

We can now combine Proposition 2.32 with [44, Theorem 1], which gives necessary conditions for achieving tractability of integration by QMC algorithms in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. This yields the following result.

Theorem 2.35. *Consider approximation in the space $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ using algorithms from the class $\Xi_{N,s,t}$. Then it is true that*

- $\sum_{j=1}^{\infty} \gamma_j^{(1)} < \infty$ *and* $\sum_{j=1}^{\infty} \gamma_j^{(2)} < \infty$ *is a necessary condition for strong polynomial tractability, and*
- $\limsup_{s\to\infty} \sum_{j=1}^{s} \frac{\gamma_j^{(1)}}{\log s} < \infty$ *and* $\limsup_{t\to\infty} \sum_{j=1}^{t} \frac{\gamma_j^{(2)}}{\log t} < \infty$ *is a necessary condition for polynomial tractability.*

Remark 2.36. Note that the algorithm defined in (2.26) lies in the class $\Xi_{N,s,t}$. Hence we cannot hope to achieve tractability using (2.26) under weaker conditions on the weights than those in Theorem 2.31.

The next proposition implies that integration in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$ is neither easier than integration in the Walsh space $H(K_{s,\alpha,\boldsymbol{\gamma}^{(1)}})$ (cf. p. 34) nor in the Korobov space $H(K_{t,\beta,\boldsymbol{\gamma}^{(2)}})$ (cf. p. 34) we defined in Subsection 2.3.1.

Proposition 2.37. *Let $s,t \in \mathbb{N}$ be given and let $e_{s+t}^{\mathrm{int}}(N)$ denote the N-th minimal worst-case error of integration using arbitrary linear algorithms in $\mathcal{H}(K_{s,t,\alpha,\beta,\boldsymbol{\gamma}})$. Furthermore, let $e_{s}^{\mathrm{int}}(N)$ denote the N-th minimal worst-case error of integration using arbitrary linear algorithms in $H(K_{s,\alpha,\boldsymbol{\gamma}^{(1)}})$, and $e_{t}^{\mathrm{int}}(N)$ denote the N-th minimal worst-case error of integration using arbitrary linear algorithms in $H(K_{t,\beta,\boldsymbol{\gamma}^{(2)}})$. Then*

(a) $e_{s}^{\mathrm{int}}(N) \le e_{s+t}^{\mathrm{int}}(N)$,

(b) $e_t^{\mathrm{int}}(N) \le e_{s+t}^{\mathrm{int}}(N)$.

Proof. We show Item (b) of the proposition. Item (a) follows by analogous reasoning. The proof is based on an inductive argument. More precisely we consider $H(K_{(s-1),t,\alpha,\gamma})$ and $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ and show, roughly speaking, that the error of integrating a function in the unit ball of $H(K_{(s-1),t,\alpha,\gamma})$ is not larger than for integrating a "corresponding" function in the unit ball of $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$. Indeed let $f(x_1,\dots,x_{s-1},y_1,\dots,y_t)$ be a function in the unit ball of $H(K_{(s-1),t,\alpha,\gamma})$, i.e., $||f||_{(s-1,t),\alpha,\gamma} \le 1$. We show that for each such $f \in H(K_{(s-1),t,\alpha,\gamma})$ there exists some $\widetilde{f}$ in the unit ball of $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ with at least equally large integration error. For $f \in H(K_{(s-1),t,\alpha,\gamma})$ we consider $\widetilde{f}(x_1,\dots,x_{s-1},x_s,y_1,\dots,y_t)$ $= f(x_1,\dots,x_{s-1},y_1,\dots,y_t) \in \mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$, i.e., $\widetilde{f}$ in fact does not depend on x_s.

It is easily checked that $\widetilde{f}$ lies in the unit ball of $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

Next we investigate the integration error. We need to show that

$$|I_{s-1+t}(f) - A_{N,s-1,t}(f)| \le |I_{s+t}(\widetilde{f}) - A_{N,s,t}(\widetilde{f})|,$$

and

$$A_{N,s-1,t}(f) = \sum_{k=1}^{N} a_k f(\boldsymbol{z}_k)$$

with some $a_k \in \mathbb{C}$ and $\boldsymbol{z}_k = (x_{k,1},\dots,x_{k,s-1},y_{k,1},\dots,y_{k,t}) \in [0,1]^{(s-1)+t}$.

Defining $\widetilde{\boldsymbol{z}_k} = (x_{k,1},\dots,x_{k,s-1},0,y_{k,1},\dots,y_{k,t}) \in [0,1]^{s+t}$ the algorithm $A_{N,s,t}(\widetilde{f})$ is given by

$$A_{N,s,t}(\widetilde{f}) = \sum_{k=1}^{N} a_k \widetilde{f}(\widetilde{\boldsymbol{z}_k}).$$

Then we have

$$\begin{aligned}
|I_{s+t}(\widetilde{f}) - A_{N,s,t}(\widetilde{f})| &= \Bigg| \int_{[0,1]^{s+t}} \widetilde{f}(x_1,\dots,x_{s-1},x_s,y_1,\dots,y_t)\,\mathrm{d}(x_1,\dots,x_{s-1},x_s,y_1,\dots,y_t) \\
&\quad - \sum_{k=1}^{N} a_k \widetilde{f}(x_{k,1},\dots,x_{k,s-1},0,y_{k,1},\dots,y_{k,t}) \Bigg| \\
&= \Bigg| \int_{[0,1]^{(s-1)+t}} f(x_1,\dots,x_{s-1},y_1,\dots,y_t)\,\mathrm{d}(x_1,\dots,x_{s-1},y_1,\dots,y_t) \\
&\quad - \sum_{k=1}^{N} a_k f(x_{k,1},\dots,x_{k,s-1},y_{k,1},\dots,y_{k,t}) \Bigg| \\
&= |I_{s-1,t}(f) - A_{N,s-1,t}(f)|.
\end{aligned}$$

Repeated application of this argument yields the result in (b). □

Proposition 2.37 implies that necessary conditions for achieving tractability of integration in $H(K_{s,\alpha,\gamma^{(1)}})$ or $H(K_{t,\beta,\gamma^{(2)}})$ are also necessary for achieving tractability of approximation using information from Λ^{std} in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$. It should be emphasized here that Proposition 2.37 allows arbitrary linear algorithms for integration in $H(K_{s,\alpha,\gamma^{(1)}})$ and $H(K_{t,\beta,\gamma^{(2)}})$, and is not restricted to QMC algorithms as in Proposition 2.34. Necessary conditions for tractability of integration by arbitrary quadratures in the Korobov space $H(K_{t,\beta,\gamma^{(2)}})$ are given in [64] (see also [30]). Combining the latter results with Proposition 2.37 yields the following theorem.

Theorem 2.38. *Consider approximation in the space $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$ using information from the class Λ^{std}. Then it is true that*

- $\sum_{j=1}^{\infty} \gamma_j^{(2)} < \infty$ *is a necessary condition for strong polynomial tractability, and*

- $\limsup_{t\to\infty} \sum_{j=1}^{t} \frac{\gamma_j^{(2)}}{\log t} < \infty$ *is a necessary condition for polynomial tractability.*

Remark 2.39. The obvious gap between the sufficient conditions in Corollary 2.31 and the necessary conditions in Theorem 2.38 stems from the lack of results on necessary conditions for integration by arbitrary linear algorithms in the Walsh space $H(K_{s,\alpha,\gamma^{(1)}})$. Such results are available if one considers only integration by QMC algorithms (see, e.g., [19]), and these even match the sufficient conditions in Corollary 2.31. However, to the author's best knowledge there are no results in the literature regarding more general integration rules. It is possible that such results could be obtained by proceeding analogously to the methods described in [30] and [64] for the Korobov space. Closing the gap between Corollary 2.31 and Theorem 2.38 remains open for future research.

2.3.5 The optimal algorithm

In this section we want to consider, once more, algorithms which use arbitrary linear functionals as information about f, that is, we study Λ^{all} as our class of information. In Section 2.3.3 we derived the results concerning approximation using Λ^{all} without specifying the algorithms we are using for approximating f. In this setting however, given an error threshold $\varepsilon > 0$, we even know the optimal algorithm (see p. 8 and 9 and also [63, Section 4.2.3]).

It has the form

$$A^{\text{opt}}_{N,s,t,\varepsilon^{-2}}(f)(\boldsymbol{x},\boldsymbol{y}) := \sum_{(\boldsymbol{k},\boldsymbol{l})\in\mathcal{A}_{\varepsilon^{-2}}} \hat{f}(\boldsymbol{k},\boldsymbol{l})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x})\mathrm{e}_{\boldsymbol{l}}(\boldsymbol{y}), \tag{2.37}$$

where we have chosen $\mathcal{A}_M = \mathcal{A}_{\varepsilon^{-2}}$, as defined in (2.25). Note that the functions $\mathrm{wal}_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{N}_0^s$, form an orthonormal basis of $L_2([0,1]^s)$, and that the functions $\mathrm{e}_{\boldsymbol{l}}$, $\boldsymbol{l} \in \mathbb{Z}^t$, form an orthonormal basis of $L_2([0,1]^t)$. From this it follows that $\mathrm{wal}_{\boldsymbol{k}}\mathrm{e}_{\boldsymbol{l}}$, $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{l} \in \mathbb{Z}^t$, is an orthonormal basis of $L_2([0,1]^{s+t})$. Furthermore, it is easily checked that the $\mathrm{wal}_{\boldsymbol{k}}\mathrm{e}_{\boldsymbol{l}}$ are mutually orthogonal in $\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})$.

Indeed

$$\langle \mathrm{wal}_{\boldsymbol{k}}\mathrm{e}_{\boldsymbol{l}}, \mathrm{wal}_{\boldsymbol{j}}\mathrm{e}_{\boldsymbol{m}}\rangle_{s,t,\alpha,\beta,\gamma} = \begin{cases} \left(\rho_{\alpha,\gamma^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\gamma^{(2)}}(\boldsymbol{l})\right)^{-1} & \text{if} \quad (\boldsymbol{k},\boldsymbol{l}) = (\boldsymbol{j},\boldsymbol{m}),\\ 0 & \text{otherwise.}\end{cases}$$

Hence, using Parseval's identity, the error can be calculated as

$$\left\|\mathrm{EMB}_{s+t}(f) - A^{\text{opt}}_{N,s,t,\varepsilon^{-2}}(f)\right\|^2_{L_2([0,1]^{s+t})} = \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_{\varepsilon^{-2}}} |\hat{f}(\boldsymbol{k},\boldsymbol{l})|^2, \tag{2.38}$$

and we obtain

$$\begin{aligned}\left\|\mathrm{EMB}_{s+t}(f) - A^{\text{opt}}_{N,s,t,\varepsilon^{-2}}(f)\right\|^2_{L_2([0,1]^{s+t})} &= \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_{\varepsilon^{-2}}} |\hat{f}(\boldsymbol{k},\boldsymbol{l})|^2\\ &= \sum_{(\boldsymbol{k},\boldsymbol{l})\notin\mathcal{A}_{\varepsilon^{-2}}} |\hat{f}(\boldsymbol{k},\boldsymbol{l})|^2 \left(\rho_{\alpha,\gamma^{(1)}}(\boldsymbol{k})\right)^{-1}\left(r_{\beta,\gamma^{(2)}}(\boldsymbol{l})\right)^{-1} \rho_{\alpha,\gamma^{(1)}}(\boldsymbol{k}) r_{\beta,\gamma^{(2)}}(\boldsymbol{l})\\ &< \varepsilon^2 ||f||^2_{\mathcal{H}(K_{s,t,\alpha,\beta,\gamma})},\end{aligned}$$

where we used the definition of the set $\mathcal{A}_{\varepsilon^{-2}}$ to see the inequality.

This means that for the algorithm given by (2.37), we always obtain

$$e^{\text{app}}_{s+t,\Lambda^{\text{all}}}(A^{\text{opt}}_{N,s,t,\varepsilon^{-2}}) < \varepsilon.$$

3 Componentwise constructions of (polynomial) lattice point sets

3.1 Introduction

Recall that in Section 2.1 we consider multivariate continuous problems. Suppose we have a function space $\mathcal{H}_s$, a normed space $\mathcal{G}$ and a solution operator $S\colon \mathcal{H}_s \to \mathcal{G}$. In Section 2.1 we were interested in how much information is needed to solve such problem at least with a given accuracy.

Now we want to consider specifically the problem of numerical integration in multivariate function spaces, i.e., the solution operater is given by $S\colon \mathcal{H}_s \to \mathbb{R}$, with

$$S(f) = \int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

This means, in particular that we study the case where $\mathcal{G} = \mathbb{R}$. For the integration problem we of course study algorithms which use information from the class Λ^{std}, i.e., we use function evaluations as information. In contrast to Section 2.1, now we are mostly interested in how to choose the information, that means at which points we evaluate the integrand, rather than in how many information evaluations are needed to reach a certain error threshold.

As mentioned before in Section 2.1, linear, non-adaptive algorithms are optimal for this type of problem. Thus we use algorithms of the form

$$\sum_{k=0}^{N-1} q_k f(\boldsymbol{p}_k), \tag{3.1}$$

with $q_k \in \mathbb{R}$ and $\boldsymbol{p}_k \in [0,1)^s$ for numerical approximation of integrals of functions over $[0,1]^s$.

Let us briefly go back to the one-dimensional problem of integrating a univariate function f over $[0,1]$, which can be approximated by an algorithm of the form

$$\sum_{k=0}^{N} t_k f(p_k),$$

with $t_k \in \mathbb{R}$ and $p_k \in [0,1)$. For example one could use the trapezoidal rule (cf. [56, Section 1.1]), which uses $t_0 = t_N = \frac{1}{2N}$ and $t_1 = \cdots = t_{N-1} = \frac{1}{N}$ and equidistant sample points $p_k = \frac{k}{N}$. For dimensions $s > 1$ one can use the Cartesian product of the trapezoidal rule (or any other one-dimensional quadrature rule). Then one ends up with a quadrature rule of the form

$$\sum_{k_1=0}^{N} \cdots \sum_{k_s=0}^{N} t_{k_1} \cdots t_{k_s} f(p_{k_1}, \ldots, p_{k_s}),$$

which can of course also be displayed as a quadrature rule of the form (3.1). A quadrature rule like this, uses $(N+1)^s$ sample points, a number that explodes with growing dimension s.

A solution to this problem is to use equal-weight quadrature rules to approximate integrals of functions over $[0,1]^s$,

$$\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \approx \frac{1}{N}\sum_{k=0}^{N-1} f(\boldsymbol{p}_k). \tag{3.2}$$

What remains is the problem as to how to choose the sample points. One possibility is to choose them randomly, which results in a method called Monte Carlo integration.

The other possibility is to choose the sample points deterministically and to try to beat Monte Carlo. In this case a quadrature rule of the form (3.2) is called quasi-Monte Carlo (QMC) algorithm. For detailed information on QMC integration see [20, 55, 56, 60].

In what follows we study QMC integration.

Here, the function f belongs to some suitable (weighted) function space or function class, and the sample points $\{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{N-1}\}$ are deterministically chosen from $[0,1)^s$. It turns out that lattice point sets are often a good choice, see, e.g., [20, 72]. Lattice point sets were introduced for the first time independently from Hlawka [32] and Korobov [41]. They are usually constructed with the aid of a generating vector $\boldsymbol{z} = (z_1, \ldots, z_s) \in \mathbb{Z}^s$ and are defined as follows.

Definition 3.1. *Let $s, N \in \mathbb{N}$ and $\boldsymbol{z} = (z_1, \ldots, z_s) \in \mathbb{Z}^s$. Then*

$$\mathcal{P}(N, \boldsymbol{z}) = \left\{ \left\{ \frac{k\boldsymbol{z}}{N} \right\} : k = 0, \ldots, N-1 \right\}$$

is the N-point lattice point set corresponding to $\boldsymbol{z}$. Here, the braces around $\frac{k\boldsymbol{z}}{N}$ indicate that we consider the fractional part of each coordinate of $\frac{k\boldsymbol{z}}{N}$.

Remark 3.2. *From [20, p. 84f.] and [56, p. 73f.] we know that we can restrict ourselves to considering only generating vectors $\boldsymbol{z} \in \{0, 1, \ldots, N-1\}^s$. Additionally, for the generating vector $\boldsymbol{z}$, one often requires $\gcd(z_j, N) = 1$ for all components z_j, with $j = 1, \ldots, s$, to achieve better distributions. Thus let*

$$\mathcal{Z}_N = \{z \in \{1, \ldots, N-1\} \colon \gcd(z, N) = 1\}. \tag{3.3}$$

We make this additional requirement throughout the rest of this thesis, so, using this notation, we study generating vectors $\boldsymbol{z} \in \mathcal{Z}_N^s$.

The goal is to construct generating vectors $\boldsymbol{z} \in \mathcal{Z}_N^s$ which yield lattice point sets that perform well in QMC algorithms such as in (3.2). In what follows we want to consider two quality criteria—the (weighted) star discrepancy criterion and the worst-case error criterion.

We are considering weighted spaces; Let $[s] = \{1, 2, \ldots, s\}$ and let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq [s]}$, with non-negative reals $\gamma_{\mathfrak{u}}$, be weights, i.e., every group of variables $\{x_i \colon i \in \mathfrak{u}\}$ is equipped with its weight $\gamma_{\mathfrak{u}}$. Roughly speaking, small weights indicate that the corresponding variables contribute little to the integration problem, whereas for large weights the opposite is true. Here we consider only product weights, as introduced in Section 2.1 on p. 10.

The weighted star discrepancy was introduced in 1998 by Sloan and Woźniakowski [74], exploiting the insight that the weights reflect the influence of different coordinates on the integration error.

Definition 3.3. *Let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq [s]}$ be a weight sequence and $\mathcal{P} = \{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{N-1}\} \subseteq [0,1)^s$ be an N-element point set. The local discrepancy $\Delta(\boldsymbol{t}, \mathcal{P})$ of the point set $\mathcal{P}$ at $\boldsymbol{t} = (t_1, \ldots, t_s) \in (0,1]^s$ is defined as*

$$\Delta(\boldsymbol{t}, \mathcal{P}) = \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{1}_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{p}_k) - \prod_{j=1}^{s} t_j, \tag{3.4}$$

where $\mathbf{1}_{[\mathbf{0},\boldsymbol{t})}$ denotes the indicator function of $[\mathbf{0},\boldsymbol{t}) = [0,t_1) \times \cdots \times [0,t_s)$. Then the weighted star discrepancy $D^*_{N,\gamma}(\mathcal{P})$ of the point set $\mathcal{P}$ is defined as

$$D^*_{N,\gamma}(\mathcal{P}) = \sup_{\boldsymbol{t}\in(0,1]^s} \max_{\emptyset\neq\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}} \left|\Delta((\boldsymbol{t}_{\mathfrak{u}},\mathbf{1}),\mathcal{P})\right|, \tag{3.5}$$

where we denote by $(\boldsymbol{t}_{\mathfrak{u}},\mathbf{1})$ the vector $(\tilde{t}_1,\ldots,\tilde{t}_s)$ with $\tilde{t}_j = \begin{cases} t_j, & \text{if } j \in \mathfrak{u} \\ 1 & \text{otherwise.} \end{cases}$

Remark 3.4. *For a lattice point set $\mathcal{P}(N,\boldsymbol{z})$ with generating vector $\boldsymbol{z}$ we often write $D^*_{N,\gamma}(\boldsymbol{z})$ instead of $D^*_{N,\gamma}(\mathcal{P}(N,\boldsymbol{z}))$ for the weighted star discrepancy of $\mathcal{P}(N,\boldsymbol{z})$.*

Remark 3.5. *One can picture star discrepancy as follows. Suppose we have a point set in the unit cube and consider boxes in the unit cube anchored in the origin. We compare the volume of these boxes to the ratio of the points inside the boxes and the overall number of points. The star discrepancy is then the supremum of all these differences.*

In this thesis we consider weighted star discrepancy. Here, as before, weights are a means to create a setting which is closer to reality. Problems in weighted spaces arise naturally from many applications and the weights reflect the fact that not all coordinates or groups of coordinates have the same influence on the problem. We study product weights in this thesis. For product weights the influence of a coordinate decreases as its index increases.

Given a function f and some point set $\mathcal{P} = \{\boldsymbol{p}_0,\ldots,\boldsymbol{p}_{N-1}\}$ the following inequality holds true. It is called weighted Koksma-Hlawka inequality (cf. [74]),

$$\left|\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{k=0}^{N-1} f(\boldsymbol{p}_k)\right| \leq D^*_{N,\gamma}(\mathcal{P})\,\|f\|_{\gamma}, \tag{3.6}$$

where $\|\cdot\|_{\gamma}$ is a norm, dependent only on the weight sequence $\boldsymbol{\gamma}$, but independent of the point set $\mathcal{P}$. The Koksma-Hlawka inequality stems from the following identity of Hlawka [31] and Zaremba [78] (see also [20, 56]), given by

$$\frac{1}{N}\sum_{k=0}^{N-1} f(\boldsymbol{p}_k) - \int_{[0,1]^s} f(\boldsymbol{x}) = \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} (-1)^{\mathfrak{u}}\gamma_{\mathfrak{u}} \int_{[0,1]^{|\mathfrak{u}|}} \Delta((\boldsymbol{x}_{\mathfrak{u}},\mathbf{1}),\mathcal{P}_N(\boldsymbol{z}))\gamma_{\mathfrak{u}}^{-1}\frac{\partial^{|\mathfrak{u}|}}{\partial \boldsymbol{x}_{\mathfrak{u}}} f(\boldsymbol{x}_{\mathfrak{u}},\mathbf{1})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}.$$

Applying Hölder's inequality to the latter identity as done in [20, 74] for integrals and sums yields (3.6). Inequality (3.6) connects the integration error of QMC algorithms to the weighted star discrepancy. Moreover, it enables us to split the problem into two parts, where the first part purely depends on the point set used in the QMC rule, and the second part shows the influence of the function f on the integration error. Obviously, it is beneficial to have lattice point sets with small weighted star discrepancy and to use them in QMC rules, and thus in Sections 3.2 and 3.3 we consider ways to construct (polynomial) lattice point sets with small weighted star discrepancy.

Another interesting aspect of the discrepancy of high dimensional point sets is the so-called tractability of discrepancy (see, e.g., [63, 64, 65] for detailed information). For $N,s \in \mathbb{N}$ let

$$\mathrm{disc}_{\infty}(N,s) := \inf_{\substack{\mathcal{P}\subseteq[0,1)^s \\ \#\mathcal{P}=N}} D^*_{N,\gamma}(\mathcal{P}),$$

be the Nth minimal star discrepancy. To introduce the concept of tractability of discrepancy we define the information complexity in this context (also called the inverse of the weighted star discrepancy) as

$$N^*(s,\varepsilon) := \min\{N \in \mathbb{N} \mid \mathrm{disc}_{\infty}(N,s) \leq \varepsilon\}.$$

Thus $N^*(s,\varepsilon)$ is the minimal number of points required to achieve a weighted star discrepancy of at most ε. Note the analogy to tractability of the worst-case error (cf. Section 2.1), where the information complexity is the minimal number of points required to achieve a worst-case error of at most ε.

Similarly to the worst-case error case, also here, to keep the construction cost of our generating vector low, it is, of course, beneficial to have a small information complexity and thus to stand a chance to have a lattice point set of small size. This is why we are interested in how fast the information complexity grows when s and ε^{-1} tend to infinity. Tractability describes this dependence of the information complexity on the dimension s and the error demand ε. The best we can hope for is the case where $N^*(s,\varepsilon)$ is independent of s and depends at most polynomially on ε^{-1}. To be more precise, we say that we achieve strong polynomial tractability if there exist constants $C,\tau>0$ such that

$$N^*(s,\varepsilon) \leq C\varepsilon^{-\tau}$$

for all $s\in\mathbb{N}$ and all $\varepsilon\in(0,1)$. Recall from Section 2.1 that a problem is considered tractable if its information complexity's dependence on s and ε^{-1} is not exponential. Taking weights into account in the definition of discrepancy can sometimes overcome the so-called curse of dimensionality, i.e., an exponential dependence of $N^*(s,\varepsilon)$ on s.

The second quality criterion we want to consider is the worst-case error criterion. As a QMC algorithm is completely determined by the underlying point set we denote the worst-case error by $e_{\mathcal{H}_s,\gamma}(\mathcal{P})$. Here, $\mathcal{H}_s$ denotes the respective function space we are working in and $\|\cdot\|_{\mathcal{H}_s}$ its norm. As before, if it is clear which function space we consider, we abbreviate our notation to $e_{s,\gamma}(\mathcal{P})$. In this context the worst-case error $e_{s,\gamma}(\mathcal{P})$ of the point set $\mathcal{P}=\{\boldsymbol{p}_0,\dots,\boldsymbol{p}_{N-1}\}$ introduced in (2.1) takes the form

$$e_{s,\gamma}(\mathcal{P}) = \sup_{\substack{f\in\mathcal{H}_\gamma \\ \|f\|_{\mathcal{H}_\gamma}\leq 1}} \left| \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \frac{1}{N}\sum_{j=0}^{N-1} f(\boldsymbol{p}_j) \right|,$$

where $\mathcal{H}_\gamma$ denotes some suitable weighted function space. For lattice point sets $\mathcal{P}_N(\boldsymbol{z})$ we often write $e_{s,N,\gamma}(\boldsymbol{z})$ instead of $e_{s,\gamma}(\mathcal{P})$. In Section 3.4 we consider the worst-case error as the quality criterion.

Now we take a brief look at how star discrepancy and worst-case error interrelate, see also [20, Section 2.4]. Let us consider the special reproducing kernel Hilbert $\bar{\mathcal{H}}_s$ space with kernel

$$\bar{K}_s(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{s} \min\{1-x_j, 1-y_j\},$$

with $\boldsymbol{x}=(x_1,\dots,x_s), \boldsymbol{y}=(y_1,\dots,y_s)\in\mathbb{R}_s$. For $s=1$ this space contains all absolutely continuous functions $f\colon[0,1]\to\mathbb{R}$ with $f(1)=0$ and square integrable first derivative. If, for example, $f_1,\dots,f_s$ are elements of $\bar{\mathcal{H}}_1$, then $f(x_1,\dots,x_s)=\prod_{i=1}^{s} f_i(x_i)$ is in $\bar{\mathcal{H}}_s$. Apart from these products and sums of these products, $\bar{\mathcal{H}}_s$ contains also its completion with respect to the norm induced by the inner product

$$\langle f,g\rangle = \int_{[0,1]^s} \frac{\partial^s f}{\partial\boldsymbol{x}}\frac{\partial^s g}{\partial\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x},$$

where $\frac{\partial^s f}{\partial\boldsymbol{x}}(x) = \frac{\partial^s f}{\partial x_1\cdots\partial x_s}(x)$.

Then for an N-point set $\mathcal{P}_N\subseteq[0,1)^s$, it is true that

$$e_{\bar{\mathcal{H}}_s,\gamma}(\mathcal{P}_N) = \left(\int_{[0,1]^s} |\Delta(\boldsymbol{x},\mathcal{P}_N)|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

Note that here $\boldsymbol{\gamma} = (1)_{j\geq 1}$, as we consider an unweighted space.

As for the weighted case consider a weight sequence $\boldsymbol{\gamma}$ and the weighted Sobolev space $W_2^{(1,\dots,1)}([0,1]^s)$, equipped with the norm $\|\cdot\|_{W_2^{(1,\dots,1)},\boldsymbol{\gamma}}$ and

$$F_{s,\boldsymbol{\gamma}} = \{f \in W_2^{(1,\dots,1)}([0,1]^s) \colon \|f\|_{W_2^{(1,\dots,1)},\boldsymbol{\gamma}} < \infty\}.$$

Then we know from [74, Theorem 1] that

$$e^{\mathrm{wor}}_{F_{s,\boldsymbol{\gamma}}}(A_{n,s}) = \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \int_{[0,1]^s} (\Delta((\boldsymbol{t}_{\mathfrak{u}}, \boldsymbol{1}), \mathcal{P}))^2 \, \mathrm{d}\boldsymbol{t}_{\mathfrak{u}} \right)^{\frac{1}{2}},$$

where $A_{n,s}$ is the QMC algorithm using the elements of the n-point set $\mathcal{P}$ as sample points.

The goal in the following sections is to construct generating vectors for (polynomial) lattice point sets with small weighted star discrepancy, and/or worst-case error, respectively. For dimensions $s = 1,2$ explicit constructions are available, see for example [4] and references cited there. For dimensions $s \geq 3$, however, this is not the case, and one usually has to resort to computer search algorithms, most commonly component-by-component (CBC) algorithms. The standard structure of a CBC construction is as follows: We start by setting the first component z_1 of the generating vector equal to 1. Then in each step one component is added until we have a full-size generating vector $\boldsymbol{z} = (z_1, \dots, z_s)$. When adding one component, all previously chosen components $z_1, \dots, z_d$ remain the same and the new component z_{d+1} is chosen from a search set, e.g., $\mathcal{Z}_N$, to minimize the weighted star discrepancy or the worst-case error, respectively, of $(z_1, \dots, z_d, z_{d+1})$ as a function of z_{d+1}. The algorithm terminates once z_s has been chosen.

It is an advantage of CBC constructions that they are extensible in the dimension. This means that if one has calculated an s-dimensional generating vector for a lattice point set with the aid of a CBC construction and wants to extend the result to an $(s+1)$-dimensional point set, he only need to do one more step of the CBC construction, rather than starting again from scratch.

In general, CBC constructions do not result in an optimal generating vector. However, the obtained vectors are in many settings of optimal order of star discrepancy and worst-case error, respectively cf., e.g., [38] and [49], and numerical results, e.g., in [49] show that they are also performing well in terms of implied constants.

As CBC constructions yield good results and a search through all generating vectors $\boldsymbol{z} \in \mathcal{Z}^s$ would be completely insurmountable even for relatively small values of N and s, we try to improve CBC constructions even more. (The number of elements in $\mathcal{Z}_N^s$ is $\phi(N)^s \geq N^{\frac{2s}{3}}$, for all $N \geq 30$, where $\phi(\cdot)$ denotes Euler's totient function. The estimate for $\phi(N)$ stems from [40])

The first CBC construction is due to Korobov [42] and has been rediscovered by Sloan and Reztsov in 2002 [73]. Sloan and Reztsov constructed lattice point sets for the integration of functions from unweighted Korobov spaces, based on the worst-case error criterion. (If we set all the weights γ_j equal to 1 in the definition of Korobov spaces on p. 34 we obtain the unweighted Korobov space.) Sloan's and Reztsov's CBC algorithm reads as follows.

Algorithm 3.6. *Let $s \in \mathbb{N}$ and let N be a prime. Determine $\boldsymbol{z} = (z_1, \dots, z_s)$ in the following way.*

1. *Set $z_1 = 1$.*
2. *For $1 \leq d < s$ assume $z_1, \dots, z_d$ to be already chosen. Find $z_{d+1} \in \mathcal{Z}_N$ as minimizer of*

$$e_{d+1,N,\boldsymbol{\gamma}}(z_1, \dots, z_d, z_{d+1})$$

as a function of z_{d+1}.

3. Increase d by 1 and repeat Step 2 while $d < s$.

The condition that N is a prime in Algorithm 3.6 is due to technical reasons and we know from Bertrand's postulate that this is not a big restriction.

Algorithm 3.6 yields a generating vector $\boldsymbol{z} = (z_1, \ldots, z_s)$ that fulfills the following worst-case error bound (cf. [73, Theorem 2.1]). Let $\beta > 1$ and N be a prime, with $N \geq 1 + 2\zeta(\beta)$, where $\zeta(\cdot)$ denotes Riemann's Zeta function. Then for all $s \in \mathbb{N}$ and all $\alpha \geq \beta$

$$e_{s,N,\gamma}(z_1, \ldots, z_s) \leq \frac{(1 + 2\zeta(\beta))^{\frac{s\alpha}{\beta}}}{N^{\frac{\alpha}{\beta}}}.$$

The computational cost of Algorithm 3.6 is of order sN^2.

It was proved in 2001 by Sloan and Woźniakowski [75] that the optimal rate of convergence in weighted Korobov spaces is given by $O(N^{-\frac{\alpha}{2}+\delta})$, with $\delta > 0$ arbitrarily small and the implied constant independent of s. The error bound quoted above does not reach this rate, as it is true for any $1 < \beta \leq \alpha$. In 2003, however, Kuo [49] proved that the same algorithm applied to weighted Korobov spaces rather than unweighted Korobov spaces as used by Sloan and Reztsov, indeed yields the optimal convergence rate. She proved the following theorem (cf. [49, Corollary 2 and Theorem 4]).

Theorem 3.7. *Let N be prime. For all $1 \leq d \leq s$ find z_d as the minimizer of $e^2_{d,N,\gamma}(z_1, \ldots, z_d)$ over the set $\mathcal{Z}_s$. Then for all $\frac{1}{\alpha} < \lambda \leq 1$*

$$e^2_{s,N,\gamma}(z_1, \ldots, z_s) \leq 2^{\frac{1}{\lambda}} N^{-\frac{1}{\lambda}} \prod_{j=1}^{s} (1 + 2\gamma_j^{\lambda} \zeta(\alpha\lambda))^{\frac{1}{\lambda}}$$

and

$$e_{s,N,\gamma}(z_1, \ldots, z_s) = O(N^{-\frac{\alpha}{2}+\delta}),$$

for all $0 < \delta \leq \frac{\alpha-1}{2}$. The implied constant is independent of s.

Remark 3.8. *Let $\varepsilon \in (0, 1)$. Then we obtain $e_{s,N,\gamma}(z_1, \ldots, z_s) < \varepsilon$ for all $N > c\varepsilon^{-\frac{2}{\alpha-2\delta}}$, with a constant $c > 0$ independent of the dimension s. Thus we have strong polynomial tractability.*

When implementing algorithms like Algorithm 3.6 one has to perform several costly matrix-vector multiplications. It turns out that the matrices involved are a special form of block matrix, consisting of identical blocks. This block structure can be exploited using fast Fourier transform (FFT) to reduce the construction cost from $O(sN^2)$ to $O(sN \log N)$. These faster versions are called fast CBC constructions. They are due to Nuyens and Cools [66, 67].

Algorithms very similar to Algorithm 3.6 have been considered by Sinescu and Joe [38, 70, 71] using the weighted star discrepancy criterion. The main difference is that in Step 2 of their algorithms the weighted star discrepancy is minimized instead of the worst-case error. The constructions of Sinescu and Joe reach the optimal order of the weighted star discrepancy, $D^*_{N,\gamma}(\boldsymbol{z}) = O(N^{-1+\delta})$, for any $\delta > 0$. Sinescu and Joe already used methods of Nuyens and Cools [66, 67] to reduce the computational cost to $O(sN \log N)$, as described above.

In 2015 Dick, Kritzer, Leobacher and Pillichshammer [13] introduced a method to speed up CBC constructions in weighted spaces even further. In their paper [13] they consider weighted Korobov spaces as defined on p. 34.

It is the nature of weighted spaces that not all coordinates of the generating vector $\boldsymbol{z}$ have equal amount of influence on the quality of the corresponding lattice point set. In what follows, for simplicity, we only consider product weights. Recall from p. 10 that these are given via a nonincreasing weight sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$. The weights $\gamma_{\mathfrak{u}}$ are then defined as $\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$. In product weighted spaces the components z_j of the generating vector have less and less influence on the quality of the approximation as their index j increases. Roughly speaking, this is due to the weights γ_j which are diminishing with increasing index j. We can exploit this property in the following way. As the components' influence is decreasing with their indices we want to make less effort and have less computational cost for choosing these components. To achieve this we choose them from ever smaller search sets, which are defined as follows. Let $w_1 \leq w_2 \leq \cdots$ be a nondecreasing sequence of non-negative integers. We sometimes assume for technical reasons that $w_1 = 0$. This sequence of w_j's is determined in accordance with the weight sequence $\boldsymbol{\gamma}$. Loosely speaking, the smaller γ_j, the bigger w_j is chosen. For $N = b^m$, with a prime b and $m \in \mathbb{N}$, the reduced search spaces $\mathcal{Z}_{N,w_j}$ are defined as

$$\mathcal{Z}_{N,w_j} = \begin{cases} \{z \in \{1,\ldots,b^{m-w_j}-1\} \colon \gcd(z,b^m)=1\}, & \text{if } w_j < m,\\ \{1\}, & \text{if } w_j \geq m. \end{cases} \tag{3.7}$$

The cardinality of these reduced search spaces is

$$\left|\mathcal{Z}_{N,w_j}\right| = \begin{cases} b^{m-w_j-1}(b-1), & \text{if } w_j < m,\\ 1, & \text{if } w_j \geq m, \end{cases}$$

as opposed to $|\mathcal{Z}_N| = b^{m-1}(b-1)$ for the full search space $\mathcal{Z}_N$. This means a reduction of the size by a factor of b^{-w_j}, if $w_j < m$.

The reduced CBC algorithm by Dick et al. [13] is then given by

Algorithm 3.9. *Let $N = b^m$, $0 = w_1 \leq w_2 \leq \ldots$ and $\mathcal{Z}_{N,w_j}$ be defined as above. Construct $\boldsymbol{z} = (b^{w_1}z_1,\ldots,b^{w_s}z_s)$ as follows.*

1. *Set $z_1 = 1$.*
2. *For $1 \leq d < s$ assume that $z_1,\ldots,z_d$ have already been found. Choose $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$ such that*

$$e_{d+1,N,\boldsymbol{\gamma}}((b^{w_1}z_1,\ldots,b^{w_d}z_d,b^{w_{d+1}}z_{d+1}))$$

is minimized as a function of z_{d+1}.

3. *Increase d by 1 and repeat the second step until $\boldsymbol{z} = (b^{w_1}z_1,\ldots,b^{w_s}z_s)$ is found.*

This algorithm can again be implemented using the fast methods of Nuyens and Cools. In this case it is often called the reduced fast CBC algorithm.

A generating vector $\boldsymbol{z}$ constructed with Algorithm 3.9 yields [13, Corollary 1]

$$e_{s,N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq c_{s,\alpha,\boldsymbol{\gamma},\delta,\boldsymbol{w}} N^{-\frac{\alpha}{2}+\delta},$$

for any $\delta \in (0, \frac{\alpha-1}{2}]$, where $\alpha > 1$ is the smoothness parameter of the weighted function space under consideration, in this case it is the smoothness parameter of the weighted Korobov space. Further $\boldsymbol{w}$ denotes the weight sequence $0 = w_1 \leq w_2 \leq \ldots$. The constant $c_{s,\alpha,\boldsymbol{\gamma},\delta,\boldsymbol{w}}$ is given by

$$c_{s,\alpha,\boldsymbol{\gamma},\delta,\boldsymbol{w}} = \left(2 \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}}^{\frac{1}{\alpha-2\delta}} \left(2\zeta\left(\frac{\alpha}{\alpha-2\delta}\right)\right)^{|\mathfrak{u}|} b^{\max_{j\in\mathfrak{u}} w_j}\right)^{\frac{\alpha}{2}-\delta}.$$

It takes

$$O\left(N\log N + \min\{s,s^*\}N + \sum_{d=1}^{\min\{s,s^*\}} (m-w_d)b^{m-w_d}\right)$$

operations to compute $\boldsymbol{z}$ with Algorithm 3.9. Here, s^* denotes the smallest j such that $w_j \geq m$. Thus, if s^* is finite and s is large enough, the construction cost is independent of the dimension.

A similar reduced fast CBC construction can be done when using the weighted star discrepancy criterion. This is the content of Section 3.2.

So far we considered means to speed up the original Algorithm 3.6 by Sloan and Reztsov so that they are feasible for large dimensions s and large N. Numerical experiments, however, show that these algorithms tend to produce generating vectors with recurring components, i.e., there exist $i, j \in [s]$, with $i \neq j$ and $z_i = z_j$. We quote from [50]:

> [...] However, it has been observed that the components start to repeat from some dimension onward for product-type weights, hence leading to a practical limit on the value of d [we remark that d has the role of s in [50]]. This side effect of the CBC algorithm is yet to be fully understood.

Gantner and Schwab write in [25]:

> [...] For large values of the worst-case error, the elements of the generating vector can repeat, leading to very bad projections in certain dimensions.

As mentioned in the quote from Kuo above this effect is not yet fully understood. It could be due to numerical problems of the algorithm, see [67, page 386]. There is, however, a way around the problem. Gantner and Schwab [25] as well as Dick and Kritzer [10] have come up with methods to avoid this problem. Gantner and Schwab call their method pruning in the CBC construction, while Dick and Kritzer name their refined version of this method projection-corrected CBC construction. The general idea is in each step of the CBC algorithm to define some exclusion set $\mathcal{E}$ whose elements cannot be selected as component of the generating vector in this step. By defining the exclusion sets as the sets consisting of all elements chosen in the previous steps one can effectively avoid components showing up several times. This method can also be used to avoid other phenomena, like for example all lattice points lying on an antidiagonal. For detailed information see [10].

The projection-corrected CBC algorithm of Dick and Kritzer is again designed for weighted Korobov spaces (cf. p. 34). It leads to generating vectors with worst-case error

$$e_{s,N,\gamma}(\boldsymbol{z}) \leq \left(\frac{1}{\phi(N)}\sum_{\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}}^{\lambda}(2\zeta(\alpha\lambda))^{|\mathfrak{u}|}\prod_{j\in\mathfrak{u}}\frac{\phi(N)}{\phi(N)-|\mathcal{E}_j|}\right)^{\frac{1}{2\lambda}},$$

for all $\frac{1}{\alpha} < \lambda < 1$, where $\mathcal{E}_j \subset \mathcal{Z}_N$ are the aforementioned exclusion sets. It is also shown in [10] that, as long as the relative size of the exclusion sets is uniformly bounded, tractability results are not affected. When using the fast matrix-vector multiplication of Nuyens and Cools the projection-corrected CBC algorithm can be implemented using $O(sN\log N)$ operations. It is the aim of Section 3.4 to combine the projection-corrected CBC construction with the reduced fast method to obtain an algorithm that is fast and yields a generating vector free of recurring components.

Up to now we only considered lattice point sets as choice of sample points in QMC algorithms. Another good choice are polynomial lattice point sets, which are defined below. They have been introduced by Niederreiter in [60, Chapter 4], [61]. In fact, it turns out that in some cases lattice point sets yield better results, whereas in other situations polynomial lattice point sets are the better choice. For instance higher-order polynomial lattice point sets work very well for smooth integrands, whereas lattice point sets turn out to be particularly well suited for smooth periodic functions. For information on higher-order polynomial lattice point sets see for example [8]. For a detailed comparison of lattice point sets and polynomial lattice point sets see, e.g., [68]. Thus it is useful to have methods at hand for the construction of good lattice point sets as well as good polynomial lattice point sets.

Dick et al. [13] considered a version of their reduced fast CBC algorithm for the construction of polynomial lattices as well. This algorithm also leads to a polynomial lattice point set with small worst-case error and a construction cost that becomes independent of the dimension eventually. In Section 3.3 we give a version of a CBC construction for polynomial lattice point sets that uses the weighted star discrepancy as the quality criterion.

Recall from p. 40 that polynomial lattice point sets are defined as follows. For a prime number p, let $\mathbb{F}_p$ be the finite field of order p. We identify $\mathbb{F}_p$ with the set $\{0, 1, \ldots, p-1\}$ equipped with the modulo p arithmetic. We denote by $\mathbb{F}_p[x]$ the set of polynomials over $\mathbb{F}_p$ and by $\mathbb{F}_p((x^{-1}))$ the field of formal Laurent series over $\mathbb{F}_p$ with elements of the form

$$L = \sum_{l=\omega}^{\infty} t_l x^{-l},$$

where $\omega \in \mathbb{Z}$ and $t_l \in \mathbb{F}_p$ for all $l \geq \omega$. For a given dimension $s \geq 2$ and an integer $m \geq 1$ we choose a so-called modulus $f \in \mathbb{F}_p[x]$ with $\deg(f) = m$, as well as polynomials $g_1, \ldots, g_s \in \mathbb{F}_p[x]$, with $\deg(g_j) < m$ for all $1 \leq j \leq s$. The vector $\boldsymbol{g} = (g_1, \ldots, g_s)$ is called the generating vector of the polynomial lattice point set. Further, we introduce the map $\phi_m : \mathbb{F}_p((x^{-1})) \to [0, 1)$ such that

$$\phi_m\left(\sum_{l=\omega}^{\infty} t_l x^{-l}\right) = \sum_{l=\max\{1,\omega\}}^{m} t_l p^{-l}.$$

With $n \in \{0, 1, \ldots, p^m - 1\}$ we associate the polynomial

$$n(x) = \sum_{r=0}^{m-1} n_r x^r \in \mathbb{F}_p[x],$$

as each such n can uniquely be written as $n = n_0 + n_1 p + \cdots + n_{m-1} p^{m-1}$ with digits $n_r \in \{0, 1, \ldots, p-1\}$ for all $r \in \{0, 1, \ldots, m-1\}$.

Definition 3.10. *With the notation above, the polynomial lattice point set $\mathcal{P}(\boldsymbol{g}, f)$ is defined as the set of $N = p^m$ points*

$$\boldsymbol{x}_n = \left(\phi_m\left(\frac{n(x)g_1(x)}{f(x)}\right), \ldots, \phi_m\left(\frac{n(x)g_s(x)}{f(x)}\right)\right) \in [0, 1)^s$$

for $0 \leq n \leq p^m - 1$.

The name polynomial lattice point sets stems from the fact that their construction resembles very much that of lattice point sets. Recall that one point of lattice point set is of the form

$$\boldsymbol{p}_n = \left(\left\{\frac{nz_1}{N}\right\}, \ldots, \left\{\frac{nz_s}{N}\right\}\right).$$

It is easy to identify the mutually corresponding parts: the map ϕ_m and the fractional part $\{\cdot\}$, $n(x)$ and n, the components of the generating vectors, g_i and z_i, and finally the moduli $f(x)$ and N, respectively.

Polynomial lattice point sets are a special case of (t, m, s)-nets, first introduced by Niederreiter [59]. For an overview on (t, m, s)-nets see also [20, Chapter 4].

For further information on polynomial lattice point sets see [20, Chapter 10].

In the following sections we discuss several CBC constructions that lead to generating vectors with different good properties.

3.2 A reduced fast component-by-component construction of lattice point sets with small weighted star discrepancy

In this section we want to consider a similar algorithm as the reduced fast CBC algorithm of Dick et al. in [13], but with the weighted star discrepancy as quality criterion instead of the worst-case error. All the results of this section are based on [47] and are joint work with Ralph Kritzinger.

Let b be an arbitrary prime number and m a positive integer. We consider lattice point sets with $N = b^m$ elements and study their weighted star discrepancy. As said before we construct a generating vector $\boldsymbol{z}$ one component at a time with the aid of a CBC construction.

When using a standard-type CBC construction as for example in [38, 70, 71], every component is chosen from $\mathcal{Z}_N = \{z \in \{1, 2, \dots, b^m - 1\} \colon \gcd(z, b^m) = 1\}$. As done in [13] for the worst-case error, we speed up the construction of such generating vectors by reducing the search space for each component according to its importance, while still achieving a small weighted star discrepancy for the corresponding lattice point set. Recall from (3.7) that the reduced search spaces are defined as

$$\mathcal{Z}_{N,w_j} = \begin{cases} \{z \in \{1, \dots, b^{m-w_j} - 1\} \colon \gcd(z, b^m) = 1\}, & \text{if } w_j < m, \\ \{1\}, & \text{if } w_j \geq m, \end{cases}$$

where we defined the sequence $0 = w_1 \leq w_2 \leq \dots$ in accordance with the weight sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$.

To illustrate how to choose the weights w_j and what can be gained from the reduced fast algorithm we start by discussing a motivating example. Consider first the standard CBC construction as treated in [38, 70, 71]. Speaking in terms of the reduced fast CBC construction, this would be the case where $w_j = 0$ for all $j \geq 0$. In this case, a sufficient condition for strong polynomial tractability is $\sum_{j=1}^{\infty} \gamma_j < \infty$, which is satisfied for instance for the special choices $\gamma_j = j^{-2}$ and $\gamma_j = j^{-1000}$. However, in the second example the weights decay much faster than in the first. We can make use of this fact by introducing the sequence $\boldsymbol{w} = (w_j)_{j \geq 0}$ such that the condition $\sum_{j=1}^{\infty} \gamma_j b^{w_j} < \infty$ holds, while still achieving strong polynomial tractability (see Corollary 3.20). This way, we can reduce the size of the search sets for the components of the generating vector if the weights γ_j decay very fast. Consider for example the weight sequence $\gamma_j = j^{-k}$ for some $k > 1$. For $w_j = \lfloor (k - \alpha) \log_p j \rfloor$ with arbitrary $1 < \alpha < k$ we find

$$\sum_{j=1}^{\infty} \gamma_j b^{w_j} \leq \sum_{j=1}^{\infty} j^{-k} j^{k-\alpha} = \sum_{j=1}^{\infty} j^{-\alpha} = \zeta(\alpha) < \infty,$$

where ζ denotes the Riemann Zeta function. Observe that for large k, i.e., fast decaying weights, we may choose smaller search sets and thereby speed up the CBC algorithm.

In what follows we denote by $\mathcal{Z}_{N,\boldsymbol{w}}^s$ the Cartesian product $b^{w_1} \mathcal{Z}_{N,w_1} \times \dots \times b^{w_s} \mathcal{Z}_{N,w_s}$, where $b^{w_j} \mathcal{Z}_{N,w_j}$ means that every element of $\mathcal{Z}_{N,w_j}$ is multiplied by b^{w_j} modulo b^m. By $\boldsymbol{z} \in \mathcal{Z}_{N,\boldsymbol{w}}^s$ we mean a vector $\boldsymbol{z} = (b^{w_1} z_1, \dots, b^{w_s} z_s)$, with $z_j \in \mathcal{Z}_{N,w_j}$ for $j \in [s]$. We study the weighted star discrepancy of lattice point sets $P_N(\boldsymbol{z})$ with generating vectors $\boldsymbol{z} \in \mathcal{Z}_{N,\boldsymbol{w}}^s$ and will see that for sufficiently fast decreasing weights we can construct lattice point sets with small weighted star discrepancy, while significantly reducing the construction cost in comparison to the standard CBC construction.

Instead of analyzing the weighted star discrepancy, we study

$$R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) = \sum_{\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}} R_N(\boldsymbol{z},\mathfrak{u}), \tag{3.8}$$

where

$$R_N(\boldsymbol{z},\mathfrak{u}) = \frac{1}{N}\sum_{k=0}^{N-1}\prod_{j\in\mathfrak{u}}\left(1+\sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0}}\frac{\mathrm{e}^{2\pi \mathrm{i} hkb^{w_j}z_j/N}}{|h|}\right)-1. \tag{3.9}$$

It is enough to consider $R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z})$, since we know from Niederreiter [60, Theorem 3.10 and Theorem 5.6] that

$$D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \sum_{\mathfrak{u}\subseteq[s]}\gamma_{\mathfrak{u}}\left(1-\left(1-\frac{1}{N}\right)^{|\mathfrak{u}|}\right)+\frac{1}{2}R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z}), \tag{3.10}$$

where the first term of the right hand side is independent of $\boldsymbol{z}$. We use this estimate to derive our results in the following sections.

3.2.1 The arithmetic mean over all $\boldsymbol{z}\in\mathcal{Z}^s_{N,\boldsymbol{w}}$

First of all we estimate the arithmetic mean of the weighted star discrepancy over all possible generating vectors

$$\boldsymbol{z} = (b^{w_1}z_1,\ldots,b^{w_s}z_s)\in\mathcal{Z}^s_{N,\boldsymbol{w}},$$

proceeding similarly to [60] and [71]. We prove that the arithmetic mean is small and thus there must exist at least one lattice point set with weighted star discrepancy smaller than or equal to the mean. This yields the existence of a lattice point set with small weighted star discrepancy. The upper bound which we obtain for the arithmetic mean is not the same as for the reduced CBC construction in the next section. Nonetheless, we need large parts of the calculations of the present section to obtain the estimate in Section 3.2.2.

Theorem 3.11. *Let $N=b^m$, $(w_j)_{j\geq 1}$ and $\mathcal{Z}^s_{N,\boldsymbol{w}}$ be as above and let $m\geq 5$. Then there exists a generating vector*

$$\boldsymbol{z} = (b^{w_1}z_1,\ldots,b^{w_s}z_s)\in\mathcal{Z}^s_{N,\boldsymbol{w}}$$

such that the weighted star discrepancy of the corresponding lattice point set satisfies

$$\begin{aligned} D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq & \sum_{\mathfrak{u}\subseteq[s]}\gamma_{\mathfrak{u}}\left(1-\left(1-\frac{1}{N}\right)^{|\mathfrak{u}|}\right) \\ &+\frac{1}{2}\left(\frac{1}{N}\prod_{j=1}^{s}(\beta_j+\gamma_j S_N)\right. \\ &\left.+\frac{1}{N}\sum_{p=0}^{m-1}b^{m-p-1}(b-1)\prod_{\substack{j=1\\ w_j\geq m-p}}^{s}(\beta_j+\gamma_j S_N)\prod_{\substack{j=1\\ w_j<m-p}}^{s}\beta_j-\prod_{j=1}^{s}\beta_j\right), \end{aligned} \tag{3.11}$$

with $\beta_j = 1+\gamma_j$ for all $j\in\mathbb{N}$ and

$$S_N = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0}}\frac{1}{|h|}. \tag{3.12}$$

Remark 3.12. *Provided that the $\gamma_j b^{w_j}$'s are summable, the bound in Theorem 3.11 is of order $N^\delta \log N$ for arbitrary $\delta \in (0,1)$ with an implied constant independent of N and s. Furthermore, note that if all weights $w_j = 0$, then we obtain the result in [71, Theorem 1 and Corollary 1].*

Proof. To prove Theorem 3.11 we calculate the arithmetic mean of the weighted star discrepancy over all possible generating vectors. This mean is smaller than or equal to the bound given in (3.11) and thus yields the existence of a lattice point set with a weighted star discrepancy not exceeding this bound.

As the first term in (3.10) is independent of $\boldsymbol{z}$, it is obviously enough to consider the mean

$$M_{N,s,\gamma} = \frac{1}{|\mathcal{Z}_{N,\boldsymbol{w}}^s|} \sum_{\boldsymbol{z} \in \mathcal{Z}_{N,\boldsymbol{w}}^s} R_{N,\gamma}^s(\boldsymbol{z}) \tag{3.13}$$

of the second term.

We have from [38, p. 186, Eq. 9]

$$\begin{aligned} R_{N,\gamma}^s(\boldsymbol{z}) &= \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{s} \left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2} < h \le \frac{N}{2} \\ h \ne 0}} \frac{e^{2\pi i h k b^{w_j} z_j / N}}{|h|} \right) - \prod_{j=1}^{s} \beta_j \\ &= \frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) + \frac{1}{N} \sum_{k=1}^{N-1} \prod_{j=1}^{s} \left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2} < h \le \frac{N}{2} \\ h \ne 0}} \frac{e^{2\pi i h k b^{w_j} z_j / N}}{|h|} \right) - \prod_{j=1}^{s} \beta_j. \end{aligned} \tag{3.14}$$

Thus

$$\begin{aligned} M_{N,s,\gamma} &= \frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) \\ &\quad + \frac{1}{N} \sum_{k=1}^{N-1} \prod_{j=1}^{s} \left(\frac{1}{|\mathcal{Z}_{N,w_j}|} \sum_{z_j \in \mathcal{Z}_{N,w_j}} \left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2} < h \le \frac{N}{2} \\ h \ne 0}} \frac{e^{2\pi i h k b^{w_j} z_j / N}}{|h|} \right) \right) - \prod_{j=1}^{s} \beta_j \\ &= \frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) \\ &\quad + \frac{1}{N} \sum_{k=1}^{N-1} \prod_{j=1}^{s} \left(\beta_j + \frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|} \sum_{z_j \in \mathcal{Z}_{N,w_j}} \sum_{\substack{-\frac{N}{2} < h \le \frac{N}{2} \\ h \ne 0}} \frac{e^{2\pi i h k b^{w_j} z_j / N}}{|h|} \right) - \prod_{j=1}^{s} \beta_j. \end{aligned}$$

To avoid lengthy formulas we use the following abbreviations:

$$T_{N,w_j}(k) = \sum_{z_j \in \mathcal{Z}_{N,w_j}} \sum_{\substack{-\frac{N}{2} < h \le \frac{N}{2} \\ h \ne 0}} \frac{e^{2\pi i h k b^{w_j} z_j / N}}{|h|}, \tag{3.15}$$

and

$$L_{N,s,\gamma} = \frac{1}{N} \sum_{k=1}^{N-1} \prod_{j=1}^{s} \left(\beta_j + \frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|} T_{N,w_j}(k) \right). \tag{3.16}$$

Then we have

$$M_{N,s,\gamma} = \frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) + L_{N,s,\gamma} - \prod_{j=1}^{s} \beta_j. \tag{3.17}$$

We study $T_{N,w_j}(k)$ distinguishing the two cases $w_j \geq m$ and $w_j < m$.

Case 1: $w_j \geq m$. This yields $\mathcal{Z}_{N,w_j} = \{1\}$ and thus

$$T_{N,w_j}(k) = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} hkb^{w_j}/N}}{|h|} = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} hkb^{w_j-m}}}{|h|} = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} = S_N. \tag{3.18}$$

Case 2: $w_j < m$. Then $\mathcal{Z}_{N,w_j} = \{z \in \{1,2,\ldots,b^{m-w_j}-1\} : \gcd(z,N)=1\}$. According to (3.16) we have to calculate $T_{N,w_j}(k)$ only for $k \in \{1,\ldots,b^m-1\}$. We display these k as $k = qb^{m-w_j}+r$ with $q \in \{0,\ldots,b^{w_j}-1\}$, $r \in \{0,\ldots,b^{m-w_j}-1\}$ and $(q,r) \neq (0,0)$. Then

$$\begin{aligned}T_{N,w_j}(k) &= \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} \sum_{z_j\in\mathcal{Z}_{N,w_j}} \mathrm{e}^{2\pi \mathrm{i} h(qb^{m-w_j}+r)b^{w_j}z_j/N} \\ &= \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} \sum_{z_j\in\mathcal{Z}_{N,w_j}} \mathrm{e}^{2\pi \mathrm{i} hqz_j}\mathrm{e}^{2\pi \mathrm{i} hrz_j/b^{m-w_j}} \\ &= \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} \sum_{z_j\in\mathcal{Z}_{N,w_j}} \mathrm{e}^{2\pi \mathrm{i} hrz_j/b^{m-w_j}}. \end{aligned} \tag{3.19}$$

If $r=0$, i.e., k is a multiple of b^{m-w_j}, this yields

$$T_{N,w_j}(k) = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} \sum_{z_j\in\mathcal{Z}_{N,w_j}} 1 = |\mathcal{Z}_{N,w_j}|S_N. \tag{3.20}$$

Next we investigate $r \in \{1,\ldots,b^{m-w_j}-1\}$. For any $z_j \in \{1,\ldots,b^{m-w_j}-1\}$ we find $\gcd(z_j,N) = \gcd(z_j,b^{m-w_j}) \in \{1,b,b^2,\ldots,b^{m-w_j-1}\}$ and hence

$$\sum_{d|\gcd(z_j,N)} \mu(d) = \sum_{d|\gcd(z_j,b^{m-w_j})} \mu(d) = \begin{cases} 1 & \text{if and only if } \gcd(z_j,N) = \gcd(z_j,b^{m-w_j}) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where μ denotes the Möbius function.

For any $z_j \in \{1,\ldots,b^{m-w_j}-1\}$ this implies $z_j \in \mathcal{Z}_{N,w_j}$ iff $\sum_{d|\gcd(z_j,b^{m-w_j})} \mu(d) = 1$. Inserting this fact into (3.19) we have

$$T_{N,w_j}(k) = \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\h\neq 0}} \frac{1}{|h|} \sum_{z_j=1}^{b^{m-w_j}-1} \mathrm{e}^{2\pi \mathrm{i} hrz_j/b^{m-w_j}} \sum_{d|\gcd(z_j,b^{m-w_j})} \mu(d). \tag{3.21}$$

Studying the two inner sums we find

$$\begin{aligned}\sum_{z_j=1}^{b^{m-w_j}-1} \mathrm{e}^{2\pi \mathrm{i} hrz_j/b^{m-w_j}} \sum_{d|\gcd(z_j,b^{m-w_j})} \mu(d) &= \sum_{d|b^{m-w_j}} \mu(d) \sum_{\substack{z_j=1\\d|z_j}}^{b^{m-w_j}-1} \mathrm{e}^{2\pi \mathrm{i} hrz_j/b^{m-w_j}} \\ &= \sum_{d|b^{m-w_j}} \mu(d) \sum_{a=1}^{\frac{b^{m-w_j}}{d}} \mathrm{e}^{2\pi \mathrm{i} hrad/b^{m-w_j}}, \end{aligned} \tag{3.22}$$

where the latter equality holds since $a \in \left\{1, \dots, \frac{b^{m-w_j}}{d}\right\}$ yields

$$ad \in \{d, 2d, \dots, b^{m-w_j}\} = \{1 \le z_j \le b^{m-w_j} - 1 : d|z_j\} \cup \{b^{m-w_j}\}$$

and

$$\sum_{d|b^{m-w_j}} \mu(d) = 0,$$

since $w_j < m$.

Changing the order of summation we obtain with (3.22)

$$\begin{aligned}\sum_{z_j=1}^{b^{m-w_j}-1} \mathrm{e}^{2\pi \mathrm{i} h r z_j / b^{m-w_j}} \sum_{d|\gcd(z_j, b^{m-w_j})} \mu(d) &= \sum_{d|b^{m-w_j}} \mu\left(\frac{b^{m-w_j}}{d}\right) \sum_{a=1}^{d} \mathrm{e}^{2\pi \mathrm{i} h r a/d} \\ &= \sum_{\substack{d|b^{m-w_j} \\ d|hr}} d\, \mu\left(\frac{b^{m-w_j}}{d}\right).\end{aligned}$$

With (3.21) this leads to

$$T_{N,w_j}(k) = \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2} \\ h\ne 0}} \frac{1}{|h|} \sum_{\substack{d|b^{m-w_j} \\ d|hr}} d\, \mu\left(\frac{b^{m-w_j}}{d}\right) = \sum_{d|b^{m-w_j}} d\, \mu\left(\frac{b^{m-w_j}}{d}\right) \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2} \\ h\ne 0 \\ d|hr}} \frac{1}{|h|}.$$

Using that $d|hr$ is equivalent to $\frac{d}{\gcd(d,r)}|h$ we display $T_{N,w_j}(k)$ as

$$T_{N,w_j}(k) = \sum_{d|b^{m-w_j}} d\, \mu\left(\frac{b^{m-w_j}}{d}\right) \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2} \\ h\ne 0 \\ \frac{d}{\gcd(d,r)}|h}} \frac{1}{|h|}. \tag{3.23}$$

To further investigate $T_{N,w_j}(k)$, we first study sums of the same type as the inner sum in (3.23). For any positive integer a we have

$$\sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2} \\ h\ne 0 \\ a|h}} \frac{1}{|h|} = \sum_{\substack{-\frac{N}{2}<ap\le\frac{N}{2} \\ p\ne 0}} \frac{1}{a|p|} = \frac{1}{a} \sum_{\substack{-\frac{N}{2a}<p\le\frac{N}{2a} \\ p\ne 0}} \frac{1}{|p|} = \frac{1}{a} S_{\frac{N}{a}}, \tag{3.24}$$

where $S_{\frac{N}{a}}$ is defined analogously to (3.12). Combining (3.24) and (3.23) and the property of μ that $\mu(1) = 1$, $\mu(b) = -1$ and $\mu(b^i) = 0$ for $i \in \mathbb{N}, i \ge 2$ we obtain

$$\begin{aligned}T_{N,w_j}(k) &= \sum_{d|b^{m-w_j}} d\, \mu\left(\frac{b^{m-w_j}}{d}\right) \frac{\gcd(d,r)}{d} S_{\frac{N}{d}\gcd(d,r)} \\ &= \sum_{d|b^{m-w_j}} \mu\left(\frac{b^{m-w_j}}{d}\right) \gcd(d,r) S_{\frac{N}{d}\gcd(d,r)} \\ &= \sum_{i=0}^{m-w_j} \mu\left(\frac{b^{m-w_j}}{b^i}\right) \gcd(b^i, r) S_{\frac{b^m}{b^i}\gcd(b^i,r)} \\ &= \gcd(b^{m-w_j}, r) S_{b^{w_j}\gcd(b^{m-w_j},r)} - \gcd(b^{m-w_j-1}, r) S_{b^{w_j+1}\gcd(b^{m-w_j-1},r)} \\ &= b^{\nu}(S_{b^{w_j+\nu}} - S_{b^{w_j+\nu+1}}),\end{aligned} \tag{3.25}$$

with $\nu \in \{0, \ldots, m - w_j - 1\}$.

Summarizing, we have for $k \in \{1, \ldots, b^m - 1\}$

$$T_{N,w_j}(k) = \begin{cases} S_N & \text{if} \quad w_j \geq m, \\ |\mathcal{Z}_{N,w_j}| S_N & \text{if} \quad w_j < m \text{ and } k \equiv 0 \pmod{b^{m-w_j}}, \\ b^\nu (S_{b^{w_j+\nu}} - S_{b^{w_j+\nu+1}}) & \\ \quad \text{with } b^\nu = \gcd(b^{m-w_j}, r) & \text{if} \quad w_j < m \text{ and } k \not\equiv 0 \pmod{b^{m-w_j}}. \end{cases} \tag{3.26}$$

Let us choose $t \in \mathbb{N}_0$ such that $w_j < m$ for all $j \leq t$ and $w_{t+1} \geq m$. (If $t = 0$, then $w_j \geq m$ for all $j \in \mathbb{N}$. In this case we obtain the generating vector $\boldsymbol{z} = (b^{w_1}, \ldots, b^{w_s})$.) With this fact we are able to write $L_{N,s,\boldsymbol{\gamma}}$ from formula (3.16) as

$$\begin{aligned} L_{N,s,\boldsymbol{\gamma}} &= \frac{1}{N} \sum_{k=1}^{N-1} \prod_{j=1}^{\min\{t,s\}} \left(\beta_j + \frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|} T_{N,w_j}(k)\right) \prod_{j=t+1}^{s} \left(\beta_j + \frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|} T_{N,w_j}(k)\right) \\ &= \frac{1}{N} \prod_{j=t+1}^{s} (\beta_j + \gamma_j S_N) \sum_{k=1}^{N-1} \prod_{j=1}^{\min\{t,s\}} \left(\beta_j + \frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|} T_{N,w_j}(k)\right). \end{aligned} \tag{3.27}$$

Next we aim at finding bounds for $\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|}$ for $w_j < m$.

If k is a multiple of b^{m-w_j} we see immediately from (3.26) that

$$\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|} = \frac{|\mathcal{Z}_{N,w_j}| S_N}{|\mathcal{Z}_{N,w_j}|} = S_N.$$

If k is not a multiple of b^{m-w_j}, we use a formula from Niederreiter [58, Lemma 1 and Lemma 2] for S_n with arbitrary $n \in \mathbb{N}$, given by

$$S_n = 2 \log n + 2\gamma - \log 4 + \varepsilon(n), \tag{3.28}$$

where γ denotes the Euler-Mascheroni constant

$$\gamma = \lim_{l \to \infty} \left(\sum_{k=1}^{l} \frac{1}{k} - \log l \right) \approx 0.577216\ldots$$

and

$$\begin{cases} -\frac{4}{n^2} < \varepsilon(n) \leq 0, & \text{if } n \text{ is even}, \\ -\frac{3}{n^2} < \varepsilon(n) < \frac{1}{n^2}, & \text{if } n \text{ is odd}. \end{cases} \tag{3.29}$$

From (3.26) we know

$$T_{N,w_j}(k) = b^\nu (S_{b^{w_j+\nu}} - S_{b^{w_j+\nu+1}}) < 0. \tag{3.30}$$

With $m \geq 5$ we find $-2 < \frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|} < 0$ for $w_j < m$ and k not a multiple of b^{m-w_j} as follows. The upper bound follows immediately from (3.30). It remains to show the lower bound. First we consider $T_{N,w_j}(k)$ using (3.28). We have

$$\begin{aligned} T_{N,w_j}(k) &= b^\nu (S_{b^{w_j+\nu}} - S_{b^{w_j+\nu+1}}) \\ &= b^\nu \left(-2 \log b + \varepsilon(b^{w_j+\nu}) - \varepsilon(b^{w_j+\nu+1})\right) \\ &= -2b^\nu \log b + b^\nu \left(\varepsilon(b^{w_j+\nu}) - \varepsilon(b^{w_j+\nu+1})\right). \end{aligned}$$

With (3.29) we obtain

$$\begin{aligned}\left|b^{\nu}\left(\varepsilon(b^{w_j+\nu})-\varepsilon(b^{w_j+\nu+1})\right)\right| &\leq \left|b^{\nu}\left(\varepsilon(b^{w_j+\nu})\right)\right|+\left|b^{\nu}\left(\varepsilon(b^{w_j+\nu+1})\right)\right| \\ &\leq 4b^{-2w_j-\nu}\left(1+\frac{1}{b^2}\right).\end{aligned}$$

Thus

$$\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|} \geq -\frac{b^{w_j-m+1}}{b-1}2b^{\nu}\log b-\frac{b^{w_j-m+1}}{b-1}4b^{-2w_j-\nu}\left(1+\frac{1}{b^2}\right).$$

Recall from (3.26) that $\nu=\log_b\left(\gcd\left(b^{m-w_j},r\right)\right)\in\{0,1,\ldots,m-w_j-1\}$. Thus

$$\begin{aligned}\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|} &\geq -2b^{w_j-m+1+m-w_j-1}\frac{\log b}{b-1}-4b^{-w_j-m+1-\nu}\frac{1}{b-1}\left(1+\frac{1}{b^2}\right) \\ &\geq -2\frac{\log b}{b-1}-4b^{-m+1}\frac{1}{b-1}\left(1+\frac{1}{b^2}\right).\end{aligned}$$

Now, with the assumption $m\geq 5$,

$$\begin{aligned}\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|} &\geq -2\frac{\log b}{b-1}-4b^{-5+1}\frac{1}{b-1}\left(1+\frac{1}{b^2}\right) \\ &\geq -2\frac{\log 2}{2-1}-4\cdot 2^{-5+1}\left(1+\frac{1}{2^2}\right)>-2,\end{aligned}$$

and hence

$$-2<\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|}<0 \quad \text{for } w_j<m \quad \text{and} \quad b^{m-w_j}\nmid k.$$

For any integer $p\in\{0,\ldots,m-1\}$ with $b^p\mid k$ and $b^{p+1}\nmid k$ the condition $b^{m-w_j}\nmid k$ is equivalent to $m-w_j>p$ or $w_j<m-p$, respectively. Thus we can display (3.27) as

$$\begin{aligned}L_{N,s,\gamma} &= \frac{1}{N}\prod_{j=t+1}^{s}\left(\beta_j+\gamma_j S_N\right) \\ &\quad\times\sum_{p=0}^{m-1}\sum_{\substack{k=1\\ b^p\mid k\\ b^{p+1}\nmid k}}^{N-1}\prod_{\substack{j=1\\ w_j\geq m-p}}^{\min\{t,s\}}\left(\beta_j+\frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|}T_{N,w_j}(k)\right)\prod_{\substack{j=1\\ w_j<m-p}}^{\min\{t,s\}}\left(\beta_j+\frac{\gamma_j}{|\mathcal{Z}_{N,w_j}|}T_{N,w_j}(k)\right) \\ &\leq \frac{1}{N}\prod_{j=t+1}^{s}\left(\beta_j+\gamma_j S_N\right)\sum_{p=0}^{m-1}\sum_{\substack{k=1\\ b^p\mid k\\ b^{p+1}\nmid k}}^{N-1}\prod_{\substack{j=1\\ w_j\geq m-p}}^{\min\{t,s\}}\left(\beta_j+\gamma_j S_N\right)\prod_{\substack{j=1\\ w_j<m-p}}^{\min\{t,s\}}\beta_j,\end{aligned}$$

where the latter estimate holds since

$$\beta_j>1, \quad -2<\frac{T_{N,w_j}(k)}{|\mathcal{Z}_{N,w_j}|}<0 \quad \text{and} \quad \gamma_j\leq 1.$$

From

$$\begin{aligned}&\left|\left\{k\in\{1,\ldots,N-1\}:b^p\mid k \quad \text{and} \quad b^{p+1}\nmid k\right\}\right| \\ &\quad=\left|\{k\in\{1,\ldots,b^m-1\}:b^p\mid k\}\right|-\left|\left\{k\in\{1,\ldots,b^m-1\}:b^{p+1}\mid k\right\}\right| \qquad (3.31)\\ &\quad=b^{m-p}-1-\left(b^{m-p-1}-1\right) \\ &\quad=b^{m-p-1}(b-1)\end{aligned}$$

we get

$$L_{N,s,\gamma} \le \frac{1}{N} \prod_{j=t+1}^{s} (\beta_j + \gamma_j S_N) \sum_{p=0}^{m-1} b^{m-p-1}(b-1) \prod_{\substack{j=1\\ w_j \ge m-p}}^{\min\{t,s\}} (\beta_j + \gamma_j S_N) \prod_{\substack{j=1\\ w_j < m-p}}^{\min\{t,s\}} \beta_j.$$

Inserting this into (3.17) we obtain for the arithmetic mean

$$\begin{aligned} M_{N,s,\gamma} = &\frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) \\ &+ \frac{1}{N} \prod_{j=t+1}^{s} (\beta_j + \gamma_j S_N) \sum_{p=0}^{m-1} b^{m-p-1}(b-1) \prod_{\substack{j=1\\ w_j \ge m-p}}^{\min\{t,s\}} (\beta_j + \gamma_j S_N) \prod_{\substack{j=1\\ w_j < m-p}}^{\min\{t,s\}} \beta_j \\ &- \prod_{j=1}^{s} \beta_j. \end{aligned} \tag{3.32}$$

This proves, with (3.10), the existence of a vector $\boldsymbol{z} \in \mathcal{Z}_{N,\boldsymbol{w}}^s$ such that the weighted star discrepancy $D_{N,\boldsymbol{\gamma}}^*(\boldsymbol{z})$ fulfills

$$\begin{aligned} D_{N,\boldsymbol{\gamma}}^*(\boldsymbol{z}) \le &\sum_{\mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{2} \left(\frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) \right. \\ &\left. + \frac{1}{N} \prod_{j=t+1}^{s} (\beta_j + \gamma_j S_N) \sum_{p=0}^{m-1} b^{m-p-1}(b-1) \prod_{\substack{j=1\\ w_j \ge m-p}}^{\min\{t,s\}} (\beta_j + \gamma_j S_N) \prod_{\substack{j=1\\ w_j < m-p}}^{\min\{t,s\}} \beta_j - \prod_{j=1}^{s} \beta_j \right) \end{aligned} \tag{3.33}$$

$$\begin{aligned} \le &\sum_{\mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{2} \left(\frac{1}{N} \prod_{j=1}^{s} (\beta_j + \gamma_j S_N) \right. \\ &\left. + \frac{1}{N} \sum_{p=0}^{m-1} b^{m-p-1}(b-1) \prod_{\substack{j=1\\ w_j \ge m-p}}^{s} (\beta_j + \gamma_j S_N) \prod_{\substack{j=1\\ w_j < m-p}}^{s} \beta_j - \prod_{j=1}^{s} \beta_j \right). \end{aligned} \tag{3.34}$$

□

3.2.2 The reduced CBC construction

In this section we give a component-by-component construction for the generating vector and an upper bound for the weighted star discrepancy of the corresponding lattice rule.

Algorithm 3.13. *Let $N = b^m$ and $(w_j)_{j \ge 1}$ be as above and construct $\boldsymbol{z} = (b^{w_1} z_1, \ldots, b^{w_s} z_s) \in \mathcal{Z}_{N,\boldsymbol{w}}^s$ as follows:*

1. *Set $z_1 = 1$.*
2. *For $d \in [s-1]$ assume $z_1, \ldots, z_d$ to be already found. Choose $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$ such that*

$$R_{N,\boldsymbol{\gamma}}^{d+1}(b^{w_1} z_1, \ldots, b^{w_d} z_d, b^{w_{d+1}} z)$$

   is minimized as a function of z.

3. *Increase d by 1 and repeat the second step until $\boldsymbol{z} = (b^{w_1} z_1, \ldots, b^{w_s} z_s)$ is found.*

In the algorithm above the search space is reduced for each coordinate of $\boldsymbol{z}$ according to its importance, as the w_j's are chosen in accordance to the γ_j's. This results in a considerable reduction of the construction cost as we will see in Section 3.2.3. This is why we call this algorithm a reduced CBC-algorithm.

The following theorem gives an upper bound for the figure of merit, $R^d_{N,\gamma}$, of lattice point sets with generating vectors obtained from the algorithm above.

Theorem 3.14. *Let $\boldsymbol{z} = (b^{w_1}z_1, \dots, b^{w_s}z_s)$ be constructed according to Algorithm 3.13. Then for every $d \in [s]$,*

$$R^d_{N,\gamma}(b^{w_1}z_1, \dots, b^{w_d}z_d) \leq \frac{1}{N} \prod_{j=1}^{d} \Big(\beta_j + \Big(1 + 2b^{\min\{w_j,m\}}\Big)\gamma_j S_N\Big). \tag{3.35}$$

Corollary 3.15. *Let $N = b^m$ and $(w_j)_{j\geq 1}$ be as above and let*

$$\boldsymbol{z} = (b^{w_1}z_1, \dots, b^{w_s}z_s) \in \mathcal{Z}^s_{N,\boldsymbol{w}}$$

be constructed using Algorithm 3.13. Then the corresponding lattice point set has a weighted star discrepancy

$$D^*_{N,\gamma}(\boldsymbol{z}) \leq \sum_{\mathfrak{u}\subseteq[s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{2N} \prod_{j=1}^{s} \Big(\beta_j + \Big(1 + 2b^{\min\{w_j,m\}}\Big)\gamma_j S_N\Big).$$

Proof. Combining (3.10), (3.12) and Theorem 3.14 we immediately obtain the result. □

To prove Theorem 3.14 we use the the following

Lemma 3.16. *Let $N = b^m$, $(w_j)_{j\geq 1}$ and $\mathcal{Z}_{N,w_j}$ be defined as above and recall from* (3.15) *the notation*

$$T_{N,w_j}(k) = \sum_{z_j \in \mathcal{Z}_{N,w_j}} \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h\neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j/N}}{|h|}.$$

Then

$$\sum_{k=1}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} \leq 2b^{\min\{w_j,m\}} S_N \quad \textit{for all } j \geq 1. \tag{3.36}$$

Proof. As before, we distinguish the two cases $w_j \geq m$ and $w_j < m$.

Case 1: $w_j \geq m$. Then (3.26) yields

$$\sum_{k=1}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} = \sum_{k=1}^{N-1} S_N = (N-1)S_N \leq 2NS_N = 2b^{\min\{w_j,m\}} S_N.$$

Case 2: $w_j < m$. We use (3.26) and (3.19) to find

$$\begin{aligned}\sum_{k=1}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} &= \sum_{\substack{k=1 \\ b^{m-w_j} \mid k}}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} + \sum_{\substack{k=1 \\ b^{m-w_j} \nmid k}}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} \\ &= (b^{w_j} - 1)S_N + b^{w_j} \sum_{r=1}^{b^{m-w_j}-1} \frac{|T_{N,w_j}(r)|}{|\mathcal{Z}_{N,w_j}|}.\end{aligned}$$

For any $r \in \{1, \ldots, b^{m-w_j}-1\}$ the condition $\gcd(r, b^{m-w_j}) = b^\nu$ is equivalent to $b^\nu \mid r$ and $b^{\nu+1} \nmid r$ simultaneously. Using this we investigate the last sum in the above equation

$$\sum_{r=1}^{b^{m-w_j}-1} \frac{|T_{N,w_j}(r)|}{|\mathcal{Z}_{N,w_j}|} = \frac{1}{|\mathcal{Z}_{N,w_j}|} \sum_{\nu=0}^{m-w_j-1} \sum_{\substack{r=1 \\ b^\nu | r \\ b^{\nu+1} \nmid r}}^{b^{m-w_j}-1} |T_{N,w_j}(r)|.$$

Once more with the aid of (3.26) this yields

$$\begin{aligned}
\sum_{r=1}^{b^{m-w_j}-1} \frac{|T_{N,w_j}(r)|}{|\mathcal{Z}_{N,w_j}|} &= \frac{1}{|\mathcal{Z}_{N,w_j}|} \sum_{\nu=0}^{m-w_j-1} \sum_{\substack{r=1 \\ b^\nu | r \\ b^{\nu+1} \nmid r}}^{b^{m-w_j}-1} \left| b^\nu (S_{b^{w_j+\nu}} - S_{b^{w_j+\nu+1}}) \right| \\
&= \frac{1}{|\mathcal{Z}_{N,w_j}|} \sum_{\nu=0}^{m-w_j-1} \sum_{\substack{r=1 \\ b^\nu | r \\ b^{\nu+1} \nmid r}}^{b^{m-w_j}-1} b^\nu (S_{b^{w_j+\nu+1}} - S_{b^{w_j+\nu}}).
\end{aligned}$$

Analogously to (3.31) we find

$$\left| \left\{ r \in \{1, \ldots, b^{m-w_j}-1\} : b^\nu \mid r \text{ and } b^{\nu+1} \nmid r \right\} \right| = b^{m-w_j-\nu-1}(b-1)$$

and hence

$$\sum_{r=1}^{b^{m-w_j}-1} \frac{|T_{N,w_j}(r)|}{|\mathcal{Z}_{N,w_j}|} = \sum_{\nu=0}^{m-w_j-1} (S_{b^{w_j+\nu+1}} - S_{b^{w_j+\nu}}) = S_N - S_{b^{w_j}}.$$

Altogether we have

$$\begin{aligned}
\sum_{k=1}^{N-1} \frac{|T_{N,w_j}(k)|}{|\mathcal{Z}_{N,w_j}|} &= (b^{w_j}-1) S_N + b^{w_j}(S_N - S_{b^{w_j}}) \\
&\leq 2 b^{w_j} S_N = 2 b^{\min\{w_j, m\}} S_N
\end{aligned}$$

and the proof is complete. □

With the aid of Lemma 3.16 we are able to prove Theorem 3.14 using induction on d.

Proof. According to Algorithm 3.13 we set $z_1 = 1$ in Step 1. We have to show that

$$R^1_{N,\gamma}(b^{w_1}) \leq \frac{1}{N} \left(\beta_1 + \left(1 + 2b^{\min\{w_1, m\}} \right) \gamma_1 S_N \right).$$

With (3.14) we have

$$\begin{aligned}
R^1_{N,\gamma}(b^{w_1}) &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\beta_1 + \gamma_1 \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h \neq 0}} \frac{e^{2\pi i h k b^{w_1}/N}}{|h|} \right) - \beta_1 \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \gamma_1 \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h \neq 0}} \frac{e^{2\pi i h k b^{w_1}/N}}{|h|}.
\end{aligned}$$

Again, we consider the two cases $w_1 \geq m$ and $w_1 < m$ separately.

Case 1: $w_1 \geq m$. Then

$$\begin{aligned} R^1_{N,\gamma}(b^{w_1}) &= \frac{1}{N}\sum_{k=0}^{N-1} \gamma_1 \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} hkb^{w_1-m}}}{|h|} = \frac{1}{N}\gamma_1 N S_N \leq \frac{1}{N}(1+\gamma_1+2N\gamma_1 S_N) \\ &= \frac{1}{N}\left(\beta_1 + 2b^{\min\{w_1,m\}}\gamma_1 S_N\right) \leq \frac{1}{N}\left(\beta_1 + \left(1+2b^{\min\{w_1,m\}}\right)\gamma_1 S_N\right), \end{aligned}$$

which is the desired result.

Case 2: $w_1 < m$. After interchanging the two sums, we once more split up the inner sum as follows,

$$\begin{aligned} R^1_{N,\gamma}(b^{w_1}) &= \frac{\gamma_1}{N} \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0}} \frac{1}{|h|}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} hk/b^{m-w_1}} \\ &= \frac{\gamma_1}{N} \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0\\ b^{m-w_1}\mid h}} \frac{1}{|h|}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} hk/b^{m-w_1}} + \frac{\gamma_1}{N} \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0\\ b^{m-w_1}\nmid h}} \frac{1}{|h|}\sum_{k=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} hk/b^{m-w_1}}. \end{aligned}$$

Now we are able to compute the inner sums. The first one sums to N, whereas the second one equals zero. Thus

$$R^1_{N,\gamma}(b^{w_1}) = \gamma_1 \sum_{\substack{-\frac{N}{2}<h\leq\frac{N}{2}\\ h\neq 0\\ b^{m-w_1}\mid h}} \frac{1}{|h|}.$$

We use (3.24) to find

$$\begin{aligned} R^1_{N,\gamma}(b^{w_1}) &= \gamma_1 \frac{1}{b^{m-w_1}} S_{\frac{N}{b^{m-w_1}}} = \frac{\gamma_1}{N} b^{w_1} S_{b^{w_1}} \\ &\leq \frac{\gamma_1}{N} b^{w_1} S_N \leq \frac{1}{N}\left(\beta_1 + 2b^{w_1}\gamma_1 S_N\right) \\ &\leq \frac{1}{N}\left(\beta_1 + \left(1+2b^{\min\{w_1,m\}}\right)\gamma_1 S_N\right), \end{aligned}$$

as claimed.

Let $d \in [s-1]$ and assume that we have a $\boldsymbol{z} \in \mathcal{Z}^d_{N,\boldsymbol{w}}$, such that

$$R^d_{N,\gamma}(b^{w_1}z_1,\ldots,b^{w_d}z_d) \leq \frac{1}{N}\prod_{j=1}^{d}\left(\beta_j + \left(1+2b^{\min\{w_j,m\}}\right)\gamma_j S_N\right).$$

We have to prove the existence of a $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$ with

$$R^{d+1}_{N,\gamma}(b^{w_1}z_1,\ldots,b^{w_d}z_d,b^{w_{d+1}}z_{d+1}) \leq \frac{1}{N}\prod_{j=1}^{d+1}\left(\beta_j + \left(1+2b^{\min\{w_j,m\}}\right)\gamma_j S_N\right).$$

Using again (3.14) we have for any $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$ that

$$R^{d+1}_{N,\gamma}(b^{w_1}z_1,\ldots,b^{w_d}z_d,b^{w_{d+1}}z_{d+1})$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_j}z_j/N}}{|h|}\right)\\
&\quad\times\left(\beta_{d+1} + \gamma_{d+1} \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_{d+1}}z_{d+1}/N}}{|h|}\right) - \beta_{d+1}\prod_{j=1}^{d}\beta_j\\
&= \beta_{d+1}R^{d}_{N,\boldsymbol{\gamma}}(b^{w_1}z_1,\dots,b^{w_d}z_d)\\
&\quad+\frac{\gamma_{d+1}}{N}\sum_{k=0}^{N-1}\left(\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_j}z_j/N}}{|h|}\right)\right)\sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_{d+1}}z_{d+1}/N}}{|h|}\\
&= \beta_{d+1}R^{d}_{N,\boldsymbol{\gamma}}(b^{w_1}z_1,\dots,b^{w_d}z_d) + \frac{\gamma_{d+1}S_N}{N}\prod_{j=1}^{d}\left(\beta_j+\gamma_j S_N\right)\\
&\quad+\frac{\gamma_{d+1}}{N}\sum_{k=1}^{N-1}\sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_{d+1}}z_{d+1}/N}}{|h|}\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_j}z_j/N}}{|h|}\right).
\end{aligned}
\tag{3.37}
$$

Next we consider the arithmetic mean of

$$R^{d+1}_{N,\boldsymbol{\gamma}}(b^{w_1}z_1,\dots,b^{w_d}z_d,b^{w_{d+1}}z) \quad \text{over all} \quad z\in\mathcal{Z}_{N,w_{d+1}}.$$

As only the third summand in (3.37) depends on the $(d+1)$-st coordinate, and thus on z_{d+1}, it suffices to investigate the mean of this summand. Clearly, if we have some upper bound for the mean over all $z\in\mathcal{Z}_{N,w_{d+1}}$ there exists $z_{d+1}\in\mathcal{Z}_{N,w_{d+1}}$ which satisfies this bound.

In fact, for technical reasons, we study the absolute value of the third term in (3.37):

$$
\begin{aligned}
&\left|\frac{1}{|\mathcal{Z}_{N,w_{d+1}}|}\sum_{z\in\mathcal{Z}_{N,w_{d+1}}}\frac{\gamma_{d+1}}{N}\sum_{k=1}^{N-1}\sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}}\frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_{d+1}}z/N}}{|h|}\right.\\
&\qquad\left.\times\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_j}z_j/N}}{|h|}\right)\right|\\
&\le\frac{\gamma_{d+1}}{N}\sum_{k=1}^{N-1}\frac{1}{|\mathcal{Z}_{N,w_{d+1}}|}\left|\sum_{z\in\mathcal{Z}_{N,w_{d+1}}}\sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}}\frac{\mathrm{e}^{2\pi\mathrm{i}hkb^{w_{d+1}}z/N}}{|h|}\right|\\
&\qquad\times\prod_{j=1}^{d}\left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2}<h\le\frac{N}{2}\\ h\neq 0}} \frac{\left|\mathrm{e}^{2\pi\mathrm{i}hkb^{w_j}z_j/N}\right|}{|h|}\right)
\end{aligned}
$$

$$\leq \frac{\gamma_{d+1}}{N} \sum_{k=1}^{N-1} \frac{|T_{N,w_{d+1}}(k)|}{|\mathcal{Z}_{N,w_{d+1}}|} \prod_{j=1}^{d} (\beta_j + \gamma_j S_N)$$

$$\leq \frac{\gamma_{d+1}}{N} 2b^{\min\{w_{d+1},m\}} S_N \prod_{j=1}^{d} (\beta_j + \gamma_j S_N),$$

where the last estimate stems from an application of Lemma 3.16. Combining this with (3.37) we have shown the existence of a $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$ such that

$$R^{d+1}_{N,\gamma}(b^{w_1}z_1, \dots, b^{w_d}z_d, b^{w_{d+1}}z_{d+1}) \leq \beta_{d+1} R^{d}_{N,\gamma}(b^{w_1}z_1, \dots, b^{w_d}z_d) + \frac{\gamma_{d+1} S_N}{N} \prod_{j=1}^{d} (\beta_j + \gamma_j S_N) + \frac{\gamma_{d+1}}{N} 2b^{\min\{w_{d+1},m\}} S_N \prod_{j=1}^{d} (\beta_j + \gamma_j S_N).$$

We use the induction hypothesis to find

$$\begin{aligned} R^{d+1}_{N,\gamma}(b^{w_1}z_1, \dots, b^{w_d}z_d, b^{w_{d+1}}z_{d+1}) &\leq \frac{\beta_{d+1}}{N} \prod_{j=1}^{d} \left(\beta_j + \left(1 + 2b^{\min\{w_j,m\}}\right) \gamma_j S_N \right) \\ &\quad + \frac{\gamma_{d+1} S_N}{N} \left(\prod_{j=1}^{d} (\beta_j + \gamma_j S_N) \right) \left(1 + 2b^{\min\{w_{d+1},m\}}\right) \\ &\leq \left(\beta_{d+1} + \left(1 + 2b^{\min\{w_{d+1},m\}}\right) \gamma_{d+1} S_N \right) \\ &\quad \times \frac{1}{N} \prod_{j=1}^{d} \left(\beta_j + \left(1 + 2b^{\min\{w_j,m\}}\right) \gamma_j S_N \right) \\ &= \frac{1}{N} \prod_{j=1}^{d+1} \left(\beta_j + \left(1 + 2b^{\min\{w_j,m\}}\right) \gamma_j S_N \right), \end{aligned}$$

which completes the proof. □

3.2.3 The reduced fast CBC construction

By now we have seen how we can construct a generating vector of a lattice point set with low weighted star discrepancy with a reduced CBC construction as in the previous section. Now we study the construction cost of this algorithm. In fact the CBC algorithm given in Section 3.2.2 can be made faster to construct generating vectors for relatively large N and s. To show this we follow closely [13] and [56].

Let $d \in [s-1]$ and assume that we have already found $(b^{w_1}z_1, \dots, b^{w_d}z_d)$. Then we have (cf. (3.14))

$$R^{d}_{N,\gamma}(b^{w_1}z_1, \dots, b^{w_d}z_d) = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h \neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j / N}}{|h|} \right) - \prod_{j=1}^{d} \beta_j.$$

Define $r(h) = \max\{1, |h|\}$. Then

$$\beta_j + \gamma_j \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h \neq 0}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j / N}}{|h|} = \beta_j + \gamma_j \left(\sum_{-\frac{N}{2} < h \leq \frac{N}{2}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j / N}}{r(h)} - 1 \right)$$

$$= 1 + \gamma_j \sum_{-\frac{N}{2} < h \le \frac{N}{2}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j / N}}{r(h)}.$$

Hence we have

$$R^d_{N,\gamma}(b^{w_1} z_1, \ldots, b^{w_d} z_d) = \frac{1}{N} \sum_{k=0}^{N-1} \prod_{j=1}^{d} \left(1 + \gamma_j \sum_{-\frac{N}{2} < h \le \frac{N}{2}} \frac{\mathrm{e}^{2\pi \mathrm{i} h k b^{w_j} z_j / N}}{r(h)} \right) - \prod_{j=1}^{d} \beta_j$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \eta_d(k) - \prod_{j=1}^{d} \beta_j, \tag{3.38}$$

where we have defined

$$\eta_d(k) = \prod_{j=1}^{d} \left(1 + \gamma_j \varphi\left(\frac{k b^{w_j} z_j}{N} \right) \right)$$

and

$$\varphi(x) = \sum_{-\frac{N}{2} < h \le \frac{N}{2}} \frac{\mathrm{e}^{2\pi \mathrm{i} h x}}{r(h)}.$$

However, this is exactly the situation, dealt with in [56, Section 4.2]. Thus we know that $\varphi\left(\frac{k b^{w_j} z_j}{N}\right)$ takes on at most N different values, namely

$$\varphi(0), \varphi\left(\frac{1}{N}\right), \ldots, \varphi\left(\frac{N-1}{N}\right),$$

which can be computed in $O(N \log N)$ operations and stored in a memory space of size $O(N)$, as demonstrated in [56, Section 4.2].

Next we investigate one actual step of the CBC construction. Assuming that we have already found $(b^{w_1} z_1, \ldots, b^{w_d} z_d) \in \mathcal{Z}^d_{N,\boldsymbol{w}}$ we have to minimize

$$R^{d+1}_{N,\gamma}(b^{w_1} z_1, \ldots, b^{w_d} z_d, b^{w_{d+1}} z)$$

as a function of $z \in \mathcal{Z}_{N,w_{d+1}}$ to find $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}}$. For $w_{d+1} \ge m$ we just set $z_{d+1} = 1$ and we are done. Therefore let $w_{d+1} < m$. Considering (3.38) we have

$$R^{d+1}_{N,\gamma}(b^{w_1} z_1, \ldots, b^{w_d} z_d, b^{w_{d+1}} z_{d+1}) = \frac{1}{N} \sum_{k=0}^{N-1} \eta_{d+1}(k) - \prod_{j=1}^{d+1} \beta_j$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \eta_d(k) \left(1 + \gamma_{d+1} \varphi\left(\frac{k b^{w_{d+1}} z_{d+1}}{N} \right) \right) - \prod_{j=1}^{d+1} \beta_j$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \eta_d(k) \left(1 + \gamma_{d+1} \varphi\left(\left\{ \frac{k b^{w_{d+1}} z_{d+1}}{N} \right\} \right) \right) - \prod_{j=1}^{d+1} \beta_j.$$

It is obviously enough to minimize $\sum_{k=0}^{N-1} \eta_d(k) \varphi\left(\left\{\frac{k b^{w_{d+1}} z_{d+1}}{N}\right\}\right)$. To do this we proceed analogously to [13]. We define the matrix

$$A = \left(\varphi\left(\left\{ \frac{k b^{w_{d+1}} z}{N} \right\} \right) \right)_{\substack{z \in \mathcal{Z}_{N,w_{d+1}} \\ k \in \{0,\ldots,N-1\}}},$$

the vector

$$\boldsymbol{\eta}_d = (\eta_d(0), \eta_d(1), \ldots, \eta_d(N-1))^\top$$

and

$$T_d(z) = \sum_{k=0}^{N-1} \eta_d(k) \varphi\left(\left\{\frac{kb^{w_{d+1}}z}{N}\right\}\right).$$

Then

$$A\boldsymbol{\eta}_d = \boldsymbol{T}_d(z) = (T_d(z))_{z\in\mathcal{Z}_{N,w_{d+1}}}.$$

We can display the matrix A as

$$A = \left(\Omega^{(m-w_{d+1})}, \ldots, \Omega^{(m-w_{d+1})}\right),$$

with

$$\Omega^{(l)} = \left(\varphi\left(\left\{\frac{kz}{b^l}\right\}\right)\right)_{\substack{z\in\mathcal{Z}_{b^l,0}\\ k\in\{0,\ldots,b^l-1\}}}.$$

Again analogously to [13] we obtain the following reduced fast CBC algorithm.

Algorithm 3.17.

a) Compute $\varphi\left(\frac{r}{N}\right)$ for all $r = 0, \ldots, N-1$.

b) Set $\eta_1(k) = 1 + \gamma_1\varphi\left(\left\{\frac{kb^{w_1}z_1}{N}\right\}\right)$ for $k = 0, \ldots, N-1$.

c) Set $z_1 = 1$. Set $d = 2$ and recall that we have defined $t = \max\{j : w_j < m\}$.
While $d \leq \min\{s,t\}$,

1. partition $\boldsymbol{\eta}_{d-1}$ into b^{w_d} vectors $\boldsymbol{\eta}_{d-1}^{(1)}, \ldots, \boldsymbol{\eta}_{d-1}^{(b^{w_d})}$ of length b^{m-w_d} and let $\boldsymbol{\eta}' = \boldsymbol{\eta}_{d-1}^{(1)} + \cdots + \boldsymbol{\eta}_{d-1}^{(b^{w_d})}$ denote their sum,

2. let $T_d(z) = \Omega^{(m-w_d)}\boldsymbol{\eta}'$,

3. let $z_d = \arg\min_z T_d(z)$,

4. let $\eta_d(k) = \eta_{d-1}(k)\left(1 + \gamma_d\varphi\left(\left\{\frac{kb^{w_d}z_d}{N}\right\}\right)\right)$ for $k = 0, \ldots, N-1$,

5. increase d by 1.

If $s > t$, then set $z_{t+1} = \cdots = z_s = 1$. Then we have

$$R_{N,\gamma}^s(b^{w_1}z_1, \ldots, b^{w_s}z_s) = \frac{1}{N}\sum_{k=0}^{N-1}\eta_s(k) - \prod_{j=1}^{s}\beta_j.$$

Remark 3.18. *Note that Algorithms 3.13 and 3.17 both yield the same generating vector $\boldsymbol{z}$.*

Using [13, 56, 66, 67] we find that Algorithm 3.17 has a construction cost of

$$O\left(N\log N + \min\{s,t\}N + N\sum_{d=1}^{\min\{s,t\}}(m - w_d)b^{-w_d}\right)$$

operations, in comparison to $O(sN\log N)$ operations for the standard CBC algorithm used for example in [71].

Remark 3.19. *As we are interested in high-dimensional problems, we also consider $s \to \infty$. In this case we always have $\min\{s,t\} = t$ and the construction cost is independent of the dimension.*

3.2.4 Conditions for strong polynomial tractability

Let $\boldsymbol{z} = (b^{w_1} z_1, \ldots, b^{w_s} z_s) \in \mathcal{Z}^s_{N,\boldsymbol{w}}$ be constructed with Algorithm 3.13 or 3.17 and consider the corresponding lattice rule. We are interested in conditions for tractability of the weighted star discrepancy of such lattice point sets. From (3.10) and (3.8) we know

$$D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \sum_{\mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{2} R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z}).$$

For now, let us assume that the $\gamma_j b^{w_j}$'s are summable, i. e.,

$$\sum_{j=1}^{\infty} \gamma_j b^{w_j} < \infty.$$

Similar to Joe and Sinescu in [38] and [71], we see that in this case

$$D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \frac{\max\{1, \Gamma\} \exp\left(\sum_{j=1}^{\infty} \gamma_j\right)}{N} + \frac{1}{2} R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z}),$$

where

$$\Gamma = \sum_{j=1}^{\infty} \frac{\gamma_j}{1+\gamma_j} < \infty.$$

In particular, considering our assumption that the $\gamma_j b^{w_j}$'s are summable, the constant

$$\max\{1, \Gamma\} \exp\left(\sum_{j=1}^{\infty} \gamma_j\right)$$

is indeed finite.

Theorem 3.14 yields

$$R^s_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) \leq \frac{1}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j S_N\right)$$

and hence we have

$$\begin{aligned} D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{z}) &\leq \frac{1 + \max\{1, \Gamma\} \exp\left(\sum_{j=1}^{\infty} \gamma_j\right)}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j S_N\right) \\ &= \frac{c_{\boldsymbol{\gamma}}}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j S_N\right), \end{aligned} \tag{3.39}$$

with $c_{\boldsymbol{\gamma}} = 1 + \max\{1, \Gamma\} \exp\left(\sum_{j=1}^{\infty} \gamma_j\right)$ independent of s.

We study the right-hand side of (3.39)

$$\begin{aligned} \frac{c_{\boldsymbol{\gamma}}}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j S_N\right) &\leq \frac{c_{\boldsymbol{\gamma}}}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j 2 \left(\log \left\lfloor \frac{N}{2} \right\rfloor + 1\right)\right) \\ &\leq \frac{c_{\boldsymbol{\gamma}}}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j 4 \log N\right) \\ &= \frac{c_{\boldsymbol{\gamma}}}{N} \prod_{j=1}^{s} \left(1 + \gamma_j \left(1 + 4 \left(1 + 2b^{\min\{w_j, m\}}\right) \log N\right)\right), \end{aligned} \tag{3.40}$$

where we have used

$$S_N = \sum_{\substack{-\frac{N}{2} < h \leq \frac{N}{2} \\ h \neq 0}} \frac{1}{|h|} \leq 2 \sum_{h=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{h} \leq 2 \log \left\lfloor \frac{N}{2} \right\rfloor + 2 \leq 4 \log N,$$

where the penultimate inequality is a well-known estimate for partial sums of the harmonic series.

Now we have

$$\begin{aligned} \frac{c_{\gamma}}{N} \prod_{j=1}^{s} \left(\beta_j + \left(1 + 2b^{\min\{w_j, m\}}\right) \gamma_j S_N\right) &\leq \frac{c_{\gamma}}{N} \prod_{j=1}^{s} \left(1 + \gamma_j \left(1 + 4(1 + 2b^{w_j}) \log N\right)\right) \\ &\leq \frac{c_{\gamma}}{N} \prod_{j=1}^{s} (1 + 13\gamma_j b^{w_j} \log N). \end{aligned}$$

Define

$$\sigma_d = 13 \sum_{j=d+1}^{\infty} \gamma_j b^{w_j} \quad \text{for} \quad d \in \mathbb{N}_0.$$

From [20, p. 222] or [29, Lemma 3] we know that

$$\prod_{j=1}^{s} (1 + 13\gamma_j b^{w_j} \log N) \leq \left(1 + \sigma_d^{-1}\right)^d N^{(\sigma_0+1)\sigma_d} \qquad \text{for all } d \in \mathbb{N}_0.$$

For $0 < \delta < 1$ choose d large enough such that $\sigma_d \leq \frac{\delta}{\sigma_0 + 1}$. Then

$$\prod_{j=1}^{s} (1 + 13\gamma_j b^{w_j} \log N) \leq \tilde{c}_{\gamma,\delta} N^{\delta},$$

where $\tilde{c}_{\gamma,\delta}$ is independent of s and N. Thus we have

$$D^*_{N,\gamma}(\boldsymbol{z}) \leq c_{\gamma,\delta} N^{\delta-1}, \tag{3.41}$$

with $c_{\gamma,\delta} = c_{\gamma} \cdot \tilde{c}_{\gamma,\delta}$ independent of s and N. We obtain $c_{\gamma,\delta} N^{\delta-1} \leq \varepsilon$ and thus

$$D^*_{N,\gamma}(\boldsymbol{z}) \leq \varepsilon \quad \text{if} \quad N \geq (c_{\gamma,\delta}\varepsilon^{-1})^{\frac{1}{1-\delta}}.$$

With this we have proved the following

Corollary 3.20. *Let $N = b^m$ and let $\boldsymbol{\gamma}$ and $\boldsymbol{w}$ be weight sequences, defined as above and consider the problem of constructing generating vectors for lattice point sets with small weighted star discrepancy. Then*

$$\sum_{j=1}^{\infty} \gamma_j b^{w_j} < \infty$$

is a sufficient condition for strong polynomial tractability.

Remark 3.21. *Whether the conditions on the γ_j's and w_j's can be mitigated while at least polynomial or weak tractability still hold remains for future research.*

3.3 A reduced fast component-by-component construction of polynomial lattice point sets with small weighted star discrepancy

As mentioned before, for many problems lattice point sets yield good results, when used as sample points for QMC algorithms. There exist situations, however, for which polynomial lattice point sets provide better approximations than lattice point sets. For example, when considering a Walsh space as defined on p. 34, polynomial lattice point sets are superior to lattice point sets, see [68]. Thus, in the following section, we extend our results for reduced fast CBC constructions of lattice point sets with small weighted star discrepancy to similar constructions of polynomial lattice point sets. For the worst-case error criterion there already exists a reduced fast construction for polynomial lattice point sets by Dick, Kritzer, Leobacher and Pillichshammer [13].

The results of this section are based on the paper [48] and have been developed in joint work with Ralph Kritzinger and Mario Neumüller.

In [15] Dick et al. construct polynomial lattice point sets using the worst-case error criterion. As for the star discrepancy criterion, standard-type CBC constructions for polynomial lattice point sets were provided in [18] for an irreducible modulus f and in [12] for a reducible f. In these papers, the authors considered the unweighted star discrepancy as well as its weighted version, which we study here. It is the aim of this section to speed up these constructions by reducing the search sets for the components of the generating vector $\boldsymbol{g}$ according to each component's importance.

In the following, by $G_{p,m}$ we denote the set of all polynomials g over $\mathbb{F}_p$ with $\deg(g) < m$. Further we define

$$G_{p,m}(f) := \{g \in G_{p,m} \mid \gcd(g, f) = 1\}. \tag{3.42}$$

Let $w_1 \leq w_2 \leq \cdots$ be a non-decreasing sequence of non-negative integers, determined in accordance with the weight sequence $\boldsymbol{\gamma}$, as described in the previous sections. Loosely speaking, the smaller γ_j, the bigger w_j is chosen. For an example as to how to choose the w_j's see p. 62. For $w \in \mathbb{N}_0$ with $w < m$ we define $G_{p,m-w}$ and $G_{p,m-w}(f)$ analogously to $G_{p,m}$ and $G_{p,m}(f)$, respectively. Further we set the reduced search spaces to

$$\mathcal{G}_{p,m-w}(f) := \begin{cases} G_{p,m-w}(f) & \text{if } w < m, \\ \{1 \in \mathbb{F}_p[x]\} & \text{if } w \geq m \end{cases}$$

for any $w \in \mathbb{N}_0$. For $w < m$ these sets have cardinality $p^{m-w} - 1$ in the case of an irreducible modulus f and $p^{m-w-1}(p-1)$ for the special case $f : \mathbb{F}_p \to \mathbb{F}_p, x \mapsto x^m$. We will consider these two cases in the following sections. The reason not to use a general reducible modulus f, but rather $f : \mathbb{F}_p \to \mathbb{F}_p, x \mapsto x^m$ is twofold. Firstly, this is what is used in practice, as in this case for $g \in \mathbb{F}_p((x^{-1}))$ computing the Laurent series g/f comes down to shifting the coefficients of g m times to the left, which saves many technicalities. Secondly, for general reducible moduli f the analysis becomes rather difficult and tedious, while it is not to be expected that the outcome is much better than for $f(x) = x^m$.

Further, for $d \in [s]$, we define $\mathcal{G}_{p,m-\boldsymbol{w}}^d(f) := \mathcal{G}_{p,m-w_1}(f) \times \cdots \times \mathcal{G}_{p,m-w_d}(f)$. The idea is to choose the ith component $x^{w_i} g_i$ of $\boldsymbol{g}$, with $g_i \in \mathcal{G}_{p,m-w_i}(f)$ instead of $g_i \in G_{p,m}(f)$, i.e., the search set for the ith component is reduced by a factor $p^{-\min\{w_i,m\}}$ in comparison to the standard CBC construction. We will show that a polynomial lattice point set constructed according to our reduced CBC algorithm has a low weighted star discrepancy of order $N^{-1+\delta}$ for all $\delta > 0$, under certain conditions on the weights $\boldsymbol{\gamma}$ and on $\boldsymbol{w}$.

For the weighted star discrepancy of a polynomial lattice point set we write $D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{g}, f)$.

3.3.1 A reduced CBC construction

In this section we present a CBC construction for the vector $(x^{w_1} g_1, \ldots, x^{w_s} g_s)$ and an upper bound for the weighted star discrepancy of the corresponding polynomial lattice point set.

First note that if $\boldsymbol{g} \in G^s_{p,m}$, then it is known (see [18]) that

$$D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{g}, f) \leq \sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f), \tag{3.43}$$

where in the case of product weights we have

$$R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f) = \sum_{\substack{\boldsymbol{h} \in G^s_{p,m} \setminus \{\boldsymbol{0}\} \\ \boldsymbol{h} \cdot \boldsymbol{g} \equiv 0 \bmod f}} \prod_{i=1}^s r_p(h_i, \gamma_i). \tag{3.44}$$

Note that (3.43) and (3.44) are in analogy to the case of lattice point sets (cf. (3.10)).

For elements $\boldsymbol{h} = (h_1, \ldots, h_s)$ and $\boldsymbol{g} = (g_1, \ldots, g_s)$ in $G^s_{p,m}$ we define the scalar product by $\boldsymbol{h} \cdot \boldsymbol{g} = h_1 g_1 + \cdots + h_s g_s$. The numbers $r_p(h, \gamma)$ for $h \in G_{p,m}$ and $\gamma \in \mathbb{R}$ are defined as

$$r_p(h, \gamma) = \begin{cases} 1 + \gamma & \text{if } h = 0, \\ \gamma r_p(h) & \text{otherwise,} \end{cases}$$

where for $h = h_0 + h_1 x + \cdots + h_a x^a \in G_{p,m}$ with $h_a \neq 0$ we set

$$r_p(h) = \frac{1}{p^{a+1} \sin^2\left(\frac{\pi}{p} h_a\right)}.$$

Thus, in order to analyze the weighted star discrepancy of a polynomial lattice point set, it suffices to investigate the quantity $R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f)$. This is due to the result of Joe [38], who proved that for any summable weight sequence $(\gamma_j)_{j \geq 1}$ we have

$$\sum_{\emptyset \neq \mathfrak{u} \subseteq [s]} \gamma_{\mathfrak{u}} \left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) \leq \frac{\max(1, \Gamma) \mathrm{e}^{\sum_{i=1}^\infty \gamma_i}}{N},$$

with $\Gamma := \sum_{i=1}^\infty \frac{\gamma_i}{1+\gamma_i}$.

Remark 3.22. *We have used the same result on p. 77 to establish tractability results for the case of lattice point sets.*

Then the reduced CBC algorithm reads as follows:

Algorithm 3.23. *Let p be a prime, $m \in \mathbb{N}$, $f \in \mathbb{F}_p[x]$ with $\deg f = m$ and let $(w_j)_{j \geq 1}$ be a non-decreasing sequence of non-negative integers. Consider product weights $(\gamma_j)_{j \geq 1}$. Construct $(g_1, \ldots, g_s) \in \mathcal{G}^s_{p,m-\boldsymbol{w}}(f)$ as follows:*

1. *Set* $g_1 = 1$.
2. *For* $d \in [s-1]$ *assume* $(g_1, \ldots, g_d) \in \mathcal{G}^d_{p,m-\boldsymbol{w}}(f)$ *to be already found. Choose* $g_{d+1} \in \mathcal{G}_{p,m-w_{d+1}}(f)$ *such that*
$$R^{d+1}_{\gamma}((x^{w_1}g_1, \ldots, x^{w_d}g_d, x^{w_{d+1}}g_{d+1}), f)$$
is minimized as a function of g_{d+1}.
3. *Increase* d *by 1 and repeat the second step until* $(g_1, \ldots, g_s)$ *is found.*

Remark 3.24. *Of course we have* $\mathcal{G}^s_{p,m-\boldsymbol{w}}(f) \subseteq G^s_{p,m}$, *and thus in Algorithm 3.23 it indeed suffices to consider* R^{d+1}_{γ} *rather than the weighted star discrepancy.*

In the algorithm above, the search sets are reduced for each coordinate of $(g_1, \ldots, g_s)$ according to their importance, as with increasing w_j the search sets become smaller, as the weights γ_j and thus their corresponding components' influence on the quality of the generating vector decrease. For this reason we call Algorithm 3.23 a reduced CBC algorithm. We will now study Algorithm 3.23 for different choices of f.

3.3.2 Polynomial lattice point sets for $f(x) = x^m$

We will now study the interesting case where $f\colon \mathbb{F}_p \to \mathbb{F}_p, x \mapsto x^m$. This is virtually the only case used in practice. Throughout the rest of this section we write x^m instead of f to emphasize our special choice of f. Note that for $g \in \mathbb{F}_p((x^{-1}))$ the Laurent series g/f can be easily computed in this case by shifting the coefficients of g m times to the left. It is the aim of this section to prove the following theorem:

Theorem 3.25. *Let* $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ *and* $\boldsymbol{w}$ *with* $0 = w_1 \leq w_2 \leq \cdots$. *Let further* $(g_1, \ldots, g_s) \in \mathcal{G}^s_{p,m-\boldsymbol{w}}(x^m)$ *be constructed using Algorithm 3.23. Then we have for every* $d \in [s]$
$$R^d_{\gamma}((x^{w_1}g_1, \ldots, x^{w_d}g_d), x^m) \leq \frac{1}{p^m}\prod_{i=1}^{d}\left(1 + \gamma_i + \gamma_i 2p^{\min\{w_i,m\}}m\frac{p^2-1}{3p}\right).$$

As a direct consequence we obtain the following discrepancy estimate.

Corollary 3.26. *Let* $N = p^m$ *and* $\boldsymbol{\gamma}$, $\boldsymbol{w}$ *and* $(g_1, \ldots, g_s)$ *as in Theorem 3.25. Then the polynomial lattice point set* $\mathcal{P}\left((x^{w_1}g_1, \ldots, x^{w_s}g_s), x^m\right)$ *has a weighted star discrepancy*
$$D^*_{N,\gamma}\left((x^{w_1}g_1, \ldots, x^{w_s}g_s), x^m\right) \leq \sum_{\substack{\mathfrak{u}\subseteq[s]\\ \mathfrak{u}\neq\emptyset}} \gamma_{\mathfrak{u}}\left(1 - \left(1 - \frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{N}\prod_{i=1}^{s}\left(1 + \gamma_i + \gamma_i 2p^{\min\{w_i,m\}}m\frac{p^2-1}{3p}\right). \tag{3.45}$$

Knowing the above discrepancy bound, we are now ready to ask about the size of the polynomial lattice point set required to achieve a weighted star discrepancy not exceeding some ε threshold. In particular, we would like to know how this size depends on the dimension s and on ε.

Corollary 3.27. *Let* $N = p^m$, $\boldsymbol{\gamma}$, *and* $\boldsymbol{w}$ *as in Theorem 3.25 and consider the problem of constructing generating vectors for polynomial lattice point sets with small weighted star discrepancy. Then*
$$\sum_{j=1}^{\infty}\gamma_j p^{w_j} < \infty$$
is a sufficient condition for strong polynomial tractability. This condition further implies $D^*_{N,\gamma}\left((x^{w_1}g_1, \ldots, x^{w_s}g_s), x^m\right) = O(N^{-1+\delta})$, *with the implied constant independent of* s, *for any* $\delta > 0$, *where* $(g_1, \ldots, g_s) \in \mathcal{G}^s_{p,m-\boldsymbol{w}}(x^m)$ *is constructed using Algorithm 3.23.*

Proof. Construct a generating vector $(g_1, \ldots, g_s) \in \mathcal{G}^s_{p,m-w}(x^m)$ applying Algorithm 3.23 and consider its weighted star discrepancy, bounded by (3.45). Following closely the lines of the argumentation in Section 3.2.4 and noticing that $2m\frac{p^2-1}{3p} = O(\log N)$ we obtain the result. More precisely, provided that the $\gamma_j p^{w_j}$'s are summable, we have a means to construct polynomial lattice point sets $\mathcal{P}(\boldsymbol{g}, f)$ with $D^*_{N,\boldsymbol{\gamma}}(\boldsymbol{g}, f) \leq \varepsilon$, whose sizes grow polynomially in ε^{-1} and are independent of the dimension. As a result the problem is strongly polynomially tractable. The discrepancy result $D^*_{N,\boldsymbol{\gamma}}((x^{w_1}g_1, \ldots, x^{w_s}g_s), x^m) = O(N^{-1+\delta})$ follows directly from [47]. It can be computed analogously to the result (3.41) in Section 3.2. □

Remark 3.28. *Recall from p. 67 that $t = \max\{j \in \mathbb{N} \colon w_j < m\}$ and note that setting $w_j = m$ for all $j > t$ does neither change the bound on the weighted star discrepancy nor the computational cost of Algorithm 3.23. It might change the generating vector though. If so, however, only components with very little influence on the quality of the point set are altered. Defining $w_j = m$ for all $j > t$, it suffices to have a summable weight sequence $\boldsymbol{\gamma}$ in order to achieve strong polynomial tractability, as long as t is finite.*

In order to show Theorem 3.25 we need several auxiliary results.

Lemma 3.29. *Let $a \in \mathbb{F}_p[x]$ be monic. Then we have*

$$\sum_{\substack{h \in G_{p,m} \setminus \{0\} \\ a | h}} r_p(h) = (m - \deg(a)) \frac{p^2-1}{3p} p^{-\deg(a)}.$$

In particular, for $a = 1$ this formula yields

$$\sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) = m \frac{p^2-1}{3p}.$$

Proof. This fact follows from [12, p. 1055] (by setting $\gamma_{d+1} = 1$). The special case $a = 1$ also follows from [18, Lemma 2.2] by setting $s = 1$. □

For our purposes, it is convenient to write $R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f)$ from (3.44) in an alternative way. To this end, we introduce some notation. For a Laurent series $L \in \mathbb{F}_p((x^{-1}))$ we denote by $c_{-1}(L)$ its coefficient of x^{-1}, i.e., its residuum. Further, we set $X_p(L) := \chi_p(c_{-1}(L))$, where χ_p is a non-trivial additive character of $\mathbb{F}_p$. One could for instance choose $\chi_p(n) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{p} n}$ for $n \in \mathbb{F}_p$ (see, e.g., [57]). It is clear (see [57, p. 78]) that $X_p(L) = 1$ if L is a polynomial and that $X_p(L_1 + L_2) = X_p(L_1)X_p(L_2)$ for $L_1, L_2 \in \mathbb{F}_p((x^{-1}))$. From [60, p. 78] we know that

$$\sum_{v \in G_{p,m}} X_p\left(\frac{v}{f} g\right) = \begin{cases} p^m & \text{if } f \mid g, \\ 0 & \text{otherwise.} \end{cases} \tag{3.46}$$

Lemma 3.30. *We have*

$$R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f) = -\prod_{i=1}^{s}(1+\gamma_i) + \frac{1}{p^m} \sum_{v \in G_{p,m}} \prod_{i=1}^{s} \left(1 + \gamma_i + \gamma_i \sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) X_p\left(\frac{v}{f} h g_i\right)\right).$$

Proof. We employ the properties of X_p as stated above to obtain from (3.44)

$$R^s_{\boldsymbol{\gamma}}(\boldsymbol{g}, f) = -\prod_{i=1}^{s}(1+\gamma_i) + \frac{1}{p^m} \sum_{\boldsymbol{h} \in G^s_{p,m}} \left(\prod_{i=1}^{s} r_p(h_i, \gamma_i)\right) \sum_{v \in G_{p,m}} X_p\left(\frac{v}{f} \boldsymbol{h} \cdot \boldsymbol{g}\right)$$

$$= -\prod_{i=1}^{s}(1+\gamma_i) + \frac{1}{p^m}\sum_{v\in G_{p,m}}\prod_{i=1}^{s}\left(\sum_{h_i\in G_{p,m}} r_p(h_i,\gamma_i)X_p\left(\frac{v}{f}h_i g_i\right)\right)$$

$$= -\prod_{i=1}^{s}(1+\gamma_i) + \frac{1}{p^m}\sum_{v\in G_{p,m}}\prod_{i=1}^{s}\left(1+\gamma_i+\gamma_i\sum_{h\in G_{p,m}\setminus\{0\}} r_p(h)X_p\left(\frac{v}{f}hg_i\right)\right),$$

and the claimed formula is verified. □

Now we study a sum which will appear later in the proof of Theorem 3.25 and show an upper bound for it.

Lemma 3.31. *Let $w\in\mathbb{N}_0$ and $v\in G_{p,m}$. Let*

$$Y_{p^m,w}(v,x^m) := \sum_{g\in\mathcal{G}_{p,m-w}(x^m)}\sum_{h\in G_{p,m}\setminus\{0\}} r_p(h)X_p\left(\frac{v}{x^m}hx^w g\right),$$

where x^w denotes the polynomial $\tilde{f}(x)=x^w$. Then we have

$$\frac{1}{\#\mathcal{G}_{p,m-w}(x^m)}\sum_{v\in G_{p,m}}|Y_{p^m,w}(v,x^m)| \le 2p^{\min\{w,m\}}m\frac{p^2-1}{3p}.$$

Proof. Let us first assume that $w\ge m$. Then we have $\mathcal{G}_{p,m-w}(x^m)=\{1\}$ and therefore

$$Y_{p^m,w}(v,x^m) = \sum_{h\in G_{p,m}\setminus\{0\}} r_p(h)X_p(vhx^{w-m}) = \sum_{h\in G_{p,m}\setminus\{0\}} r_p(h) = m\frac{p^2-1}{3p}$$

with Lemma 3.29. This leads to

$$\frac{1}{\#\mathcal{G}_{p,m-w}(x^m)}\sum_{v\in G_{p,m}}|Y_{p^m,w}(v,x^m)| = p^m m\frac{p^2-1}{3p} \le 2p^{\min\{w,m\}}m\frac{p^2-1}{3p}$$

in this case. For the rest of the proof let $w<m$ and additionally we abbreviate $\#\mathcal{G}_{p,m-w}(x^m)$ by $\#\mathcal{G}$. We write

$$\frac{1}{\#\mathcal{G}}\sum_{v\in G_{p,m}}|Y_{p^m,w}(v,x^m)| = \frac{1}{\#\mathcal{G}}\sum_{\substack{v\in G_{p,m}\\ x^{m-w}|v}}|Y_{p^m,w}(v,x^m)| + \frac{1}{\#\mathcal{G}}\sum_{\substack{v\in G_{p,m}\\ x^{m-w}\nmid v}}|Y_{p^m,w}(v,x^m)|.$$

In what follows, we refer to the latter sums as

$$S_1 := \frac{1}{\#\mathcal{G}}\sum_{\substack{v\in G_{p,m}\\ x^{m-w}|v}}|Y_{p^m,w}(v,x^m)| \quad\text{and}\quad S_2 := \frac{1}{\#\mathcal{G}}\sum_{\substack{v\in G_{p,m}\\ x^{m-w}\nmid v}}|Y_{p^m,w}(v,x^m)|.$$

We may uniquely write any $v\in G_{p,m}\setminus\{0\}$ in the form $v=qx^{m-w}+\ell$, where $q,\ell\in\mathbb{F}_q[x]$ with $\deg(q)<w$ and $\deg(\ell)<m-w$. Using the properties of X_p it is clear that $Y_{p^m,w}(v,x^m)=Y_{p^m,w}(\ell,x^m)$ and hence

$$S_1 = \frac{1}{\#\mathcal{G}}\sum_{\substack{v\in G_{p,m}\\ x^{m-w}|v}}|Y_{p^m,w}(0,x^m)| = \sum_{\substack{v\in G_{p,m}\\ x^{m-w}|v}}\frac{1}{\#\mathcal{G}}\sum_{g\in\mathcal{G}_{p,m-w}(x^m)}\sum_{h\in G_{p,m}\setminus\{0\}} r_p(h)$$

$$= \sum_{\substack{v\in G_{p,m}\\ x^{m-w}|v}} m\frac{p^2-1}{3p} = p^{\min\{w,m\}}m\frac{p^2-1}{3p}.$$

We move on to S_2. Let for $\ell \in \mathbb{F}_q[x]$ with $\deg(\ell) < m-w$, $e(\ell) = \max\{k \in \{0,1,\dots,m-w-1\} : x^k \mid \ell\}$. With this definition we may display S_2 as

$$S_2 = \frac{p^w}{\#\mathcal{G}} \sum_{k=0}^{m-w-1} \sum_{\substack{\ell \in G_{p,m-w}\setminus\{0\} \\ e(\ell)=k}} |Y_{p^m,w}(\ell, x^m)|. \tag{3.47}$$

In the following, we compute $Y_{p^m,w}(\ell, x^m)$ for $\ell \in G_{p,m-w} \setminus \{0\}$ with $e(\ell) = k$. Let μ_p be the Möbius function on the set of monic polynomials over $\mathbb{F}_p$, i.e., $\mu_p : \mathbb{F}_p[x] \to \{-1,0,1\}$ and

$$\mu_p(h) = \begin{cases} (-1)^\nu & \text{if } h \text{ is squarefree and has } \nu \text{ irreducible factors,} \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $\mu_p(1) = 1$, $\mu_p(x) = -1$ and $\mu_p(x^i) = 0$ for $i \in \mathbb{N}$, $i \geq 2$, yields the equivalence of $\sum_{t \mid \gcd(x^{m-w},g)} \mu_p(t) = 1$ and $\gcd(x^{m-w}, g) = 1$. Therefore we can write

$$\begin{aligned}
Y_{p^m,w}(\ell, x^m) &= \sum_{h \in G_{p,m}\setminus\{0\}} r_p(h) \sum_{g \in G_{p,m-w}} X_p\left(\frac{\ell}{x^{m-w}} hg\right) \sum_{t \mid \gcd(x^{m-w},g)} \mu_p(t) \\
&= \sum_{h \in G_{p,m}\setminus\{0\}} r_p(h) \sum_{t \mid x^{m-w}} \mu_p(t) \sum_{\substack{g \in G_{p,m-w} \\ t \mid g}} X_p\left(\frac{\ell}{x^{m-w}} hg\right) \\
&= \sum_{h \in G_{p,m}\setminus\{0\}} r_p(h) \sum_{t \mid x^{m-w}} \mu_p(t) \sum_{a \in G_{p,m-w-\deg(t)}} X_p\left(\frac{\ell}{x^{m-w}} hat\right) \\
&= \sum_{h \in G_{p,m}\setminus\{0\}} r_p(h) \sum_{t \mid x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) \sum_{a \in G_{p,\deg(t)}} X_p\left(\frac{a}{t} h\ell\right) \\
&= \sum_{h \in G_{p,m}\setminus\{0\}} r_p(h) \sum_{\substack{t \mid x^{m-w} \\ t \mid h\ell}} \mu_p\left(\frac{x^{m-w}}{t}\right) p^{\deg(t)} \\
&= \sum_{t \mid x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) p^{\deg(t)} \sum_{\substack{h \in G_{p,m}\setminus\{0\} \\ t \mid h\ell}} r_p(h).
\end{aligned}$$

The equivalence of the conditions $t \mid h\ell$ and $\frac{t}{\gcd(t,\ell)} \mid h$ yields

$$Y_{p^m,w}(\ell, x^m) = \sum_{t \mid x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) p^{\deg(t)} \sum_{\substack{h \in G_{p,m}\setminus\{0\} \\ \frac{t}{\gcd(t,\ell)} \mid h}} r_p(h).$$

We investigate the inner sum and use Lemma 3.29 with $a = \frac{t}{\gcd(t,\ell)}$ to find

$$\sum_{\substack{h \in G_{p,m}\setminus\{0\} \\ \frac{t}{\gcd(t,\ell)} \mid h}} r_p(h) = \left(m - \deg\left(\frac{t}{\gcd(t,\ell)}\right)\right) \frac{p^2-1}{3p} p^{-\deg\left(\frac{t}{\gcd(t,\ell)}\right)}.$$

Now we have

$$Y_{p^m,w}(\ell, x^m) = \frac{p^2-1}{3p} \sum_{t \mid x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) \left(m - \deg\left(\frac{t}{\gcd(t,\ell)}\right)\right) p^{\deg(\gcd(t,\ell))}$$

$$=\frac{p^2-1}{3p} m \sum_{t|x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) p^{\deg(\gcd(t,\ell))}$$
$$-\frac{p^2-1}{3p} \sum_{t|x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) \deg\left(\frac{t}{\gcd(t,\ell)}\right) p^{\deg(\gcd(t,\ell))}.$$

From the facts that $\deg(l) < m - w$ and that $e(\ell) = k \leq m - w - 1$ we obtain $\gcd(x^{m-w}, \ell) = \gcd(x^{m-w-1}, \ell) = x^k$. This observation leads to

$$\sum_{t|x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) p^{\deg(\gcd(t,\ell))} = p^{\deg(\gcd(x^{m-w},\ell))} - p^{\deg(\gcd(x^{m-w-1},\ell))} = 0$$

and

$$\sum_{t|x^{m-w}} \mu_p\left(\frac{x^{m-w}}{t}\right) \deg\left(\frac{t}{\gcd(t,\ell)}\right) p^{\deg(\gcd(t,\ell))}$$
$$= \deg\left(\frac{x^{m-w}}{\gcd(x^{m-w},\ell)}\right) p^{\deg(\gcd(x^{m-w},\ell))} - \deg\left(\frac{x^{m-w-1}}{\gcd(x^{m-w-1},\ell)}\right) p^{\deg(\gcd(x^{m-w-1},\ell))}$$
$$=(m-w-k)p^k - (m-w-k-1)p^k = p^k.$$

Altogether we have

$$Y_{p^m,w}(\ell, x^m) = -\frac{p^2-1}{3p} p^k.$$

Inserting this result into (3.47) yields

$$S_2 = \frac{p^w}{\#\mathcal{G}} \frac{p^2-1}{3p} \sum_{k=0}^{m-w-1} p^k \sum_{\substack{\ell \in G_{p,m-w} \setminus \{0\} \\ e(\ell)=k}} 1.$$

Since

$$\#\{\ell \in G_{p,m-w} \setminus \{0\} : e(\ell) = k\}$$
$$=\#\{\ell \in G_{p,m-w} \setminus \{0\} : x^k \mid \ell\} - \#\{\ell \in G_{p,m-w} \setminus \{0\} : x^{k+1} \mid \ell\}$$
$$=p^{m-w-k} - 1 - (p^{m-w-k-1} - 1) = p^{m-w-k-1}(p-1),$$

we have

$$S_2 = \frac{p^w}{p^{m-w-1}(p-1)} \frac{p^2-1}{3p} \sum_{k=0}^{m-w-1} p^k p^{m-w-k-1}(p-1)$$
$$= p^w \frac{p^2-1}{3p}(m-w) \leq p^{\min\{w,m\}} m \frac{p^2-1}{3p}.$$

Summarizing, we have shown

$$\frac{1}{\#\mathcal{G}} \sum_{v \in G_{p,m}} |Y_{p^m,w}(v, x^m)| = S_1 + S_2 \leq 2p^{\min\{w,m\}} m \frac{p^2-1}{3p},$$

which completes the proof.

□

Now we are ready to prove Theorem 3.25 using induction on d.

Proof. We show the result for $d = 1$. From Lemma 3.30 we have

$$\begin{aligned} R^1_{\boldsymbol{\gamma}}((x^{w_1}), x^m) =& -(1+\gamma_1) + \frac{1}{p^m} \sum_{v \in G_{p,m}} \left(1 + \gamma_1 + \gamma_1 \sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) X_p\left(\frac{v}{x^m} h x^{w_1}\right)\right) \\ =& \frac{\gamma_1}{p^m} \sum_{v \in G_{p,m}} \sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) X_p\left(\frac{v}{x^m} h x^{w_1}\right). \end{aligned}$$

If $w_1 \geq m$, then

$$\begin{aligned} R^1_{\boldsymbol{\gamma}}((x^{w_1}), x^m) =& \frac{\gamma_1}{p^m} \sum_{v \in G_{p,m}} \sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) = \frac{\gamma_1}{p^m} p^{\min\{w_1,m\}} m \frac{p^2-1}{3p} \\ \leq& \frac{1}{p^m}\left(1+\gamma_1+\gamma_1 2 p^{\min\{w_1,m\}} m \frac{p^2-1}{3p}\right). \end{aligned}$$

If $w_1 < m$, then we can write

$$\begin{aligned} R^1_{\boldsymbol{\gamma}}((x^{w_1}), x^m) =& \frac{\gamma_1}{p^m} \sum_{v \in G_{p,m}} \sum_{h \in G_{p,m} \setminus \{0\}} r_p(h) X_p\left(\frac{v}{x^m} h x^{w_1}\right) \\ =& \frac{\gamma_1}{p^m} \sum_{\substack{h \in G_{p,m} \setminus \{0\} \\ x^{m-w_1} \mid h}} r_p(h) \sum_{v \in G_{p,m}} X_p\left(\frac{v}{x^m} h x^{w_1}\right) \\ &+ \frac{\gamma_1}{p^m} \sum_{\substack{h \in G_{p,m} \setminus \{0\} \\ x^{m-w_1} \nmid h}} r_p(h) \sum_{v \in G_{p,m}} X_p\left(\frac{v}{x^m} h x^{w_1}\right) \\ =& \gamma_1 \sum_{\substack{h \in G_{p,m} \setminus \{0\} \\ x^{m-w_1} \mid h}} r_p(h), \end{aligned}$$

where we used (3.46) in the latter step. We regard Lemma 3.29 with $a = x^{m-w_1}$ to compute

$$\sum_{\substack{h \in G_{p,m} \setminus \{0\} \\ x^{m-w_1} \mid h}} r_p(h) = \frac{1}{p^m} p^{w_1} w_1 \frac{p^2-1}{3p} \leq \frac{1}{p^m} p^{\min\{w_1,m\}} m \frac{p^2-1}{3p},$$

which leads to the desired result in this case as well.
Now let $d \in [s-1]$. Assume that we have $(g_1, \ldots, g_d) \in \mathcal{G}^d_{p,m-\boldsymbol{w}}(x^m)$ such that

$$R^d_{\boldsymbol{\gamma}}((x^{w_1} g_1, \ldots, x^{w_d} g_d), x^m) \leq \frac{1}{p^m} \prod_{i=1}^d \left(1 + \gamma_i + \gamma_i 2 p^{\min\{w_i,m\}} m \frac{p^2-1}{3p}\right).$$

Let $g^* \in \mathcal{G}_{p,m-w_{d+1}}(x^m)$ be such that $R^{d+1}_{\boldsymbol{\gamma}}((x^{w_1} g_1, \ldots, x^{w_d} g_d, x^{w_{d+1}} g_{d+1}), x^m)$ is minimized as a function of g_{d+1} for $g_{d+1} = g^*$. Then we have, using Lemma 3.30

$$R^{d+1}_{\boldsymbol{\gamma}}((x^{w_1} g_1, \ldots, x^{w_d} g_d, x^{w_{d+1}} g^*), x^m) = -(1+\gamma_{d+1}) \prod_{i=1}^d (1+\gamma_i)$$

$$+\frac{1}{p^m}\sum_{v\in G_{p,m}}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i\sum_{h\in G_{p,m}\setminus\{0\}}r_p(h)X_p\left(\frac{v}{x^m}hx^{w_i}g_i\right)\right)$$
$$\times\left(1+\gamma_{d+1}+\gamma_{d+1}\sum_{h\in G_{p,m}\setminus\{0\}}r_p(h)X_p\left(\frac{v}{x^m}hx^{w_{d+1}}g^*\right)\right)$$
$$=(1+\gamma_{d+1})R^d_{\boldsymbol{\gamma}}((x^{w_1}g_1,\dots,x^{w_d}g_d),x^m)+L(g^*), \tag{3.48}$$

where

$$L(g^*)=\frac{\gamma_{d+1}}{p^m}\sum_{v\in G_{p,m}}\sum_{h\in G_{p,m}\setminus\{0\}}r_p(h)X_p\left(\frac{v}{x^m}hx^{w_{d+1}}g^*\right)$$
$$\times\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i\sum_{u\in G_{p,m}\setminus\{0\}}r_p(u)X_p\left(\frac{v}{x^m}ux^{w_i}g_i\right)\right).$$

A minimizer g^* of $R^{d+1}_{\boldsymbol{\gamma}}((x^{w_1}g_1,\dots,x^{w_d}g_d,x^{w_{d+1}}g_{d+1}),x^m)$ is also a minimizer of $L(g_{d+1})$. Combining (3.44) and (3.48) we obtain that $R^d_{\boldsymbol{\gamma}}(\boldsymbol{g},f)\in\mathbb{R}$ for all $d\in[s]$. Moreover with equation (3.51), established later on in Section 3.3.3, and the fact that $r_p(h,\gamma)>0$ for all $h\in G_{p,m}$ and $\gamma\in(0,1]$, we get that $L(g)\in\mathbb{R}^+$ for all $g\in\mathcal{G}_{p,m-w_{d+1}}(x^m)$. Thus we may bound $L(g^*)$ by the mean over all $g\in\mathcal{G}_{p,m-w_{d+1}}(x^m)$. Hence

$$L(g^*)\le\frac{1}{\#\mathcal{G}_{p,m-w_{d+1}}(x^m)}\sum_{g_{d+1}\in\mathcal{G}_{p,m-w_{d+1}}(x^m)}L(g_{d+1})$$
$$\le\frac{\gamma_{d+1}}{p^m}\sum_{v\in G_{p,m}}\frac{1}{\#\mathcal{G}_{p,m-w_{d+1}}(x^m)}$$
$$\times\left|\sum_{g_{d+1}\in\mathcal{G}_{p,m-w_{d+1}}(x^m)}\sum_{h\in G_{p,m}\setminus\{0\}}r_p(h)X_p\left(\frac{v}{x^m}hx^{w_{d+1}}g_{d+1}\right)\right|$$
$$\times\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i\sum_{u\in G_{p,m}\setminus\{0\}}r_p(u)\left|X_p\left(\frac{v}{x^m}ux^{w_i}g_i\right)\right|\right)$$
$$\le\frac{\gamma_{d+1}}{p^m}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i m\frac{p^2-1}{3p}\right)\sum_{v\in G_{p,m}}\frac{|Y_{p^m,w_{d+1}}(v,x^m)|}{\#\mathcal{G}_{p,m-w_{d+1}}(x^m)},$$

where we used the estimate $\left|X_p\left(\frac{v}{x^m}hx^{w_i}g_i\right)\right|\le 1$ in the last step. With the induction hypothesis and Lemma 3.31 this leads to

$$R^{d+1}_{\boldsymbol{\gamma}}((x^{w_1}g_1,\dots,x^{w_d}g_d,x^{w_{d+1}}g^*),x^m)$$
$$\le(1+\gamma_{d+1})\frac{1}{p^m}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i 2p^{\min\{w_i,m\}}m\frac{p^2-1}{3p}\right)$$
$$+\frac{\gamma_{d+1}}{p^m}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i m\frac{p^2-1}{3p}\right)2p^{\min\{w_{d+1},m\}}m\frac{p^2-1}{3p}$$
$$\le\frac{1}{p^m}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i 2p^{\min\{w_i,m\}}m\frac{p^2-1}{3p}\right)\left(1+\gamma_{d+1}+\gamma_{d+1}2p^{\min\{w_{d+1},m\}}m\frac{p^2-1}{3p}\right)$$
$$=\frac{1}{p^m}\prod_{i=1}^{d+1}\left(1+\gamma_i+\gamma_i 2p^{\min\{w_i,m\}}m\frac{p^2-1}{3p}\right).$$

□

The reduced fast CBC construction
So far we have seen how to construct a generating vector $\boldsymbol{g}$ of the point set $\mathcal{P}(\boldsymbol{g}, x^m)$. In fact Algorithm 3.23 can be made much faster using results from [13, 66, 67]. In this section we are investigating and improving Algorithm 3.23 and additionally analyzing the computational cost of the improved algorithm.

As explained in the following lines, Walsh functions are a suitable tool for analyzing the computational cost of CBC algorithms for constructing polynomial lattice point sets. Recall from p. 33 that Walsh functions are defined as follows. Let $\omega = \mathrm{e}^{2\pi i/p}$, $x \in [0,1)$ and h a non-negative integer with base p representations $x = x_1/p + x_2/p^2 + \ldots$ and $h = h_0 + h_1 p + \ldots + h_r p^r$, respectively. Then we define

$$\mathrm{wal}_h : [0,1) \to \mathbb{C}, \mathrm{wal}_h(x) := \omega^{h_0 x_1 + \ldots + h_r x_{r+1}}.$$

The Walsh function system $\{\mathrm{wal}_h \mid h = 0, 1, \ldots\}$ is a complete orthonormal basis in $L_2([0,1))$ which has been used in the analysis of the discrepancy of digital nets (an important class of low-discrepancy point sets which contains polynomial lattice point sets) several times before, see for example [18, 27, 54]. For further information on Walsh functions see [20, Appendix A].
Let $d \geq 1$, $N = p^m$. For $P(\boldsymbol{g}, f) = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{p^m-1}\}$ with $\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)})$ we have the formula (see [18, Section 4])

$$\frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{s} \mathrm{wal}_{h_i}(x_n^{(i)}) = \begin{cases} 1 & \text{if } \boldsymbol{g} \cdot \boldsymbol{h} \equiv 0 \pmod{f}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.49}$$

where h_i are non-negative integers with base p representation $h_i = h_0^{(i)} + h_1^{(i)} p + \cdots + h_r^{(i)} p^r$. We identify these non-negative integers h_i with the polynomials $h_i(x) = h_0^{(i)} + h_1^{(i)} x + \cdots + h_r^{(i)} x^r$. These polynomial are elements of $\mathcal{G}_{p,m}$. The vectors $\boldsymbol{h}$ in (3.49) are then from $\mathcal{G}_{p,m}^s$ such that $\boldsymbol{h} = (h_1(x), \ldots, h_s(x))$.

Equation (3.49) allows us to rewrite $R_{\boldsymbol{\gamma}}^d((x^{w_1} g_1, \ldots, x^{w_d} g_d), x^m)$ in the following way

$$R_{\boldsymbol{\gamma}}^d((x^{w_1} g_1, \ldots, x^{w_d} g_d), x^m) = -\prod_{i=1}^{d}(1+\gamma_i) + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{d} \sum_{h=0}^{p^m-1} r_p(h, \gamma_i) \mathrm{wal}_h\left(\phi_m\left(\frac{n x^{w_i} g_i}{x^m}\right)\right).$$

Note that $r_p(h, \gamma)$ is defined as in (3.44) and we identify the integer in base p representation $h = h_0 + h_1 p + \ldots + h_r p^r$ with the polynomial $h(x) = h_0 + h_1 x + \ldots + h_r x^r$. If we set $\psi(\frac{n x^{w_i} g_i}{x^m}) := \sum_{h=1}^{p^m-1} r_p(h) \mathrm{wal}_h(\phi_m(\frac{n x^{w_i} g_i}{x^m}))$ we get that

$$\begin{aligned} R_{\boldsymbol{\gamma}}^d((x^{w_1} g_1, \ldots, x^{w_d} g_d), x^m) &= -\prod_{i=1}^{d}(1+\gamma_i) + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \prod_{i=1}^{d} \left(1 + \gamma_i + \gamma_i \psi\left(\frac{n x^{w_i} g_i}{x^m}\right)\right) \\ &= -\prod_{i=1}^{d}(1+\gamma_i) + \frac{1}{p^m} \sum_{n=0}^{p^m-1} \eta_d(n), \end{aligned} \tag{3.50}$$

where $\eta_d(n) = \prod_{i=1}^{d} \left(1 + \gamma_i + \gamma_i \psi(\frac{n x^{w_i} g_i}{x^m})\right)$.
In [18, Section 4] it is proved that we can compute the at most N different values of $\psi(\frac{r}{x^m})$ for $r \in G_{p,m}$ in $O(N)$ operations.
Let us now analyze one step of the reduced CBC Algorithm 3.23. Assuming that we already found $(g_1, \ldots, g_d) \in \mathcal{G}_{p,m-\boldsymbol{w}}^d(x^m)$ we have to minimize

$$R_{\boldsymbol{\gamma}}^{d+1}((x^{w_1} g_1, \ldots, x^{w_{d+1}} g_{d+1}), x^m)$$

as a function of $g_{d+1} \in \mathcal{G}_{p,m-w_{d+1}}(x^m)$. If $w_{d+1} \geq m$ then $g_{d+1} = 1$ and we are done. Let now $w_{d+1} < m$. From (3.50) we have that

$$\begin{aligned} R_\gamma^{d+1}((x^{w_1}g_1, \ldots, x^{w_{d+1}}g_{d+1}), x^m) &= -\prod_{i=1}^{d+1}(1+\gamma_i) + \frac{1}{p^m}\sum_{n=0}^{p^m-1}\eta_{d+1}(n) \\ &= -\prod_{i=1}^{d+1}(1+\gamma_i) + \frac{1}{p^m}\sum_{n=0}^{p^m-1}\Bigg(1+\gamma_{d+1} \\ &\quad + \gamma_{d+1}\psi\left(\frac{nx^{w_{d+1}}g_{d+1}}{x^m}\right)\Bigg)\eta_d(n). \end{aligned}$$

In order to minimize $R_\gamma^{d+1}((x^{w_1}g_1, \ldots, x^{w_{d+1}}g_{d+1}), x^m)$ it is enough to minimize

$$T_d(g) := \sum_{n=0}^{p^m-1}\psi(\frac{nx^{w_{d+1}}g}{x^m})\eta_d(n).$$

As in [13, Section 4] we can represent this quantity using some specific $(p^{m-w_{d+1}-1}(p-1)\times N)$-matrix A and exploiting its additional structure. Let, to this end,

$$A = \left(\psi\left(\frac{nx^{w_{d+1}}g}{x^m}\right)\right)_{\substack{g\in G_{p,m-w_{d+1}}(x^m),\\ n\in\{0,\ldots,N-1\}}} \text{ and } \boldsymbol{\eta}_d = (\eta_d(0), \ldots, \eta_d(N-1))^\top.$$

First of all observe that we get $(T(g))_{g\in G_{p,m-w_{d+1}}(x^m)} = A\boldsymbol{\eta}_d$. Secondly the matrix A is a block matrix and can be written in the following form

$$A = \left(\Omega^{(m-w_{d+1})}\ldots\Omega^{(m-w_{d+1})}\right), \text{ where } \Omega^{(l)} = \left(\psi\left(\frac{nx^{w_{d+1}}g}{x^m}\right)\right)_{\substack{n\in\{0,\ldots p^l-1\}\\ g\in G_{p,m-w_{d+1}}(x^m)}}.$$

If $\boldsymbol{x}$ is any vector of length p^m then we compute

$$A\boldsymbol{x} = \Omega^{(m-w_{d+1})}\boldsymbol{x}_1 + \ldots + \Omega^{(m-w_{d+1})}\boldsymbol{x}_{p^{w_{d+1}}} = \Omega^{(m-w_{d+1})}(\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_{p^{w_{d+1}}}).$$

With this representation we can apply the machinery of [66, 67] and get that multiplication with $\Omega^{(m-w_{d+1})}$ can be done in $O((m-w_{d+1})p^{m-w_{d+1}})$ operations. Summarizing we have:

Algorithm 3.32.

1. *Compute* $\psi(\frac{r}{x^m})$ *for* $r \in G_{p,m}$.
2. *Set* $\eta_1(n) = \psi(\frac{nx^{w_1}g_1}{x^m})$ *for* $n = 0, \ldots, p^m - 1$.
3. *Set* $g_1 = 1,\ d = 2$ *and* $t = \max\{j \in [s] \mid w_j < m\}$.
   While $d \leq \min\{s, t\}$,
   (a) *Partition* η_{d-1} *into* p^{w_d} *vectors* $\eta_{d-1}^{(1)}, \ldots, \eta_{d-1}^{(p^{w_d})}$ *of length* p^{m-w_d} *and let* $\eta' = \sum_{i=1}^{p^{w_d}}\eta_{d-1}^{(i)}$.
   (b) *Let* $(T_d(g))_{g\in G_{p,m-w_d}} = \Omega^{(m-w_d)}\eta'$.
   (c) *Let* $g_d = \operatorname{argmin}_g T_d(g)$.
   (d) *Let* $\eta_d(n) = \eta_{d-1}(n)\left(1+\gamma_d+\gamma_d\psi(\frac{nx^{w_d}g_d}{x^m})\right)$
   (e) *Increase* d *by 1.*

4. If $s \geq t$ then set $g_t = g_{t+1} = \ldots = g_s = 1$.

Similar to [13] we obtain the following theorem from the observations in this section:

Theorem 3.33. *The cost of Algorithm 3.32 is*

$$O\left(N + \min\{s,t\}N + \sum_{d=1}^{\min\{s,t\}} (m - w_d)Np^{-w_d}\right).$$

3.3.3 Polynomial lattice point sets for irreducible f

Finally we want to consider the case where f is an irreducible polynomial. So, for this section let f be an irreducible polynomial over $\mathbb{F}_p$ with $\deg(f) = m$.

Theorem 3.34. *Let $\boldsymbol{\gamma}$ and $\boldsymbol{w}$ as in Theorem 3.25 and let $f \in \mathbb{F}_p[x]$ be an irreducible polynomial with* $\deg(f) = m$. *Let further $(g_1, \ldots, g_s) \in \mathcal{G}^s_{p,m-\boldsymbol{w}}(f)$ be constructed according to Algorithm 3.23. Then we have for every $d \in [s]$*

$$R^d_{\boldsymbol{\gamma}}((x^{w_1}g_1, \ldots, x^{w_d}g_d), f) \leq \frac{1}{p^m}\prod_{i=1}^{d}\left(1 + \gamma_i + \gamma_i p^{\min\{w_i,m\}} m\frac{p+1}{3}\right).$$

Proof. We will prove this result by induction on d. According to Algorithm 3.23 we know that $g_1 = 1$ for $d = 1$. Therefore $R^1_{\boldsymbol{\gamma}}((x^{w_1}g_1), f) = 0$ since for all $h \in G_{p,m}$ we have $\deg(h) < m$ and hence the congruence $hx^{w_1} \equiv 0 \pmod f$ has no solutions.
Let $d \in [s-1]$ and assume that we have already found $(g_1, \ldots, g_d) \in \mathcal{G}^d_{p,m-\boldsymbol{w}}(f)$. For $\boldsymbol{g} = (x^{w_1}g_1, \ldots, x^{w_d}g_d)$ we have from (3.44) that

$$R^{d+1}_{\boldsymbol{\gamma}}((\boldsymbol{g}, x^{w_{d+1}}g_{d+1}), f) = (1 + \gamma_{d+1})R^d_{\boldsymbol{\gamma}}(\boldsymbol{g}, f) + \theta(g_{d+1}), \tag{3.51}$$

where we proceeded similarly as in the proof of Theorem 3.25. Here we have

$$\theta(g_{d+1}) = \sum_{h_{d+1}\in G_{p,m}\setminus\{0\}} r_p(h_{d+1}, \gamma_{d+1}) \sum_{\substack{\boldsymbol{h}\in G^d_{p,m} \\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv -h_{d+1}x^{w_{d+1}}g_{d+1} \pmod f}} \prod_{i=0}^{d} r_p(h_i, \gamma_i).$$

Let $g^* \in \mathcal{G}_{p,m-w_{d+1}}(f)$ be a minimizer of $R^{d+1}_{\boldsymbol{\gamma}}((\boldsymbol{g}, x^{w_{d+1}}g_{d+1}), f)$ as a function of g_{d+1}. Therefore g^* also minimizes $\theta(g_{d+1})$. Bounding $\theta(g^*)$ by its mean we obtain

$$\begin{aligned}\theta(g^*) \leq & \frac{1}{\#\mathcal{G}_{p,m-w_{d+1}}(f)} \sum_{h_{d+1}\in G_{p,m}\setminus\{0\}} r_p(h_{d+1}, \gamma_{d+1}) \\ & \times \sum_{\boldsymbol{h}\in G^d_{p,m}} \left(\prod_{i=1}^{d} r_p(h_i, \gamma_i)\right) \sum_{\substack{g_{d+1}\in G_{p,m-w_{d+1}}(f) \\ \boldsymbol{h}\cdot\boldsymbol{g}\equiv -h_{d+1}x^{w_{d+1}}g_{d+1} \pmod f}} 1.\end{aligned}$$

Observe that $\gcd(f, h_{d+1}x^{w_{d+1}}) = 1$. Therefore the congruence $h_{d+1}x^{w_{d+1}}g_{d+1} \equiv -\boldsymbol{h}\cdot\boldsymbol{g} \pmod f$ has a unique solution in $G_{p,m}$ but not necessarily in $\mathcal{G}_{p,m-w_{d+1}}(f)$. In the case that $-\boldsymbol{h}\cdot\boldsymbol{g} \not\equiv 0 \pmod f$ we conclude that the congruence has at most one solution in $\mathcal{G}_{p,m-w_{d+1}}(f)$. If $-\boldsymbol{h}\cdot\boldsymbol{g} \equiv 0 \pmod f$ the congruence has no solution in $\mathcal{G}_{p,m-w_{d+1}}(f)$ since $0 \notin \mathcal{G}_{p,m-w_{d+1}}(f)$. Hence we find by an application of [18, Lemma 3.3] that

$$\theta(g^*) \le \frac{1}{\#\mathcal{G}_{p,m-w_{d+1}}(f)} \sum_{h_{d+1}\in G_{p,m}\setminus\{0\}} r_p(h_{d+1},\gamma_{d+1}) \sum_{\boldsymbol{h}\in G_{p,m}^d} \prod_{i=1}^{d} r_p(h_i,\gamma_i)$$
$$= \frac{1}{\#\mathcal{G}_{p,m-w_{d+1}}(f)} \left[\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i m\frac{p^2-1}{3p}\right)\right]\left(\gamma_{d+1} m \frac{p^2-1}{3p}\right).$$

By (3.51) and the induction hypothesis we have that

$$\begin{aligned}
R_\gamma^{d+1}((\boldsymbol{g}, x^{w_{d+1}}g_{d+1}), f) &= (1+\gamma_{d+1})R_\gamma^d(\boldsymbol{g}, f) + \theta(g_{d+1}) \\
&\le \frac{1}{p^m}\prod_{i=1}^{d}\left(1+\gamma_i+\gamma_i p^{\min\{w_i,m\}} m \frac{p+1}{3}\right) \\
&\quad\times\left(1+\gamma_{d+1}+\gamma_{d+1}\frac{p^m}{\#\mathcal{G}_{p,m-w_{d+1}}(f)} m\frac{p^2-1}{3p}\right) \\
&\le \frac{1}{p^m}\prod_{i=1}^{d+1}\left(1+\gamma_i+\gamma_i p^{\min\{w_i,m\}} m \frac{p+1}{3}\right),
\end{aligned}$$

where we used in the latter step that $\frac{p^m}{\#\mathcal{G}_{p,m-w_{d+1}}(f)} \le \frac{p}{p-1}p^{\min\{w_{d+1},m\}}$. This follows from the fact that $\#\mathcal{G}_{p,m-w_{d+1}}(f) = p^{m-w_{d+1}} - 1$ if $w_{d+1} < m$ and $\#\mathcal{G}_{p,m-w_{d+1}}(f) = 1$ if $w_{d+1} \ge m$. This finishes the proof of Theorem 3.34. □

As an immediate consequence of (3.43) and Theorem 3.34 we obtain the following result.

Corollary 3.35. *Let $N = p^m$, $(w_j)_{j\ge1}$ be a non-decreasing sequence of non-negative integers and let $(g_1,\dots,g_s) \in \mathcal{G}^s_{p,m-\boldsymbol{w}}(f)$ for irreducible $f \in G_{p,m}$ be constructed using Algorithm 3.23. Then the polynomial lattice point set $\mathcal{P}((x^{w_1}g_1,\dots,x^{w_s}g_s), f)$ has a weighted star discrepancy*

$$\begin{aligned}
&D^*_{N,\boldsymbol{\gamma}}((x^{w_1}g_1,\dots,x^{w_s}g_s), f) \\
&\quad\le \sum_{\substack{\mathfrak{u}\subseteq[s]\\ \mathfrak{u}\ne\emptyset}} \gamma_{\mathfrak{u}}\left(1-\left(1-\frac{1}{N}\right)^{|\mathfrak{u}|}\right) + \frac{1}{N}\prod_{i=1}^{s}\left(1+\gamma_i+\gamma_i p^{\min\{w_i,m\}} m\frac{p+1}{3}\right).
\end{aligned}$$

Remark 3.36. *Using the same argumentation as in Corollary 3.27 we again obtain the sufficient condition $\sum_{j=1}^{\infty}\gamma_j p^{w_j} < \infty$ for strong polynomial tractability and for the discrepancy bound $D^*_{N,\boldsymbol{\gamma}}((x^{w_1}g_1,\dots,x^{w_s}g_s), f) = O(N^{-1+\delta})$, with the implied constant independent of s, for any $\delta > 0$.*

The CBC constructions presented in the previous sections all have the aim to speed up the construction in order to be able to tackle large dimensions s. As already mentioned in the introduction (see p. 59), with all these constructions, there is another issue that leads to a practical limit on the dimension s. Numerical experiments [50] show that from some dimension onward the components produced by the CBC construction tend to recur. This could be due to rounding errors that occur when implementing the CBC construction, but the definite reason is yet unknown. However, there is a way around it—the projection-corrected CBC construction by Dick and Kritzer [10].

In the present section we combine the reduced and the reduced fast, respectively, with the projection-corrected CBC construction to get a construction which pools the advantages of these two constructions, that is, being considerably faster than the standard CBC construction and being free of recurring components. As the quality criterion we consider the worst-case error in this section. All results presented here are based on [52].

Recall from (3.7) the definition of the reduced search spaces as

$$\mathcal{Z}_{N,w_j} = \begin{cases} \{z \in \{1,\ldots,b^{m-w_j}-1\} \colon \gcd(z,b^m)=1\} & \text{if } w_j < m, \\ \{1\} & \text{otherwise,} \end{cases}$$

where $0 = w_1 \leq w_2 \leq \ldots$ is a nondecreasing sequence of non-negative integers, defined in accordance to the weight sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ as done before.

3.4.1 Definition of the function space

With our combined CBC algorithm we would like to construct generating vectors for lattice point sets used in QMC algorithms applied to functions in certain weighted Korobov spaces. The Korobov spaces we want to consider are the ones defined on p. 34. As we change the names of the parameters in this section, in the following lines, we briefly recapitulate the definition of the spaces.

Let $\alpha > 1$. The product-weighted Korobov spaces $H(K_{s,\alpha,\boldsymbol{\gamma}})$ we want to consider are defined as follows. They are reproducing kernel Hilbert spaces of functions defined on $[0,1]^s$, with their reproducing kernel given by

$$K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{h}\in\mathbb{Z}^s} r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) \exp(2\pi \mathrm{i} \boldsymbol{h}\cdot(\boldsymbol{x}-\boldsymbol{y})), \quad \boldsymbol{x},\boldsymbol{y} \in [0,1)^s,$$

where, for $\boldsymbol{h} = (h_1,\ldots,h_s) \in \mathbb{Z}^s$, we have

$$r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}) = \prod_{j=1}^{s} r_\alpha(\gamma_j,h_j), \tag{3.52}$$

with

$$r_\alpha(\gamma_j, h_j) = \begin{cases} 1 & \text{if } h_j = 0, \\ \frac{\gamma_j}{|h_j|^\alpha} & \text{otherwise.} \end{cases} \tag{3.53}$$

For $f, g \in H(K_{s,\alpha,\boldsymbol{\gamma}})$ the inner product is then given by

$$\langle f, g \rangle_{H(K_{s,\alpha,\boldsymbol{\gamma}})} = \sum_{\boldsymbol{h} \in \mathbb{Z}^s} (r_\alpha(\boldsymbol{\gamma}, \boldsymbol{h}))^{-1} \hat{f}(\boldsymbol{h}) \overline{\hat{g}(\boldsymbol{h})},$$

where $\hat{f}(\boldsymbol{h}) = \int_{[0,1]^s} f(\boldsymbol{x}) \exp(-2\pi \mathrm{i} \boldsymbol{h} \cdot \boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$ denotes the $\boldsymbol{h}$-th Fourier coefficient of f. Note that we changed the notation of the Fourier coefficients here. On p. 34, where we first introduced them, the where denoted by $\hat{f}_{\text{trig}}$, whereas here, for simplicity, we abbreviate this notation to $\hat{f}$. The norm in $H(K_{s,\alpha,\boldsymbol{\gamma}})$ is the norm induced by this inner product.

As a quality measure for a generating vector constructed with our algorithm we want to consider the (squared) worst-case error of integration in $H(K_{s,\alpha,\boldsymbol{\gamma}})$ by a QMC rule using the lattice point set as integration nodes. Recall from (2.1) that the worst-case error of $\boldsymbol{z} = (z_1, \ldots, z_s)$ is given by

$$e_{s,N,\boldsymbol{\gamma}}(\boldsymbol{z}) = e_{s,N,\boldsymbol{\gamma}}(z_1, \ldots, z_s) = \sup_{\substack{f \in H(K_{s,\alpha,\boldsymbol{\gamma}}) \\ \|f\|_{H(K_{s,\alpha,\boldsymbol{\gamma}})} \leq 1}} \left| \int_{[0,1]^s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - \frac{1}{N} \sum_{j=0}^{N-1} f(\boldsymbol{p}_j) \right|,$$

where $\{\boldsymbol{p}_0, \ldots, \boldsymbol{p}_{N-1}\}$ denotes the lattice point set, generated by $\boldsymbol{z}$. Here, again, we write $e_{s,N,\boldsymbol{\gamma}}(\boldsymbol{z})$ instead of $e_{s,\boldsymbol{\gamma}}(A_N)$ as a QMC algorithm is fully determined by the generating vector of the underlying lattice point set.

It is known (see for example [13, 22]) that for a generating vector $\boldsymbol{z} \in \{0, \ldots, N-1\}^s$ the squared worst-case error in the weighted Korobov space $H(K_{s,\alpha,\boldsymbol{\gamma}})$ is given by

$$e^2_{s,N,\boldsymbol{\gamma}}(\boldsymbol{z}) = \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \setminus \{\boldsymbol{0}\} \\ \boldsymbol{z} \cdot \boldsymbol{h} \equiv 0 \pmod{N}}} r_\alpha(\boldsymbol{\gamma}, \boldsymbol{h}). \tag{3.54}$$

3.4.2 The combined CBC algorithm

Before we describe the combined CBC construction let us first introduce a little more notation. We denote by $t_1 = \max\{j \in \mathbb{N} \colon w_j = 0\}$ the index up to which we consider the whole set $\mathcal{Z}_N$ as the search space and by $t_2 = \min\{j \in \mathbb{N} \colon w_j \geq m\}$ the first index for which the search space is reduced to $\{1\}$. Note that $t_1 \geq 1$, $t_2 \geq 2$, and $t_1 < t_2$. Furthermore, let for sets $\mathcal{E} \subseteq \mathbb{Z}$, $|\mathcal{E}|$ denote their cardinality.

Now we are ready to state the combined CBC algorithm. Recall from p. 59 that the idea to avoid recurrence of the components is to define exclusion sets, for each step of the CBC construction, whose elements cannot be chosen in this step.

Algorithm 3.37. *Let $s \in \mathbb{N}$, $b \in \mathbb{P}$, $m \in \mathbb{N}$, $N = b^m$, $0 = w_1 \leq w_2 \leq \cdots$, and $\mathcal{Z}_N, \mathcal{Z}_{N,w_j}, t_1$ and t_2 as above.*

1. *Set $z_1 = 1$ and set $\mathcal{E}_1 = \emptyset$. Set $\tilde{z}_1 = z_1$.*
2. *For $d \in \{1, \ldots, \min\{t_1 - 1, s - 1\}\}$ do the following: Assume that $z_1, \ldots, z_d$ have already been found and choose $\mathcal{E}_{d+1} \subsetneq \mathcal{Z}_N$. (If no coordinates are to be excluded in this step, we define $\mathcal{E}_{d+1} = \emptyset$.) Now choose $z_{d+1} \in \mathcal{Z}_N \setminus \mathcal{E}_{d+1}$ such that*

$$e^2_{d+1,N\boldsymbol{\gamma}}(z_1, \ldots, z_d, z_{d+1})$$

is minimized as a function of z_{d+1}. Set $\tilde{z}_{d+1} = z_{d+1}$.

3. *Increase d by 1 and repeat Step 2 until $d = \min\{t_1 - 1, s - 1\}$. (The last repetition of Step 2 is for $d = \min\{t_1 - 1, s - 1\}$.)*

4. *If $t_1 \geq s$ the algorithm terminates with $d + 1 = s$. Else, for $d \in \{t_1, \ldots, \min\{t_2 - 2, s - 1\}\}$ do the following: Assume that $z_1, \ldots, z_{t_1}, z_{t_1+1}, \ldots, z_d$ have already been found and choose $\mathcal{E}_{d+1} \subsetneq \mathcal{Z}_{N,w_{d+1}}$. (If no coordinates are to be excluded in this step, we define $\mathcal{E}_{d+1} = \emptyset$.) Now choose $z_{d+1} \in \mathcal{Z}_{N,w_{d+1}} \setminus \mathcal{E}_{d+1}$ such that*

$$e^2_{d+1,N,\gamma}(z_1, \ldots, z_{t_1}, b^{w_{t_1+1}} z_{t_1+1}, \ldots, b^{w_d} z_d, b^{w_{d+1}} z_{d+1})$$

is minimized as a function of z_{d+1}. Set $\tilde{z}_{d+1} = b^{w_{d+1}} z_{d+1}$.

5. *Increase d by 1 and repeat Step 4 until $d = \min\{t_2 - 2, s - 1\}$. (The last repetition of Step 2 is for $d = \min\{t_2 - 2, s - 1\}$.)*

6. *If $t_2 > s$ the algorithm terminates with $d + 1 = s$. Else, for $d \in \{t_2 - 1, \ldots, s - 1\}$ set $z_{d+1} = 1$. (The corresponding exclusion set is the empty set.) Set $\tilde{z}_{d+1} = z_{d+1}$.*

7. *Increase d by 1 and repeat Step 6 until $d = s - 1$.*

To avoid lengthy case analyses let us here and, if not stated otherwise, for the rest of this section, assume that $t_2 \leq s$. The proofs of Theorem 3.39 for the cases where t_2 or even $t_1 > s$ are easy modifications of the proof stated below.

Remark 3.38. *Algorithm 3.37 produces a vector*

$$\tilde{\boldsymbol{z}} = (\tilde{z}_1, \ldots, \tilde{z}_s) = (z_1, \ldots, z_{t_1}, b^{w_{t_1+1}} z_{t_1+1}, \ldots, b^{w_{t_2-1}} z_{t_2-1}, 1, \ldots, 1)$$

with the last $s - t_2 + 1$ components equal to 1. That means, $z_{t_1}, \ldots, z_{t_2-1}$ are multiplied by the factors $b^{w_{t_1}}, \ldots, b^{w_{t_2-1}}$ in the generating vector. The reason is the following: The reduced search sets $\mathcal{Z}_{N,w_j}$ contain only elements of $\mathcal{Z}_N$ which are smaller than b^{w_j}. So, roughly speaking, all elements of $\mathcal{Z}_{N,w_j}$ lie on the "left side" of $\mathcal{Z}_N$, whereas the elements in $b^{w_j}\mathcal{Z}_{N,w_j}$ are spread all over $\mathcal{Z}_N$, where the notation $b^{w_j}\mathcal{Z}_{N,w_j}$ means, as already before in Section 3.2, that each element of $\mathcal{Z}_{N,w_j}$ is multiplied by b^{w_j} modulo N.

As mentioned before we consider the squared worst-case error $e^2_{s,N,\gamma}(\tilde{z}_1, \ldots, \tilde{z}_s)$ defined above as a quality measure for the generating vector $\tilde{\boldsymbol{z}}$ produced by Algorithm 3.37. Thus we would like to find upper bounds for $e^2_{s,N,\gamma}(\tilde{z}_1, \ldots, \tilde{z}_s)$. Note that the coordinates $\tilde{z}_j$ of $\tilde{\boldsymbol{z}}$ do not necessarily belong to $\mathcal{Z}_N$. However, the formula (3.54) for the squared worst-case error used in [10] is also true for arbitrary coordinates $\tilde{z}_j \in \{0, \ldots, N - 1\}$, see for example [13]. Thus we end up with the following theorem.

Theorem 3.39. *Let $s \in \mathbb{N}$, $b \in \mathbb{P}$, $m \in \mathbb{N}$, $N = b^m$, $0 = w_1 \leq w_2 \leq \cdots$, and $\mathcal{Z}_N, \mathcal{Z}_{N,w_j}, t_1$ and t_2 as above. Further let $\tilde{\boldsymbol{z}} = (\tilde{z}_1, \ldots, \tilde{z}_s)$ be constructed by Algorithm 3.37, with exclusion sets $\mathcal{E}_j$. Then for all $1 \leq d \leq s$ and $\frac{1}{\alpha} < \lambda \leq 1$ we have*

$$e^2_{d,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d) \leq \left(\sum_{\mathfrak{u} \subseteq [d]} \frac{\gamma_{\mathfrak{u}}^{\lambda} (4\zeta(\alpha\lambda))^{|\mathfrak{u}|}}{\phi(b^{\max\{0, m - \max_{j \in \mathfrak{u}}\{w_j\}\}})} \prod_{j \in \mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \right)^{\frac{1}{\lambda}},$$

where we set $\max \emptyset = 0$.

Proof. The proof of Theorem 3.39 is inspired by the proof in [10]. We use induction on d to show the result. Recall our assumptions that $w_1 = 0$ and $z_1 = 1$, and that we consider product weights

$\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} \gamma_j$. Then (3.52), (3.53), and (3.54) together with Jensen's inequality, $(\sum_k a_k)^\lambda \le \sum_k a_k^\lambda$ for non-negative a_k and $0 < \lambda \le 1$, yield

$$e^2_{1,N,\boldsymbol{\gamma}}(z_1) = \gamma_1 \sum_{h\in\mathbb{Z}\setminus\{0\}} |Nh|^{-\alpha} = \frac{\gamma_1}{N^\alpha} 2\zeta(\alpha) \le \left(\frac{\gamma_1^\lambda}{b^m} 4\zeta(\alpha\lambda)\right)^{\frac{1}{\lambda}} \le \left(\frac{\gamma_{\{1\}}^\lambda (4\zeta(\alpha\lambda))^{|\{1\}|}}{\phi(b^{\max\{0,m-\max_{j\in\{1\}} w_j\}})}\right)^{\frac{1}{\lambda}}$$
$$\le \left(\sum_{\mathfrak{u}\subseteq[1]} \frac{\gamma_{\mathfrak{u}}^\lambda (4\zeta(\alpha\lambda))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}}\{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}\right)^{\frac{1}{\lambda}},$$

as claimed.

Now let $d \in \{1,\dots,s-1\}$, and let $\tilde{z}_1,\dots,\tilde{z}_d$ be chosen with Algorithm 3.37 and assume that

$$e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\dots,\tilde{z}_d) \le \left(\sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^\lambda (4\zeta(\alpha\lambda))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}}\{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}\right)^{\frac{1}{\lambda}}$$

holds for any $\lambda \in (\frac{1}{\alpha}, 1]$. We distinguish two cases, namely

1. $d+1 \in \{2,\dots,t_1\}$,
2. $d+1 \in \{t_1+1,\dots,s\}$.

Let us start with the first case where $d+1 \in \{2,\dots,t_1\}$. As then $w_1 = \dots = w_{d+1} = 0$, we effectively consider the case of the projection-corrected CBC construction as in [10]. Note that for $w_j = 0$, we have $\mathcal{Z}_{N,w_j} = \mathcal{Z}_N$ and thus $\phi(N) = \left|\mathcal{Z}_{N,w_j}\right|$. Using this, we already know that

$$e^2_{d+1,N\boldsymbol{\gamma}}(\tilde{z}_1,\dots,\tilde{z}_d,\tilde{z}_{d+1}) \le \left(\frac{1}{\phi(N)} \sum_{\mathfrak{u}\subseteq[d+1]} \gamma_{\mathfrak{u}}^\lambda (2\zeta(\alpha\lambda))^{|\mathfrak{u}|} \prod_{j\in\mathfrak{u}} \frac{\phi(N)}{\phi(N) - |\mathcal{E}_j|}\right)^{\frac{1}{\lambda}}$$
$$\le \left(\sum_{\mathfrak{u}\subseteq[d+1]} \frac{\gamma_{\mathfrak{u}}^\lambda (4\zeta(\alpha\lambda))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}}\{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}\right)^{\frac{1}{\lambda}}$$

for any $\lambda \in (\frac{1}{\alpha}, 1]$ and we are done with this case.

Next we deal with the second case where we have $d+1 \in \{t_1+1,\dots,s\}$. Using (3.54) we easily obtain that for any $z \in \mathcal{Z}_N$

$$e^2_{d+1,N,\boldsymbol{\gamma}}(\tilde{z}_1,\dots,\tilde{z}_d,z) = e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\dots,\tilde{z}_d) + \theta_{N,d+1,\alpha,\boldsymbol{\gamma}}(z),$$

where

$$\theta_{N,d+1,\alpha,\boldsymbol{\gamma}}(z) = \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^{d+1} \\ h_{d+1}\neq 0 \\ \boldsymbol{h}\cdot(\tilde{z}_1,\dots,\tilde{z}_d,z)\equiv 0 \pmod{N}}} r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}).$$

By setting $\beta_j = 1$ in [7, Eq. (5)], we obtain

$$\theta_{N,d+1,\alpha,\boldsymbol{\gamma}}(z) = 2\gamma_{d+1}\zeta(\alpha)N^{-\alpha}(1 + e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\dots,\tilde{z}_d)) + \gamma_{d+1}\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z), \tag{3.55}$$

with

$$\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z) = \sum_{\substack{h_{d+1}\in\mathbb{Z} \\ N\nmid h_{d+1}}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d \\ \boldsymbol{h}\cdot(\tilde{z}_1,\ldots,\tilde{z}_d)\equiv -h_{d+1}z \pmod N}} |h_{d+1}|^{-\alpha}\, r_\alpha(\boldsymbol{\gamma},\boldsymbol{h}). \tag{3.56}$$

Thus we have

$$\begin{aligned} e^2_{d+1,N,\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d,z) &= e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d) + 2\gamma_{d+1}\zeta(\alpha)N^{-\alpha}(1+e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d)) + \gamma_{d+1}\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z) \\ &= (1+2\gamma_{d+1}\zeta(\alpha)N^{-\alpha})e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d) + 2\gamma_{d+1}\zeta(\alpha)N^{-\alpha} + \gamma_{d+1}\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z). \end{aligned} \tag{3.57}$$

Recall that we want to show

$$e^2_{d+1,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d,\tilde{z}_{d+1}) \le \left(\sum_{\mathfrak{u}\subseteq[d+1]} \frac{\gamma^\lambda_{\mathfrak{u}}(4\zeta(\alpha\lambda))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}}\{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}\right)^{\frac{1}{\lambda}} \tag{3.58}$$

for any $\lambda \in (\frac{1}{\alpha},1]$.

Now choose $\lambda^* \in (\frac{1}{\alpha},1]$ such that the right hand side of (3.58) is minimized as a function of λ. Applying Jensen's inequality to (3.57) we obtain

$$\begin{aligned} &(e^2_{d+1,N,\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d,z))^{\lambda^*} \\ &\quad\le (1+2\gamma_{d+1}\zeta(\alpha)N^{-\alpha})^{\lambda^*}(e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d))^{\lambda^*} + 2^{\lambda^*}\gamma^{\lambda^*}_{d+1}\zeta(\alpha)^{\lambda^*}N^{-\alpha\lambda^*} + \gamma^{\lambda^*}_{d+1}(\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z))^{\lambda^*} \\ &\quad\le (1+2^{\lambda^*}\gamma^{\lambda^*}_{d+1}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*})(e^2_{d,N\boldsymbol{\gamma}}(\tilde{z}_1,\ldots,\tilde{z}_d))^{\lambda^*} + 2^{\lambda^*}\gamma^{\lambda^*}_{d+1}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*} + \gamma^{\lambda^*}_{d+1}(\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z))^{\lambda^*}. \end{aligned} \tag{3.59}$$

Next we apply Jensen's inequality to (3.56) and find

$$\frac{1}{\left|\mathcal{Z}_{N,w_{d+1}}\right|}\sum_{l\in\mathcal{Z}_{N,w_{d+1}}} (\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(l))^{\lambda^*} \le \frac{1}{\left|\mathcal{Z}_{N,w_{d+1}}\right|}\sum_{l\in\mathcal{Z}_{N,w_{d+1}}} \kappa_{N,d+1,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}(l) =: \overline{\kappa}_{N,d+1,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}},$$

where we used the notation $\boldsymbol{\gamma}^{\lambda^*} = (\gamma_j^{\lambda^*})_{j\ge1}$.

In the following we use methods similar to [21, 22]. Recall that Markov's inequality states that for a non-negative random variable X with $\mathbb{E}(X) < \infty$ and any real number $c \ge 1$ we have $\mathbb{P}(X < c\mathbb{E}(X)) > 1 - \frac{1}{c}$. We use the normalized counting measure μ on $\mathcal{Z}_{N,w_{d+1}}$ as the probability measure and apply Markov's inequality as follows. For $c_{d+1} \ge 1$ let

$$\begin{aligned} G_{c_{d+1}} &:= \left\{z \in \mathcal{Z}_{N,w_{d+1}} \colon (\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z))^{\lambda^*} \le c_{d+1}\overline{\kappa}_{N,d+1,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}}\right\} \\ &\supseteq \left\{z \in \mathcal{Z}_{N,w_{d+1}} \colon (\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z))^{\lambda^*} \le \frac{c_{d+1}}{\left|\mathcal{Z}_{N,w_{d+1}}\right|}\sum_{l\in\mathcal{Z}_{N,w_{d+1}}}(\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(l))^{\lambda^*}\right\} =: A_{c_{d+1}}. \end{aligned}$$

Then Markov's inequality yields

$$\mu(G_{c_{d+1}}) = \frac{|G_{c_{d+1}}|}{\left|\mathcal{Z}_{N,w_{d+1}}\right|} \ge \mu(A_{c_{d+1}}) = \frac{|A_{c_{d+1}}|}{\left|\mathcal{Z}_{N,w_{d+1}}\right|} > 1 - \frac{1}{c_{d+1}},$$

that is, for any $c_{d+1} \ge 1$, there exists a subset $G_{c_{d+1}} \subseteq \mathcal{Z}_{N,w_{d+1}}$ of size strictly bigger than $\left|\mathcal{Z}_{N,w_{d+1}}\right|\left(1 - \frac{1}{c_{d+1}}\right)$, such that

$$(\kappa_{N,d+1,\alpha,\boldsymbol{\gamma}}(z))^{\lambda^*} \le c_{d+1}\overline{\kappa}_{N,d+1,\alpha\lambda^*,\boldsymbol{\gamma}^{\lambda^*}} \text{ for all } z \in G_{c_{d+1}}.$$

By choosing $c_{d+1} \geq 1$ such that

$$\left|\mathcal{Z}_{N,w_{d+1}}\right| \left(1 - \frac{1}{c_{d+1}}\right) = |\mathcal{E}_{d+1}|,$$

it is ensured that the set $G_{c_{d+1}} \setminus \mathcal{E}_{d+1}$ is not empty. Thus we have

$$c_{d+1} = \frac{\left|\mathcal{Z}_{N,w_{d+1}}\right|}{\left|\mathcal{Z}_{N,w_{d+1}}\right| - |\mathcal{E}_{d+1}|}.$$

As $\tilde{z}_{d+1}$ is chosen by Algorithm 3.37 such that the error $e^2_{d+1,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d, \tilde{z}_{d+1})$ is minimal, we obtain together with (3.59)

$$\begin{aligned}
&\Big(\ e^2_{d+1,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d, \tilde{z}_{d+1})\Big)^{\lambda^*} \\
&\quad \leq (1 + 2^{\lambda^*}\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*})(e^2_{d,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d))^{\lambda^*} + 2^{\lambda^*}\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*} + \gamma_{d+1}^{\lambda^*}\overline{\kappa}_{N,d+1,\alpha\lambda^*,\gamma^{\lambda^*}} \\
&\quad \leq (1 + c_{d+1}2\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*})(e^2_{d,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d))^{\lambda^*} + c_{d+1}2\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*} \\
&\qquad + c_{d+1}\gamma_{d+1}^{\lambda^*}\overline{\kappa}_{N,d+1,\alpha\lambda^*,\gamma^{\lambda^*}}.
\end{aligned} \tag{3.60}$$

Using the induction assumption with $\lambda = \lambda^*$, we obtain

$$\Big(e^2_{d,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d)\Big)^{\lambda^*} \leq \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}}\{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}. \tag{3.61}$$

Furthermore, from the proof of [7, Lemma 5], we obtain

$$\begin{aligned}
\overline{\kappa}_{N,d+1,\alpha\lambda^*,\gamma^{\lambda^*}} &\leq 2\zeta(\alpha\lambda^*)(1 - N^{-\alpha\lambda^*})\phi(N)^{-1} \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \gamma_{\mathfrak{u}}^{\lambda^*}(2\zeta(\alpha\lambda^*))^{|\mathfrak{u}|} \\
&\leq 2\zeta(\alpha\lambda^*)(1 - N^{-\alpha\lambda^*}) \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(2\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}} w_j\}})} \\
&\leq 2\zeta(\alpha\lambda^*)(1 - N^{-\alpha\lambda^*}) \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|}.
\end{aligned} \tag{3.62}$$

Plugging (3.61) and (3.62) into (3.60) we have

$$\begin{aligned}
&\Big(\ e^2_{d+1,N\gamma}(\tilde{z}_1, \ldots, \tilde{z}_d, \tilde{z}_{d+1})\Big)^{\lambda^*} \leq \\
&\quad \leq (1 + c_{d+1}2\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*}) \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
&\qquad + c_{d+1}2\gamma_{d+1}^{\lambda^*}\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*} \\
&\qquad + c_{d+1}\gamma_{d+1}^{\lambda^*}2\zeta(\alpha\lambda^*)(1 - N^{-\alpha\lambda^*}) \sum_{\emptyset\neq\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
&\quad = \sum_{\mathfrak{u}\subseteq[d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*}(4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0,m-\max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
&\qquad + c_{d+1}\gamma_{d+1}^{\lambda^*}2\zeta(\alpha\lambda^*)N^{-\alpha\lambda^*}\frac{1}{\phi(N)}
\end{aligned}$$

$$
\begin{aligned}
&+ c_{d+1}\gamma_{d+1}^{\lambda^*} 2\zeta(\alpha\lambda^*) N^{-\alpha\lambda^*} \\
&+ c_{d+1}\gamma_{d+1}^{\lambda^*} 2\zeta(\alpha\lambda^*) \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*} (4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0, m - \max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
\leq & \sum_{\mathfrak{u} \subseteq [d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*} (4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0, m - \max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
&+ \frac{\left|\mathcal{Z}_{N,w_{d+1}}\right|}{\left|\mathcal{Z}_{N,w_{d+1}}\right| - |\mathcal{E}_{d+1}|} 4\gamma_{d+1}^{\lambda^*} \zeta(\alpha\lambda^*) N^{-\alpha\lambda^*} \\
&+ \frac{\left|\mathcal{Z}_{N,w_{d+1}}\right|}{\left|\mathcal{Z}_{N,w_{d+1}}\right| - |\mathcal{E}_{d+1}|} 4\gamma_{d+1}^{\lambda^*} \zeta(\alpha\lambda^*) \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*} (4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0, m - \max_{j\in\mathfrak{u}} w_j\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|} \\
\leq & \sum_{\mathfrak{u} \subseteq [d+1]} \frac{\gamma_{\mathfrak{u}}^{\lambda^*} (4\zeta(\alpha\lambda^*))^{|\mathfrak{u}|}}{\phi(b^{\max\{0, m - \max_{j\in\mathfrak{u}} \{w_j\}\}})} \prod_{j\in\mathfrak{u}} \frac{\left|\mathcal{Z}_{N,w_j}\right|}{\left|\mathcal{Z}_{N,w_j}\right| - |\mathcal{E}_j|},
\end{aligned}
$$

as claimed. Thus the result holds for the special case of λ^*. As we have chosen λ^* such that the right hand side of (3.58) is minimized the estimate is true for arbitrary $\lambda \in (\frac{1}{\alpha}, 1]$ as well. □

Theorem 3.39 enables us to combine the reduced with the projection-corrected CBC construction, while still achieving a small worst-case error. The reduced fast CBC construction can be used here as well. Indeed, in this case one has to perform the additional step of checking whether a component is in the respective exclusion set. Hence, in the process of choosing component d one has to carry out $|\mathcal{E}_d|$ checks for exclusions at most, that is a total of at most

$$
\sum_{j=2}^{\min\{s,t_2\}} |\mathcal{E}_j| \leq \min\{s, t_2\} N
$$

checks for the entire process of finding a generating vector, where one only has to sum up to $j = \min\{s, t_2\}$ as for all subsequent steps the search space is reduced to $\{1\}$. Hence the overall complexity of the reduced fast CBC algorithm, which is (cf. [13])

$$
O\left(N \log N + \min\{s, t_2\} N + \sum_{j=1}^{\min\{s,t_2\}} (m - w_j) N b^{-w_d} \right),
$$

is not increased. This proves the following corollary.

Corollary 3.40. *Let $s \in \mathbb{N}$, $b \in \mathbb{P}$, $m \in \mathbb{N}$, $N = b^m$, $0 = w_1 \leq w_2 \leq \cdots$, and $\mathcal{Z}_N, \mathcal{Z}_{N,w_j}, t_1$ and t_2 as above. Then Algorithm 3.37 takes at most*

$$
O\left(N \log N + \min\{s, t_2\} N + \sum_{j=1}^{\min\{s,t_2\}} (m - w_j) N b^{-w_d} \right)
$$

steps to construct a generating vector $\tilde{\boldsymbol{z}}$ which satisfies the error bound of Theorem 3.39.

4 Conclusion and Outlook

The last section of this thesis consists of a brief summary of our main results as well as concluding remarks and ideas for further research projects.

In the first part of this thesis we studied tractability theory. In particular we considered two different settings for which we tried to find necessary and sufficient conditions for several tractability notions to hold.

In Section 2.2 we considered integration in a Hermite space of analytic functions and found necessary and sufficient conditions for SPT, as well as sufficient conditions for PT, QPT, UWT, (t_1, t_2)-WT and WT.

In Section 2.3 we studied a hybrid function space which is the tensor product of a Walsh and a Korobov space. For this space we found necessary and sufficient conditions for the standard tractability notions of $\mathbb{L}$-approximation using information from Λ^{all} and Λ^{std}, respectively.

Concerning the sections on tractability there remain two unresolved problems within close proximity of the problems studied in Sections 2.2 and 2.3.

To the author's best knowledge, necessary conditions for integration in the Hermite spaces for all tractability notions, except strong polynomial tractability, are yet unknown.

As for hybrid functions spaces, the problem of finding necessary conditions for the standard tractability notions for approximation in the Walsh spaces remains unresolved. Once these conditions are found one could complete the alternative approach to find necessary conditions in the hybrid function spaces, as described in Section 2.3.4. It is to be expected that these necessary conditions would match the sufficient conditions already found.

These problems remain to be solved.

In the second part of this thesis we studied the construction of (polynomial) lattice point sets as sample points in QMC algorithms for integration.

In Section 3.2 we managed to apply the reduced fast CBC construction to finding lattice point sets with small weighted star discrepancy, while having a small computational cost. Previously this construction was used for finding lattice point sets with small worst-case error. The reduced CBC construction uses the fact that in weighted function spaces not all components of the generating vector have the same amount of influence on the quality of the corresponding lattice point set. The idea is to reduce the size of the search set for each component according to its importance.

In Section 3.3 we extended this to constructing polynomial lattice point sets with small weighted star discrepancy.

Finally, in Section 3.4 we studied the following problem. As numerical experiments of Kuo, Gantner and Schwab show, the components of generating vectors obtained from CBC constructions tend to have recurring components from some dimension onwards. In [25] Gantner and Schwab presented numerical experiments with a CBC construction which avoids such recurrences. In [10] Dick and Kritzer showed that a generating vector constructed with such an algorithm yields a lattice point set with good worst-case error. In this thesis we combined this construction with the reduced fast concept and found a CBC construction for lattice point sets free of recurring components with small worst-case error and small computational cost.

When it comes to construction of lattice point sets, there is one possible future project which we would like to describe in a greater detail. Recently Ebert, Leövey and Nuyens [23] have come up with a whole different approach to the problem of constructing lattice point sets.

In a CBC construction we determine the generating vector $\boldsymbol{z} = (z_1, \ldots, z_s)$ one component at a time. This means that we start with (z_1) and in each step of the algorithm we add one component of our generating vector until we end up with a full-size generating vector $\boldsymbol{z} = (z_1, \ldots, z_s)$. When adding the d-th component of the generating vector, we minimize the worst-case error of the d-dimensional integration problem to choose z_d.

Ebert, Leövey and Nuyens in contrast consider a successive coordinate search algorithm which works as follows. They choose an s-dimensional starting vector $\boldsymbol{z}^0 = (z_1^0, \ldots, z_s^0)$ and in each step of the algorithm one component of $\boldsymbol{z}^0$ is altered. In the first step z_1 is chosen as the minimizer of the s-dimensional worst-case integration error as a function of z_1^0 when all other components are fixed. Similarly in the d-th step z_d is chosen as

$$z_d = \operatorname{argmin}_{z \in \mathcal{Z}_s} e_{\mathcal{H}_s,N}(z_1, \ldots, z_{d-1}, z, z_{d+1}^0, \ldots, z_s^0).$$

This process terminates after s steps once $\boldsymbol{z} = (z_1, \ldots, z_s)$ has been chosen. The crucial point in this algorithm is how to choose the starting vector $\boldsymbol{z}^0$. It can be shown (cf. [23]) that the successive coordinate search algorithm and a CBC construction yield the same generating vector if the starting vector is the zero vector. Numerical experiments [23] show that the successive coordinate search algorithm provides better results then CBC constructions if the starting vector is good. (For exampe, one idea would be to choose a generating vector obtained by a CBC construction as starting vector in the successive coordinate search algorithm.)

As a possible future project one could now try to speed up the successive coordinate search algorithm by reducing the search spaces similarly as for the reduced fast CBC constructions presented in this thesis. The hope would be to find a fast algorithm which produces better results than CBC.

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Curriculum Vitae

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EDUCATION

2014-	Doctorate programme in natural sciences at JKU Linz
2012-2014	Master's programme in mathematics at the university of Salzburg
2008-2012	Bachelor's programme in mathematics at the university of Salzburg
2000-2008	High school in Salzburg
1996-2000	Elementary school in Oberndorf

AWARDS

2014	Hans-Stegbuchner award of the mathematics department of the university of Salzburg
2012	Scholarship for excellence of the university of Salzburg

TALKS

11/2016	Componentwise constructions of (polynomial) lattice point sets, Guest lecture, Salzburg, Austria
8/2016	Approximation in hybrid function spaces, IBC on the 70th anniversary of Henryk Woźniakowski, Będlewo, Poland
8/2016	On combined component-by-component constructions of lattice point sets, MCQMC 2016, Stanford, California
7/2016	Reduced fast component-by-component constructions of lattice point sets, 5th international conference on uniform distribution theory, Sopron, Hungary
6/2016	Multivariate approximation in hybrid function spaces, Workshop on discrepancy theory, Varenna, Italy
7/2015	A reduced fast component-by-component construction of lattice point sets with small weighted star discrepancy, Tenth IMACS seminar on Monte Carlo Methods, Linz, Austria
4/2015	On standard tractability notions of integration in Hermite spaces, Conference on Information-based complexity, Będlewo, Poland